Fictitious Currents as a Source of Electromagnetic Field

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Abstract

In this paper we show that, by modifying the topology of space-time in a small region of space, the vector potential for the electromagnetic field will assume a configuration which is either vanishing (not allowed by the uncertainty principle) or non conservative leading to a magnetic moment in space-time similar to the one of a charged particle. There are other examples in literature of such an effect (e.g. Aharonov-Bohm effect), however, this paper is an attempt to show that the topologies of compact 3D-manifolds behave like and can be mapped to particles.

Key Words: electrodynamics, gauge theory, magnetic moment, half spin particles.

1 Introduction

The Standard Model of particle physics is a very successful theory describing three out of the four known forces of nature. Its final formulation it is heavily based on the use of gauge fields. In gauge theories the Lagrangian of the system (i.e. its dynamics) does not change under local transformations acting in a region of space-time which is homeomorphic to an open set of $\mathbb{R}^2$.

However in the standard model, particle are point objects with no dimension being, as a matter of fact, space-time singularities with fields around them having sometimes infinite values. For example, the classical version of electric field around an electron goes to infinity as $1/r^2$, and in the Standard Models we start from the classical Lagrangian before quantizing.

In this paper we study what happens to a gauge field when we introduce a topological "singularity" in space, such that the space is not simply connected any more. The major result is that, depending on the topology of the "singularity", fictitious currents may arise as a manifestation of the inertia of the system in changing topology. In the example we studied, dealing only with the $U(1)$ symmetry of the Standard Model, these currents may be seen as a sources for the magnetic moment of particles. However, we believe that the same principle may be applied to the other symmetries of particle physics.

There are other examples in literature where topology induces electromagnetic measurable effects (e.g. Aharonov-Bohm effect). However, this paper is an attempt to show that the topologies of compact 3D-manifolds behave like and can be mapped to particles.

In this paper we are not saying that topology can explain the nature of particle. We are saying that there are some mathematical links between particle characteristics and topology of space that may be exploited in some particle theories such as QFT.

In section 2 we derive fictitious currents generated by a singularity in space this section gives a motivation for the paper but it is not essential to the final result and may be completely skipped by the reader interested in the main result. In sections 3 we show how magnetic moment of a particle can arise as a consequence of fictitious currents. This section contains the main result of the paper. In the sections 4 and 5 we give additional thoughts and conclusions.

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2 Inhomogeneous Maxwell's Equation - Fictitious Currents

The Maxwell's equations notoriously are:

$$\begin{aligned} I) \qquad & \partial_\mu F_{\nu\sigma} + \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} = 0 \\ II) \qquad & \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \end{aligned} \tag{1}$$

In this paragraph we derive the inhomogeneous Maxwell's equations (i.e. $J^\nu \neq 0$) from the homogeneous ones (i.e. $J^\nu = 0$) in an unusual way. What we find is nothing new. However, in the process of deriving the inhomogeneous Maxwell's equations, we give a possible new interpretation for the currents J^ν.

We start from the Lagrangian density of the electromagnetic field in units where $\mu_0 = 1$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_s^\nu A_\nu \tag{2}$$

where J_s^ν is the source four-vector of real current, and where $F_{\mu\nu}$ is equal to:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

and it is the electromagnetic strength tensor. Finally, in the above equation A_μ, is the four-potential. Moreover, we know that $F_{\mu\nu}$ is associated to a gauge field θ and that its Lagrangian is invariant with respect to the symmetry:

$$A_\mu \to A_\mu + \partial_\mu \theta \tag{4}$$

where $\theta(x^\mu)$ is any smooth function in a simply connected open region of space-time Ω.

In the general case, the Lagrangian density $\mathcal{L}(A, \partial A)$, given by Eq. (2), depends on both the four-vector potential and its derivatives. We will consider now the case of the free electromagnetic field (i.e. $J^\nu = 0$) in which the Lagrangian density $\mathcal{L}(\partial A)$ depends only on the derivatives and it can be written as (see Appendix A.1):

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{5}$$

Let $\Lambda_\mu = \partial_\mu \theta$ be the gradient of a smooth function in Ω. In this section we will apply the symmetry $A_\mu \to A_\mu + \Lambda_\mu$ in order to see what happens to the Lagrangian density.

For the first term in (5) we have;

$$\partial_\mu (A_\nu + \Lambda_\nu) \partial^\mu (A^\nu + \Lambda^\nu) = \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\mu \Lambda^\nu + \partial_\mu \Lambda_\nu \partial^\mu A^\nu + \overbrace{\partial_\mu \Lambda_\nu \partial^\mu \Lambda^\nu}^{\text{not needed}} \tag{6}$$

where the last term is not needed and can be omitted because we are interested in the equation of motion and that term does not depend on A_μ and therefore has not effect on the variation of the action with respect to the fields.

For the second term we have:

$$\partial_\nu (A_\mu + \Lambda_\mu) \partial^\mu (A^\nu + \Lambda^\nu) = \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\mu \Lambda^\nu + \partial_\nu \Lambda_\mu \partial^\mu A^\nu + \overbrace{\partial_\nu \Lambda_\mu \partial^\mu \Lambda^\nu}^{\text{not needed}} \tag{7}$$

where the last term once again can be omitted if we are interested in the equation of motion. Putting the two equation above back together, swapping some terms and rearranging the names of dummy indices of the third term in parenthesis below, we have:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu \partial^\mu \Lambda^\nu + \partial^\mu A^\nu \partial_\mu \Lambda_\nu - \partial_\mu A_\nu \partial^\nu \Lambda^\mu - \partial^\mu A^\nu \partial_\nu \Lambda_\mu) \tag{8}$$

Applying the Leibniz rule (i.e. $f'g' = (fg')' - fg''$) to the terms in parenthesis above, we have:

$$\partial_\alpha A_\beta \partial^\gamma \Lambda^\delta = \overbrace{\partial_\alpha (A_\beta \partial^\gamma \Lambda^\delta)}^{\text{not needed}} - A_\beta \partial_\alpha \partial^\gamma \Lambda^\delta \tag{9}$$

where in this case the first term is not needed, if we are interested in the equation of motion, because it is a total derivative and therefore it depends only on the value of the tensors on the boundary of Ω and has no effect on the variation of the action with respect to the fields. Eq. (8) becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu\partial_\mu\partial^\mu\Lambda^\nu - A^\nu\partial^\mu\partial_\mu\Lambda_\nu + A_\nu\partial_\mu\partial^\nu\Lambda^\mu + A^\nu\partial^\mu\partial_\nu\Lambda_\mu) \tag{10}$$

If we are in nice flat Minkowski space (or if we use covariant derivatives) we can raise and lower indices at will, also on the derivative symbols because the metric commutes with the derivatives. We have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu\partial_\mu\partial^\mu\Lambda^\nu - A_\nu\partial_\mu\partial^\mu\Lambda^\nu + A_\nu\partial_\mu\partial^\nu\Lambda^\mu + A_\nu\partial_\mu\partial^\nu\Lambda^\mu) \tag{11}$$

and eventually:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\nu \overbrace{\partial_\mu(\partial^\nu\Lambda^\mu - \partial^\mu\Lambda^\nu)}^{\text{we call this term } J^\nu} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\nu A_\nu = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\nu A^\nu \end{aligned} \tag{12}$$

In the last equation we have put the J in both contravariant and covariant form where the contravariant form is given by:

$$J^\nu = \partial_\mu(\partial^\nu\Lambda^\mu - \partial^\mu\Lambda^\nu) \tag{13}$$

and the covariant form is given by:

$$J_\nu = \partial^\mu(\partial_\nu\Lambda_\mu - \partial_\mu\Lambda_\nu) \tag{14}$$

From the definition of $F^{\mu\nu}$ we see immediately that Eq. (13) can be written as:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{15}$$

where we have reintroduced μ_0, previously set to 1. We started from the homogeneous Maxwell's equations and we arrived to the inhomogeneous ones just by a shift of the gauge field.

We note immediately that, although Eq. (13) have been derived with the assumption of flat space, since they agree with the Maxwell's equations, we can conclude that they are valid also in curved space, provided that covariant derivatives are used instead of partial derivatives.

Moreover, we note that:

$$\mathcal{L}(\partial A) \neq \mathcal{L}(\partial(A + \Lambda)) \tag{16}$$

only because we have removed terms from the Lagrangian. Otherwise, the two Lagrangians would have been identical since we are applying a symmetry. However, the way we have removed the terms has left the equation of motion unchanged, and in fact, if Ω is simply connected, we have $J^\nu = 0$ and this restores the correctness of the equation of motion and leaves the field unchanged.

However, if Ω is not simply connected, there may be a field Λ, which corresponds to a vanishing electric and magnetic field but that could be equivalent to the presence in space of electromagnetic currents.

We have called these currents **Fictitious Currents** (Pseudo Currents). Note that these are electric fictitious currents not to be confused with the (unreal) magnetic fictitious currents sometimes used in computational electrodynamics as a trick for solving complex problems.

The reason why we call J^ν fictitious, is due to an analogy with the way we derive it and the way fictitious forces are derived in classical mechanics (see Appendix A.2).

The analogy is not perfect though. For classical mechanics systems, fictitious forces appear when we have a change in the symmetry during the evolution of the system. For the electromagnetic field, fictitious currents appear when we have a change of the gauge symmetry due to a change of space topology.

With abuse of terminology, we may say that fictitious currents are due to the inertia of the system in changing space topology.

3 A Two Dimensional Example

We will introduce the main idea of this paper with a two dimensional example. Let the manifold M be a real projective space $\mathbf{RP}^2$, and let Λ be a vector field on M. Since the Euler characteristic of M is $\chi = 1$, for the Poincare-Hopf theorem, the configurations of Λ with the minimum number of critical points will have a single critical point P with index of Λ at P equal to 1. We can always find a closed curve γ on M such that γ spits M in two manifolds with boundaries, Γ which contains P and it is homeomorphic to a disk and Ω which is homeomorphic to a Mobius strip.

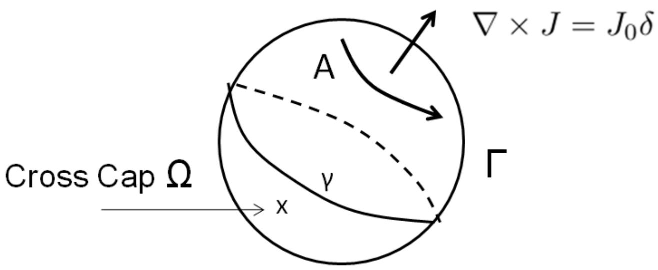

Figure 1: Manifold M

We have two possibilities for the configuration of Λ: the critical point P is a sink/source or P is a vortex. It is possible to show that the configuration where P is a sink/source is not possible and that P has to be a vortex. This is because the Poincare-Hopf theorem is a necessary condition for the field Λ to exist but it is not a sufficient condition. To show that the sink/source configuration is not possible, we suppose P to be a sink/source and we choose a curve $\gamma = \partial\Omega$ orthogonal to Λ at each point. We can see immediately that it is not possible to draw a vector field on Ω, which is a Mobius strip, which is orthogonal to the boundary and that has no critical points.

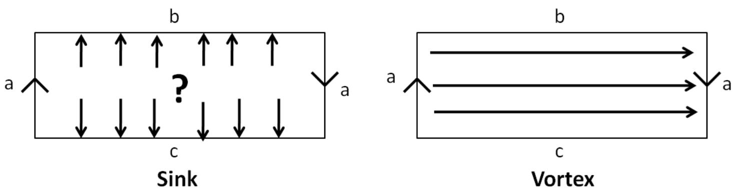

Figure 2: Field on a Mobius Strip

Furthermore, we define $M' = M \times \mathbb{R}$ to be a three-dimensional manifold representing space-time (this is a model with one time coordinate and only two space coordinates). We want the vector field Λ to be a symmetry of the free field Lagrangian as per Eq. (4) which is:

$$\Lambda = \partial_\mu \theta \tag{17}$$

Since M, with the point P removed, is not an open subset of $\mathbb{R}^2$, it is not possible to find a potential $\theta(t)$ for $\Lambda(t)$ which is single valued in M. However, we can always find a good cover for M, made of a finite number of open sets $\Omega_i \in M$, where we can find θ_i,θ_j such that $\Lambda = \partial_\mu \theta_i$ in Ω_i (and the same for j) and $\theta_i = \theta_j$ in $\Omega_i \cap \Omega_j$.

We suppose now that θ is independent from time and we go back to a two dimensional problem. We know that, since Λ is the gradient of θ, it has vanishing curl in $M \backslash \{P\}$ and it has discrete curl in P. It can be therefore represented by a Dirac delta function on M as: $\nabla \times \Lambda = -\Lambda_0 \delta(x - P)$, where the minus sign is just a convention since Λ_0 can have any sign. For the Stoke's theorem, its line integral along the boundary of Γ which is $\partial\Gamma = \gamma$ is equal to:

$$\oint_{\partial\Gamma} \Lambda dl = \int_{\Gamma} (\nabla \times \Lambda) ds = -\Lambda_0 \tag{18}$$

The line integral on the boundary $\partial\Omega = -\partial\Gamma = -\gamma$ of Ω is:

$$\oint_{\partial\Omega} \Lambda dl = -\oint_{\partial\Gamma} \Lambda dl = \Lambda_0 \tag{19}$$

and this is although the integral of $\nabla \times \Lambda$ on Ω is vanishing. This is because Ω is not orientable and the Stokes's theorem does not work unless it is reformulated in terms of differential forms.

3.1 Equivalent Currents for Vector Potential

In this section and in the sections below we will use the following notation: Λ is the vector potential in all space, $\bar{\Lambda}$ is the vector potential Λ which is set to zero inside the non simply connected space (Ω or N defined below) and $\tilde{\Lambda}$ is the special part of the vector potential $\bar{\Lambda}$. This notation applies also for other objects like F and J.

We give the following proposition:

Proposition. *Let N be a 3-dimensional compact subset of $\mathbb{R}^3$, $N' \times \mathbb{R}$ a subset of space-time, A the 4-vector potential for a solution of the Maxwell's equations for which the currents J^μ are all inside N' and $u(x^\mu)$ the following function:*

$$u(x) = \begin{cases} 0 & for\, x \in N' \\ 1 & for\, x \notin N' \end{cases} \tag{20}$$

We define also $\bar{A}(x) = u(x)A(x)$ to be a four-vector potential which vanishes inside N', F to be the field strength tensor associated to A and $\bar{F}$ the field strength tensor associated with $\bar{A}$.

With the above definitions we have then the two four-currents:

$$J^\nu = -\frac{1}{\mu_0}\partial_\mu F^{\mu\nu} \tag{21}$$

and

$$\bar{J}^\nu = -\frac{1}{\mu_0}\partial_\mu \bar{F}^{\mu\nu} \tag{22}$$

are equivalent meaning they generate the same electromagnetic field. Note that $\bar{J}$ and $\bar{F}$ have to be evaluated in the space of generalised functions (e.g. Dirac delta functions) since $\bar{A}$ is a step discontinuous tensors.

Proof. Once we accept that the solutions of Maxwell's equations can be found in the space of generalised function, the proof of the proposition is trivial. Most of us recognise that this is the case, but many find themselves uncomfortable with it and prefer to use boundary conditions for discontinuous fields and surface currents to take into account of the discontinuity. However, this would make the statement and proof of this proposition more complex.

To prove the proposition we note that A and $\bar{A}$ give the same electromagnetic field F outside N' since they are identical. Moreover, $\bar{J}$ vanish outside N' because F and $\bar{F}$ are identical and F is the solution for a problem without currents outside N'. Finally, $\bar{J}$ vanish inside N' because $\bar{A} = 0$. The only place where $\bar{J} \neq 0$ is on the boundary of N' where $\bar{J}$ are described by Dirac delta functions.

3.2 Magnetic Monopole of a Half-Spin Particle

Now, from our non simply connected manifold $M = \Omega \cup \Gamma$ with a singular point $P \in \Gamma$ for the field Λ, defined in one of the sections above, we want Γ to be a small disk, with boundary γ, around the singular point P (the vertex for Λ) and we remove it from M. We are left with Ω (which is a Mobius strip) and we stretch it on a plane $\mathbb{R}^2$ with $\partial\Omega$ that goes to infinity and the region where Ω is not homeomorphic to $\mathbb{R}^2$ confined in a tiny small region around the origin of $\mathbb{R}^2$. By doing so, Ω becomes a flat space identified with $\mathbb{R}^2$ almost everywhere apart from a small region around the origin. At the same time, we want Λ still be a solution of the free field (no currents) Maxwell's equations in Ω.

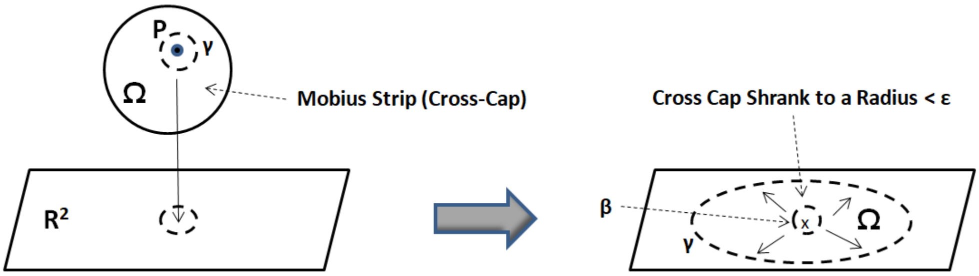

Figure 3: Half-Spin Particle Configuration

We want to show now that the above configuration of space-time produces electromagnet fields similar to the one of a half-spin particle. We are not saying that in reality a particle is a compact 3-dimensional manifold concentrated in a point. We are saying that, some topological configurations of space-time happen to have mathematical connections with classical fields produced by particle, and this is at the very least an interesting fact to study.

Let β be a small circle of radius ϵ around the topological "singularity" concentrated in the origin which, as described above, is a Mobius strip attached by direct sum to a flat plane. In QFT, the gauge field $\theta(x)$ of a half-spin particle is the phase of the particle field $\psi = \psi_0^{j\theta(x,t)}$. The field Λ may be vanishing on Ω, which is not allowed by the uncertainty principle, or non vanishing. In the latter case, we know from Eq. (19) that the circulation of Λ along β, equal to the circulation of Λ along γ, is non vanishing. In particular, since the removed singular point had index 1 for Λ, then the field will complete a full circle on the Gauss map going around β and, if ϵ is big with respect to the dimension of the cross cap, we may suppose Λ to be constant in module.

In polar coordinates, we can write the value of θ on β as:

$$\lim_{r\to\epsilon} \theta(r, \phi, z) = \phi + \theta_0 \tag{23}$$

and it is worth to note that θ is a multivalued function.

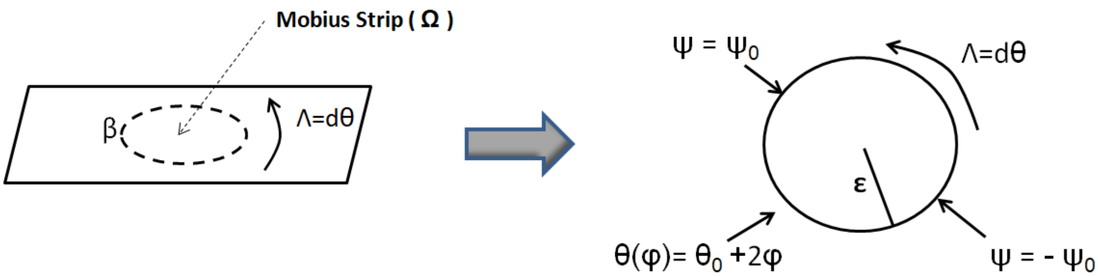

Figure 4: Boundary Conditions in the Singularity

Since now we know θ on β, we can easily evaluate $\Lambda = d\theta$, (which we can assume independent from the time variable) on it. Using Eq. (19) we can also evaluate $\bar{\Lambda}$ in Ω outside β (meaning $r > \epsilon$), which is:

$$\bar{\Lambda} = \left(\frac{1}{r}u(r-\epsilon)\right)\hat{i}_\phi \tag{24}$$

We have now all we need to evaluate the fictitious currents induced by the topology of Ω. If we call $\tilde{\Lambda}$ the special components of $\bar{\Lambda}$, from the Maxwell's equations we have that $\nabla \times \tilde{\Lambda} = B$ and $\nabla \times B = \mu_0 J$, in the static case). Putting all together we have:

$$\bar{J} = \frac{1}{\mu_0}\nabla \times \nabla \times \tilde{\Lambda} \tag{25}$$

We can evaluate the curls adding a third special coordinate with the fields independent from it. Using Eq. (49), we have:

$$\bar{J} = \nabla \times \nabla \times \tilde{\Lambda} = \nabla \times \nabla \times \left(\frac{1}{\mu_0 r} u(r-\epsilon) \hat{i}_\phi \right) = \frac{1}{\mu_0} \mathring{\delta}_{z,\epsilon}(x,y) \quad [A][m^{-2}] \tag{26}$$

where the definition of $\mathring{\delta}$ is given in Appendix A.3. In the above equation we have taken into account that $\hat{\Lambda}$ has the same units of A^μ ($[kg][m][s^{-2}][A^{-1}]$). Moreover, each curl adds an $[m^{-1}]$ units, the field $\mathring{\delta}_t$ has units of $[m^{-3}]$ and the quantity $1/\mu_0$ has units of $[A][m]$. The above equation is a new nice interpretation of μ_0 in terms of the currents circulating around a topological non trivial space.

The above equation represents a current circulating around β. Its units are wrong because the example is in 2-dimension. This current generates a magnetic moment m, independent from ϵ and, again with wrong units, given by Eq. (50). We have:

$$m = \frac{2\pi}{\mu_0} \;\; [A][m^{-2}] \tag{27}$$

3.3 Charge of One Half Spin Particle

We want now to see if the same topological configuration induces also electric charge. We need to introduce the time component. Let $t = 0$ be the time when the particle is created. Before $t = 0$ space is just regular $\mathbb{R}^2$ flat space. After $t = 0$ we have a tiny Mobius strip attached to the origin by means of a direct sum. The gauge filed can be written as follows:

$$\theta = \theta(x)u(t) \tag{28}$$

Which gives the following vector potential:

$$\bar{\Lambda} = \partial_\mu \theta = \left(-\frac{\Phi}{c} \; , \; \tilde{\Lambda} \right) = \left(-\theta(x)\delta(t) \; , \; \frac{1}{r} u(r-\epsilon)u(t)\hat{i}_\phi \right) \tag{29}$$

from which we can evaluate the electric field:

$$E = -\nabla\Phi - \frac{\partial\tilde{\Lambda}}{\partial t} = -\frac{2}{r} u(r-\epsilon)\delta(t)\hat{i}_\phi \tag{30}$$

As before, to proceed we need to introduce a third space component and set the value of the fields to be independent from it. The expression of the divergence in cylindrical coordinates is:

$$\nabla \cdot E = \frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \tag{31}$$

from which we have:

$$\rho = \epsilon_0 \nabla \cdot E = 0 \tag{32}$$

We conclude that the model does not work for the charge.

4 The Three Dimensional Model

We can easily make an example with three special dimensions. The example will go exactly as for the two dimension. We start from a compact three-dimensional manifold with Euler characteristic $\chi = 1$. We see that only the configuration with a vortex (or the equivalent in three dimensions) is allowed. We open the manifold in $\mathbb{R}^3$ keeping the non simply connected part of the manifold in a small region around the origin. And we find that the configuration gives a magnetic moment. The only difference with the two-dimensional case are the calculations which are slightly more complex and that are left to the reader.

5 Conclusions

In this paper, we have shown that a sufficient condition to have magnetic moment in a point in space is to introduce a topological "singularity" (i.e. small region of non simply connected space) designed to twist the fields in a specific way. Although the paper has been dealing with the $U(1)$ symmetry only, the very same approach may be used to explore what happens with the other two symmetries of the Standard Model. We believe that it is worth to further research in order to trying to match particle characteristics to topological characteristics of space "singularity" (i.e. 3D compact manifolds connected by direct sum to space) with respect to the three symmetries of QFT, in an effort to make a one-to-one correspondence between particles and manifolds (or topological singularities).

Although we know that this approach is very unlikely to be a theory that fully describe particles, however, there is also a chance that things may partially match, end this would allow to exploit the huge variety of 3D-compact manifold to explain some of the complex characteristics of particles.

Appendix

A.1 Lagrangian Density of Electromagnetic Field

Given the Lagrangian density of the free electromagnetic field:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{33}$$

we have:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) && (34) \\
&= \frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\mu A_\nu \partial^\mu A^\nu) && (35)
\end{aligned}$$

the above terms are equal in pairs, we have:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{36}$$

A.2 Fictitious Forces

Given the Lagrangian of a one variable discrete system:

$$L = L(q, \dot{q}) \tag{37}$$

having the following symmetry:

$$q \to q + \phi \;\Rightarrow\; \Delta L = 0 \tag{38}$$

then from the Noether's theorem we know that the quantity $\frac{\partial L}{\partial \dot{q}}$ is conserved.

The above quantity is conserved when we let the system evolve without applying the symmetry to it. However, if we apply the symmetry by changing some symmetry parameter (e.g. for a mechanical system this parameter may be position) while the system is evolving (e.g. for a mechanical system it may correspond to a shift of the whole system in space), then $\frac{\partial L}{\partial \dot{q}}$ is not a conserved quantity any more. This physically corresponds to having fictitious forces (pseudo forces) in the system depending from the way we change the symmetry parameter as a function of time.

To address the above case, we need to change the Lagrangian in order to take into account the dependency from symmetry parameters (see [2]). To illustrate that, we will use the same example of [2], where it is shown how to get a new Lagrangian just with a change of coordinates from q_0, the coordinate of the inertial system, to $q = q_0 + \phi$, the coordinate of the non inertial one:

Given the Lagrangian of a particle in an external field:

$$L = \frac{1}{2} m (\dot{x}_0)^2 - U(x_0) \tag{39}$$

where x_0 is the coordinate in the inertial frame and x is the coordinate of the moving frame with velocity $\dot{\phi}$, we consider the change of variable $x_0 = x + \phi(t)$. We have:

$$L'(x + \phi, \dot{x} + \dot{\phi}) = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{\phi} + \frac{1}{2}m\dot{\phi}^2 - U(x) \tag{40}$$

The term $\frac{1}{2}m\dot{\phi}$ does not depends on x, gives no contribution to $\frac{\partial \mathcal{S}}{\partial q}$ and, if we are interested to the equation of motion, it can be dropped. Regarding the term $m\dot{x}\dot{\phi}$, using the Leibniz rule we have:

$$m\dot{x}\dot{\phi} = m\frac{d}{dt}(x\dot{\phi}) - mx\ddot{\phi} \tag{41}$$

the term $m\frac{d}{dt}(x\dot{\phi})$ is the derivative of a function. Its contribution to the action depends only on its value at the two ends of the time integral. Once again, it gives no contribution to $\frac{\partial \mathcal{S}}{\partial q}$ and it can be dropped. We are left with the following Lagrangian:

$$L' = \frac{1}{2}m\dot{x}^2 - mx\ddot{\phi} - U(x) \tag{42}$$

which, by using the Euler-Lagrange equation, gives the following equation of motion:

$$m\ddot{x} = -\frac{\partial U}{\partial x} - m\ddot{\phi} \tag{43}$$

The term $m\ddot{\phi}$ is what we call a fictitious force (pseudo force) and it is a force experienced by the particle m because it is in a non inertial reference frame.

A.3 Definition of the $\mathring{\delta}$ Function

Let $\tilde{\Lambda}(r, \phi, z)$ be the following vector field in the (r, ϕ, z) cylindrical coordinates:

$$\tilde{\Lambda} = \frac{1}{r}u(r - \epsilon)\hat{i}_\phi \tag{44}$$

where $u(r)$ is the Heaviside unitary step discontinuous function, $\hat{i}_\phi$ is the unitary vector (versor) in the ϕ coordinate direction and ϵ is any positive real number.

Using the expression of the curl in cylindrical coordinates we have:

$$\nabla \times \hat{\Lambda} = \frac{1}{r}\delta(r - \epsilon)\hat{i}_z \tag{45}$$

Given Eq. (45) and using the expression of the curl in cylindrical coordinates we have:

$$\nabla \times (\nabla \times \hat{\Lambda}) = \left(\frac{1}{r^2}\delta(r - \epsilon) - \frac{1}{r}\delta'(r - \epsilon)\right)\hat{i}_\phi = \frac{2}{r^2}\delta(r - \epsilon)\hat{i}_\phi \tag{46}$$

Where the above is true because, although it is hard to believe, we have that:

$$\frac{1}{r^2}\delta(r - \epsilon) \approx -\frac{1}{r}\delta'(r - \epsilon) \tag{47}$$

Where "$\approx$" is the association relation in a Colombeau algebra that, in our case, can be understood by all means as the "$=$" symbol (see [4]). We will prove Eq. (47) at the end of the section. For the time being, inspired by Eq. (45), we define the field $\mathring{\delta}_{z,\epsilon}$ as follows:

$$\mathring{\delta}_{z,\epsilon}(x, y) = \frac{2}{\epsilon^2}\delta(r - \epsilon)\hat{i}_\phi \tag{48}$$

which is a field circulating on the (x, y) plane, discrete with respect to the distance from the z axis and present on a circle centred in the origin and having radius ϵ. From Eq. (46) and Eq. (48) we have:

$$\nabla \times \nabla \times \left(\frac{1}{r}u(r - \epsilon)\hat{i}_z\right) = \mathring{\delta}_{z,\epsilon}(x, y) \tag{49}$$

We want now to give some physical interpretation of $\mathring{\delta}_{z,\epsilon}$. Suppose $J_0\mathring{\delta}_{z,\epsilon}$ represents a current circulating around the z axis. We want to evaluate the magnetic moment m generated by this current. The magnetic moment of a single coil of a solenoid is given by the intensity of the current multiplied by the area enclosed by the coil. Given Eq. (48), we have that the magnetic moment per unit length along z is given by:

$$m = \pi\epsilon^2 J_0 \frac{2}{\epsilon^2}|\delta(r-\epsilon)|\hat{i}_z = 2\pi J_0 |\mathring{\delta}_{z,\epsilon}(x,y)|\hat{i}_z \tag{50}$$

where with $|\delta(r-\epsilon)|$, we mean the amplitude of the δ. We note that the above results is independent from ϵ and therefore we may say that a $\mathring{\delta}_z(x,y)$ is a distribution of current, as close to the z axis as we like (i.e. $\epsilon \to 0$), and giving a magnetic moment of 2π.

We finish this paragraph by proving Eq. (47). Let $g(r)$ be a smooth function with $g(r) = 0$ for $r < 0$, $\lim_{r\to\infty} rg(r) = 0$ and such that:

$$\int_0^\infty g(r)dr = 1 \quad \Rightarrow \quad \begin{cases} \delta(r-\epsilon) = \lim_{n\to\infty} ng(nr-\epsilon) \\ \\ \delta'(r-\epsilon) = \lim_{n\to\infty} n^2 g'(nr-\epsilon) \end{cases} \tag{51}$$

Let $T(r)$ be the generalised function $\frac{1}{r}\delta'(r-\epsilon)$. We have:

$$r^2 T(r) = r^2 \lim_{n\to\infty} \frac{1}{r} n^2 g'(nr-\epsilon) = \lim_{n\to\infty} n[(nr)g'(nr-\epsilon)] \tag{52}$$

Clearly $r^2T(r)$ is a delta function $A\delta(r-\epsilon)$ centred in ϵ, where the amplitude A is given by:

$$A = \int_\epsilon^\infty rg'(r-\epsilon)dr = -\int_\epsilon^\infty g(r-\epsilon)dr = 1 \tag{53}$$

where we have used integration by parts. This proves Eq. (47) because we have proven that:

$$r^2 T \approx r^2 \frac{1}{r}\delta'(r-\epsilon) \approx -r^2 \frac{1}{r^2}\delta(r-\epsilon) \tag{54}$$

where the above equation has to be understood in terms of association relation in a Colombeau algebra (see [4]).

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