Cantor's illusion
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abstract
This analysis shows Cantor's diagonal argument published in 1891 neglected the effects of orientation of a sequence, and did not prove the cardinality of his infinite set $M$ is greater than the set of integers $N$.

the argument
Translation from Cantor's 1891 paper [1]:

Namely, let $m$ and $n$ be two different characters, and consider a set $[\text{Inbegriff}] M$ of elements

$$E = (x_1, x_2, \ldots, x_v, \ldots)$$

which depend on infinitely many coordinates $x_1, x_2, \ldots, x_v, \ldots$, and where each of the coordinates is either $m$ or $w$. Let $M$ be the totality $[\text{Gesamtheit}]$ of all elements $E$.

To the elements of $M$ belong e.g. the following three:

$$E^I = (m, m, m, m, \ldots),$$
$$E^{II} = (w, w, w, w, \ldots),$$
$$E^{III} = (m, w, m, w, \ldots).$$

I maintain now that such a manifold $[\text{Mannigfaltigkeit}] M$ does not have the power of the series $1, 2, 3, \ldots, v, \ldots$.

This follows from the following proposition:
"If $E_1, E_2, \ldots, E_v, \ldots$ is any simply infinite $[\text{einfach unendliche}]$ series of elements of the manifold $M$, then there always exists an element $E_0$ of $M$, which cannot be connected with any element $E_v$.

For proof, let there be

$$E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, \ldots)$$
$$E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots)$$
$$E_u = (a_{u,1}, a_{u,2}, \ldots, a_{u,v}, \ldots)$$

..............................

where the characters $a_{uv}$ are either $m$ or $w$. Then there is a series $b_1, b_2, \ldots b_v, \ldots$, defined so that $b_v$ is also equal to $m$ or $w$ but is different from $a_{uv}$.

Thus, if $a_{uv} = m$, then $b_v = w$.

Then consider the element

$$E_0 = (b_1, b_2, b_3, \ldots)$$
of $M$, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer $u$, otherwise for that $u$ and for all values of $v$.

$$b_v = a_{u,v}$$

and so we would in particular have

$$b_u = a_{u,u}$$

which through the definition of $b$, is impossible. From this proposition it follows immediately that the totality of all elements of $M$ cannot be put into the sequence $\textit{Reihenform}$: $E_1, E_2, \ldots, E_v, \ldots$ otherwise we would have the contradiction, that a thing $\textit{Ding}$ $E_0$ would be both an element of $M$, but also not an element of $M$.

(end of translation)

**the list**

A sequence is defined as a one dimensional pattern of characters from the set $\{m, w\}$. Each sequence occupies one row $u$. The character $m$ or $w$, for each position $v$ is determined by a random process such as a coin toss. Each sequence extends horizontally and is infinite in length, having a first character but no last character. The list is a visual aid to comprehend the properties of the infinite set $M$ as envisioned by Cantor. The list extends vertically and is infinite in length, having a first row but no last row. Cantor presents the list as an infinite array of characters, each with a unique $(u, v)$ coordinate location, with $u$ and $v$ from the set of integers $\mathbb{N}$.

**the diagonal form**

![fig.1](image-url)
Using the initial A-list in fig.1, Cantor selects the six blue characters with coordinates (1, 1) to (6, 6) as part of his definition of the b series. They are now all characters for a single sequence b as shown in the B-list. The negation of a sequence requires all characters differ. The negation of b (u20) is shown as E₀ (u21). The sequences in the B-list differ from those in the A-list only by their rotation of 45°, extending in two dimensions.

\[
\begin{array}{ccccccc}
 & & & & & & \\
\text{A2-list} & v & & & & & \\
1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
1 & m & m & w & w & m & w \\
2 & w & w & m & m & w & m \\
3 & & & & & & \\
4 & & & & & & \\
5 & & & & & & \\
6 & & & & & & \\
\vdots & & & & & & \\
\end{array}
\]

\[\text{fig.2}\]

If the sequences in the B-list are rotated ccw 45°, they are aligned with the (u, v) coordinate grid, and transformed to an A type list A2 as shown in fig.2. Since they are parallel in both the B-list and A2, they can coexist in either form.

**Conclusion**

Cantor states E₀ is excluded from the A-list as a horizontal sequence since it will differ from each diagonal element. This would be obvious if he substituted E₀ for b, without any rotation applied. By definition, in the A-list the coordinates for Eu are (u, 1), (u, 2), (u, 3), (u, 4), (u, 5), ...

All sequences are parallel and coexist with no intersections and no contradictions. The problem originates when Cantor defines the diagonal b from the A-list. Coordinates for b are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), ...

Since the A-list is complete before b is defined, all coordinates are already assigned to specific sequences of the A-list. No coordinate can have multiple characters. The sequence b exists in the A-list but would not be detected since its character at the intersection with the diagonal would not differ. Cantor’s method does not provide a means of comparing positions for duplication. The b sequence would not be new.

The fact of complementary sequences b and E₀ argues against the exclusion. If one exists both exist. The simplest example being (mmm...) and (www...).

The comparing of sequences in the A-list with different orientations is the cause of the apparent exclusion of E₀. In a truly random list, all sequences are independent of each other, and there is no factor that imposes any degree of order. The contradiction he describes is of his own making. He ignores the factor of orientation and complementarity. The rotation does not alter the order of the characters within a sequence, so the sequence retains its identity, but it does allow an interaction of sequences which can impose an order which restricts their location within a list. All sequences are independent within their respective
lists, horizontal A type, or diagonal B type. His contradiction only has meaning in terms of subsets $M_m$ (all sequences beginning with $m$), and $M_w$ (all sequences beginning with $w$). Sequence $b$ is a member of $M_m$, while $E_0$ is a member of $M_w$ but not a member of $M_m$, which is not a contradiction.

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