A direct, simple, and basic computation of a difference of two dilogarithms

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Abstract

The computation of $\text{Li}_2(\sqrt{2} - 1) - \text{Li}_2(1 - \sqrt{2})$ is performed.

Introduction

Several papers ([1] [2] [3]) contain the proof that,

$$\text{Li}_2(\sqrt{2} - 1) - \text{Li}_2(1 - \sqrt{2}) = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(\sqrt{2} - 1)$$

Following is a direct, simple, and basic proof of this equality.

Proof:

Let,

$$\alpha = \sqrt{2} - 1$$

Observe that,

$$\frac{1 - \alpha}{1 + \alpha} = \alpha$$

Since, for $u$ real,

$$\text{Li}_2(u) = -\int_0^u \frac{\ln(1 - t)}{t} dt$$

1
Then,

\[
J = \text{Li}_2(\alpha) - \text{Li}_2(-\alpha)
\]

\[
= -\int_0^\alpha \frac{\ln(1-t)}{t} \, dt + \int_0^{-\alpha} \frac{\ln(1-t)}{t} \, dt
\]

\[
= -\int_0^\alpha \frac{\ln(1-t)}{t} \, dt - \int_0^{-\alpha} \frac{\ln(1-t)}{t} \, dt
\]

\[
= -\int_0^\alpha \frac{\ln(1-t)}{t} \, dt + \int_0^\alpha \frac{\ln(1+w)}{w} \, dw
\]

\[
= -\int_0^\alpha \frac{\ln\left(\frac{1-t}{1+t}\right)}{t} \, dt = -2\int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{1-z^2} \, dz
\]

\[
J = -2\int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{1-z^2} \, dz + 2\int_0^\alpha \frac{\ln\left(\frac{1-z}{1+z}\right)}{1-z^2} \, dz
\]

\[
= \left[\ln\left(\frac{1-z}{1+z}\right) \ln z\right]_0^1 - \int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} \, dz - \left[\ln\left(\frac{1-z}{1+z}\right) \ln z\right]_0^\alpha
\]

\[
\int_0^\alpha \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} \, dz = -J
\]

\[
= -\frac{1}{2} \int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} \, dz - \frac{1}{2} \ln^2 \alpha
\]

\[
= \frac{\text{Li}_2(1)}{2} + \frac{1}{2} \left(\int_0^1 \frac{\ln(1-z^2)}{z} \, dz + \text{Li}_2(1)\right) - \frac{1}{2} \ln^2 \alpha
\]

\[
= \frac{3\text{Li}_2(1)}{4} - \frac{1}{2} \ln^2 \alpha = \frac{\pi^2}{8} - \frac{1}{2} \ln^2 \alpha
\]

NB: I assume that,

\[
\text{Li}_2(1) = \frac{\pi^2}{6}
\]
References

