Proof of the Collatz conjecture,

3x+1

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Abstract

We will proof that the 3x+1 conjecture is true, using modular arithmetic and a new approach based on an ancient symbol THE ENNEAGRAMMA. We will show that for every integer n, $n \equiv 1 \pmod 2$ if and only if $3n+1 \equiv 4 \pmod 6$. With the help of directed graphs, flow and block diagrams we will find 1 equation which, applying the 2 conditions, links all the odd numbers and consequently the positive integers to the powers of 2. We will find the analytical equation of the function. We will show how "numerical gravity" arises from the deterministic divisibility that the combinations of integers allow. We will go up the Collatz graph represented by the inverse function which forms a tree with the exception of the cycle 1-4-2-1… We will show how all positive integers are present in the tree, that is connected to the number 1, making extensive use of graphs, tables and colors to represent the beauty of mathematics. We will follow the exact chronology of the insights. Careful observation of the numbers will return an elementary (-a)rithmetic (double logical negation equals affirmation). We will not omit steps that are obvious, since these are the substrate on which the approach is based. We hope you can appreciate the extreme simplicity, harmony and rhythm that the numbers manifest.

1 Introduction

Let's analyse the algorithm algebraically:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is EVEN,} \\ 3n+1, & \text{if n is ODD.} \end{cases}$$

The constituent elements will be: 1,2,3,4,n, the distinction between EVEN and ODD numbers, the powers of 2 and modular arithmetic.
It is evident that the second condition, thanks to +1, "forces" the ODD numbers to become EVEN in order to make them divisible.
Inverse function:

$$R(n) = \begin{cases} \left\{2n, \frac{n-1}{3}\right\}, & \text{if } n \equiv 4 \pmod 6, \\ \left\{ 2n \right\}, & \text{otherwise.} \end{cases}$$

Definition 1.1. Methodological criterion assumed. The observation of the numbers that arise from what has been introduced will lead us to detect repetitive patterns that will be formalized through equations. We will demonstrate by induction the properties stated using directed graphs which will promptly report the data resulting from the reference equations and the numerical sequences used. Expressed the inductive hypothesis, represented by the first scheme of the directed graph, we will identify the mathematical relationship that links the subsequent inductive steps to the inductive basis, highlighting how the basic scheme is perpetuated ad infinitum. In the following directed graphs, we will sometimes omit repeating the arrows that indicate the direction of flow so as not to burden perception, considering them existing given their repetitiveness in the various schemes.

2 Modular arithmetic

By placing integers in an array according to a given module, they will be arranged in columns with peculiar characteristics that will return useful information. We will use this method to understand the 3x+1 algorithm.

Statement 2.1.

2.1.1. All positive integers are representable using any modulo.

2.1.2. We report the reflective property of congruencies: every number is congruent to itself modulus n, for every n different from 0. $a \equiv a \pmod{n}, \quad \forall a \in \mathbb{N}, \ \forall n \in \mathbb{N}_{>0}$ (set $\mathbb{N}$ excluding 0).
Proof: we have a-a=0 and every non-zero integer is a divisor of 0. So n divides a-a.
$a = a \ \Rightarrow \ a \equiv a \pmod{n}, \quad \forall a \in \mathbb{N}, \ \forall n \in \mathbb{N}_{>0}$

Statement 2.2. **Module 2**: distinguishes ODD and EVEN numbers:

Analytical expression of ODD positive integers:

$$a_n = 1+2n, \ n \in \mathbb{N}$$

Analytical expression of EVEN positive integers:

$$a_n = 2+2n, \ n \in \mathbb{N}$$

ODD	EVEN
1	
3	2
5	4
7	6
9	8
11	10
13	12
15	14
17	16
19	18

We add beside the basic numbers 10 the corresponding number expressed with the binary positional numbering system:

Statement 2.2.1.
Power numbers of 2 (highlighted in yellow) have a peculiarity:
expressed by the binary system they are represented by a 1 followed by a number of zeros. The number of zeros present in the binary representation is the exponent of 2 that inserted in the formula, expressed with the decimal system, returns 1:

$$\frac{\text{number power of 2}}{2^t} = 1$$

All EVEN numbers, expressed using the binary system, have 1 or more zeros as the least significant digit.

All ODD numbers, expressed using the binary system, have 1 as the least significant digit.

DISPARI	N° Binario		PARI	N° Binario
1	1		2	10
3	11		4	100
5	101		6	110
7	111		8	1000
9	1001		10	1010
11	1011		12	1100
13	1101		14	1110
15	1111		16	10000
17	10001		18	10010
19	10011		20	10100
21	10101		22	10110
23	10111		24	11000
25	11001		26	11010
27	11011		28	11100
29	11101		30	11110
31	11111		32	100000
33	100001		34	100010
35	100011		36	100100
37	100101		38	100110
39	100111		40	101000
41	101001		42	101010
43	101011		44	101100
45	101101		46	101110
47	101111		48	110000
49	110001		50	110010
51	110011		52	110100
53	110101		54	110110
55	110111		56	111000
57	111001		58	111010
59	111011		60	111100
61	111101		62	111110
63	111111		64	1000000
127	1111111		128	10000000
255	11111111		256	100000000
511	111111111		512	1000000000

Theorem 2.3.

By multiplying the positive integers by 2 and iteratively multiplying the products by 2 we get all the numbers EVEN>0, which is equivalent to saying: multiplying the ODD numbers by 2^t, with t>0, we obtain the numbers EVEN>0.

Proof

Lemma 2.3.1 Powers of 2 "connect" the number 1, the EVEN numbers and the ODD numbers. If we multiply the ODD numbers by powers of 2, we get the following formula: $\mathbf{(1+2m)*2^t}$, $m,t \in \mathbb{N}$

which has as possible solutions:

- ODD numbers with $t = 0$
- **1**= power of 2 with $m,t = 0$
- powers of 2 with $m = 0$
- EVEN numbers >0 with $t > 0$

it follows the equation with t >0:

$$\boxed{\text{ODD numbers} * 2^t = \text{EVEN numbers}>0}$$

$$(1+2m)*2^t = 2n \Rightarrow 2^t+2^{t+1}*m=2n$$

equations respecting the arithmetic of the numbers EVEN and ODD.

Proof If we multiply *3+1 both members we get:

$(1+2m)*2^t*3+1=2n*3+1$, by varying t the first member becomes:

$t=0, \Rightarrow \{4+ 6m\}$
$t=1, \Rightarrow \{7+12m\} =\{7,19,31,43,55,67,79...\}$
$t=2, \Rightarrow \{13+24m\}=\{13,37,61,85,109,133,157...\}$
$t=3, \Rightarrow \{25+48m\}=\{25,73,121,169,217,264,313...\}$
$t=4, \Rightarrow \{49+96m\}=\{49,145,241,337,433,529,625...\}$
...

$\{(1+2m)*2^t*3+1\}=\{1+6*2^t\}$, $t \in \mathbb{N}_{>0}$, $m \in \mathbb{N}$

$\{7,13,19,25,31,37...\} \equiv 1(\text{mod}6)$

$\{12,24,48,96...\}= \{6*2^t\} \Rightarrow \{6*2^t\} \equiv 0(\text{mod}6)$

the numbers generated with t>0 are $\equiv$ 1 (mod6) being [1]+[0]=[1] $\Rightarrow$

$$6n+1=2n*3+1 \Rightarrow (1+2m)*2^t*3+1=2n*3+1 \Rightarrow$$

$$(1+2m)*2^t=2n, \quad m \in \mathbb{N},\ t,n \in \mathbb{N}_{>0}$$ □

Lemma 2.3.1.1. **The set of positive integers** can be divided into 2 subsets: EVEN and ODD. Even numbers are divisible by 2 by definition. By multiplying an ODD number or an EVEN number by 2 we obtain an EVEN number, in accordance with what is established by the arithmetic of EVEN and ODD numbers. Thus, if we divide an EVEN number by 2 and the quotient is EVEN, we can repeat the operation until we generate an ODD number. In fact, if the number 2 is present in the prime factorization of a positive integer, with any exponent other than 0, the number is EVEN. The product of prime numbers, excluding 2, is always ODD. All this respects the Fundamental Theorem of Arithmetic: every natural number greater than 1 is either a prime number, which is ODD, or it can be represented as a product of prime numbers. Therefore, EVEN numbers are composed of the product of an ODD number multiplied by a power of 2 greater than 0.

Assuming t = tmax and tmax equal to the number of least significant zeros of the EVEN number expressed by the binary positional numbering system with $n,t \in \mathbb{N}_{>0}$, $m \in \mathbb{N}$, we can write the following equation:

$$\mathbf{1 + 2m = \frac{2n}{2^{tmax}}}$$

Proof Following the rule of conversion from the decimal to binary positional system we will perform the following operations: $N_{10}/2$ and then continuing to divide the obtained quotients by 2 until we find a quotient =0. The remainders of the divisions are the binary digits starting with the least significant one. **It is trivial to observe that if the quotient is EVEN the remainder will be 0.**

$$2^f <= N_{10} \text{ e } 2^{(f+1)} > N_{10}$$

$$N_{10} = \sum_{a=0}^{f} 2^a * x_a \,,\ \ x_a \in \{0,1\}$$

$$\mathbf{t}(N_{\{10\}}) := \min\{a \in \{0,1,\ldots,f\} : x_a = 1\} = \mathbf{tmax}$$

If $t <$ tmax the equation becomes: $\Rightarrow$ $$\mathbf{2p = \frac{2n}{2^t}} \ ,\ \ p,n,t \in \mathbb{N}_{>0}$$ □

Lemma 2.3.1.2. Given Statement 2.2. positive integers can be written as $\{1+2n,2+2n\}$. Multiplying the 2 subsets by 2 we obtain $\{2+4n,4+4n\}$ which returns all EVEN numbers >0 as deduced from the following tables which continue ad infinitum:

Table 2.3.1.3.

2+4n	4+4n
2	4
6	8
10	12
14	16
18	20
22	24

$\{(1+2n)*2,(2+2n)*2\}=\{2+4n,4+4n\}$

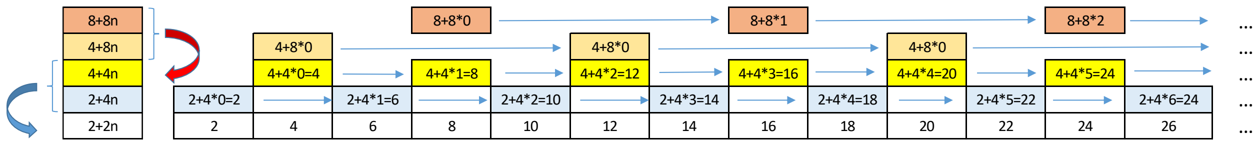
$\{2+4n,4+4n\}\equiv 0(\mathrm{mod}2),\ n\in\mathbb{N}$

$\{2+2n\}=\{2p\},\ \ n\in\mathbb{N},\ p\in\mathbb{N}_{>0}$

$\{2p\}=\{2+4n,4+4n\},\ n\in\mathbb{N},\ p\in\mathbb{N}_{>0}$

Directed graph 2.3.2.

Let's assume the set $\{2+4n,4+4n\}$. Representing the two subsets using the 2 sequences, as in the following graph, it is clear that we reach all numbers EVEN>0.



Having seen Directed graph 2.3.2. we can say:

Table 2.3.2.1.

2+4n	4+8n	8+8n
2	4	8
6	12	16
10	20	24
14	28	32
18	36	40
22	44	48
26	52	56
30	60	64
34	68	72
38	76	80
42	84	88

$\{4+4n\}=\{4p\}\ \Rightarrow\ \{4+8n,8+8n\}=\{4p\}\ \Rightarrow$

$\{2p\}=\{2+4n,4+8n,8+8n\},\ \ n\in\mathbb{N},\ p\in\mathbb{N}_{>0}\Rightarrow$

$\{(4+8n)*2,(8+8n)*2\}=\{8+16n,16+16n\}\Rightarrow$

$\{8+8n\}=\{8+16n,16+16n\}\ \Rightarrow$

$\{2p\}=\{2+4n,4+8n,8+16n,16+16n\ \}\ \Rightarrow$

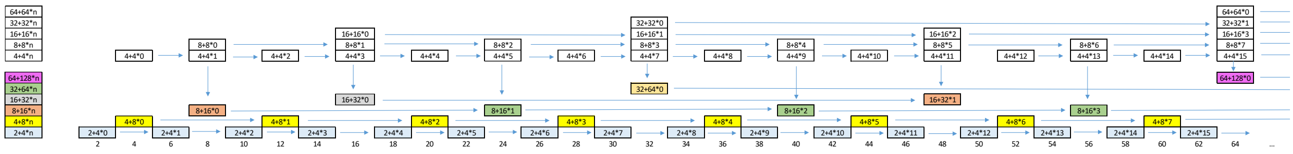
$\{2p\}=\{2^1+2^2n,2^2+2^3n,2^3+2^4n,2^4+2^4n\}\ \Rightarrow$

We can generalize:

$\{2p\}=\{2^2+2^3n,2^3+2^4n,2^4+2^5n,\ \ldots 2^t+2^{t+1}n,2^{t+1}+2^{t+1}n\},\ p,t\in\mathbb{N}_{>0},\ n\in\mathbb{N}$

If we end the series of subsets at the top, the last subset will always be $\{2^{t+1}+2^{t+1}n\}$.

Directed graph 2.3.2.2. We highlight how the numbers reached by the successions 4+4n, 8+8n, 16+16n… $2^{t+1+}2^{t+1}n$ which fill the gaps left by those that show a higher periodicity, are reached by the subsequent successions following this scheme:



…

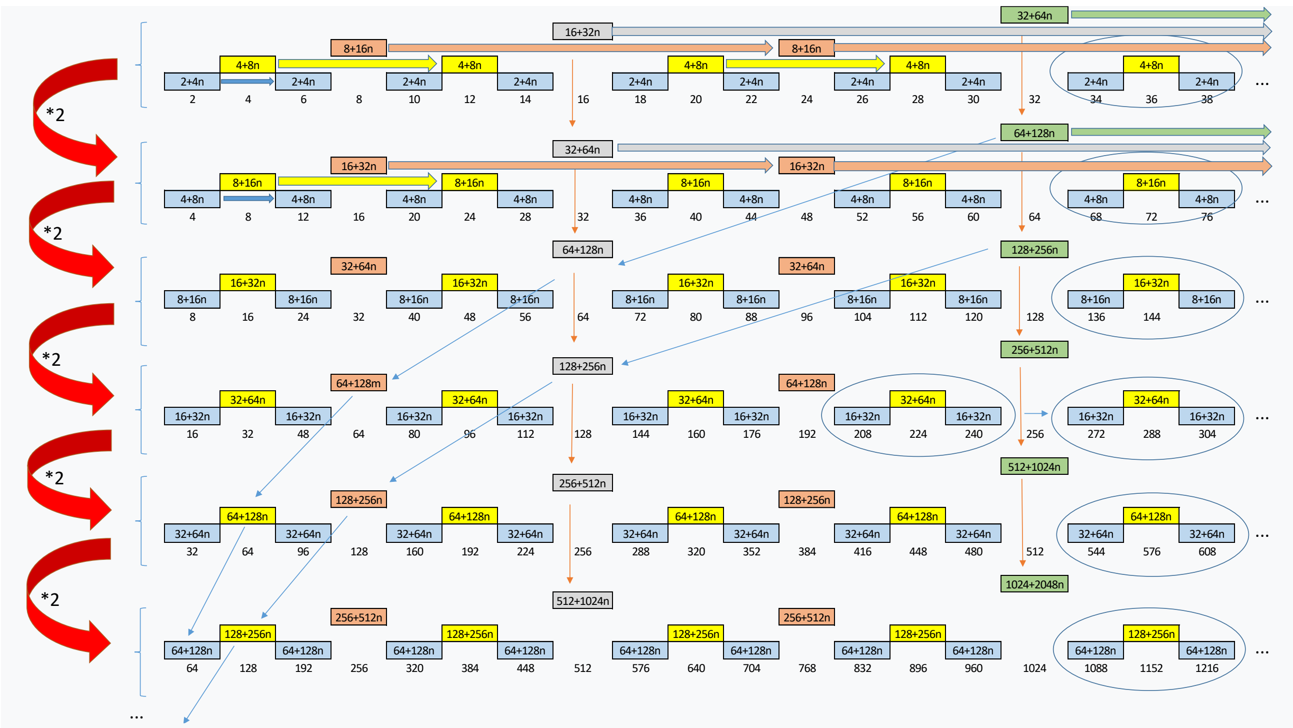
If we leave the series of subsets open at the top we obtain:

$$\left.\begin{array}{l}(1+2n)*2=2+4n\\(2+4n)*2=4+8n\\(4+8n)*2=8+16n\\(8+16n)*2=16+32n\end{array}\right\} \Rightarrow \{(1+2n)*2^t\}=\{2+4n,4+8n,8+16n,\ldots\} \Rightarrow \{(1+2n)*2^t\}=\{2^t+2^{t+1}n\},\ n\in\mathbb{N},\ t\in\mathbb{N}_{>0}$$

…

We place in the following directed graph some of the first sequences that generate {2p} in 5 different rows, we multiply each node by 2 and report the product in the correspondent row of the next scheme. We repeat the operation to generate the patterns that follow one another infinitely. By doing so we basically eliminate the sequence that occurs most frequently, it will appear infinitely alternating a value with a gap. The following inductive step will only highlight the gaps which will then be filled.

Directed graph 2.3.2.3. $\{2+4n,4+8n,8+16n\ldots\}=\{2p\}$



The sequences 2^1+2^2n, 2^2+2^3n, 2^3+2^4n, 2^4+2^5 do not reach the points expressible by the sequence 2^4+2^4*n, because the "available" module is 2^5 which is double the necessary one. This happens in the following lines and to infinity since the sequences have double the modulus of the root and also have double the modulus and root of the previous one. The addition of sequences to infinity allows $2^t+2^{t+1}*n$ to reach all numbers EVEN>0 as can be deduced from Direct Graph 2.3.2.3.

Table 2.3.2.3.1.

DISPARI =			PARI =	2+4m	4+8m	8+16m	16+32m	32+64m	64+128m	128+256m
1+2m	m	t	$(1+2m)*2^t$	t=1	t=2	t=3	t=4	t=5	t=6	t=7
				$\frac{\text{PARI}}{2^t}$ → $2^t+2^{t+1}m$						
1	0	1	2	1						
1	0	2	4	2	1					
3	1	1	6	3						
1	0	3	8	4	2	1				
5	2	1	10	5						
3	1	2	12	6	3					
7	3	1	14	7						
1	0	4	16	8	4	2	1			
9	4	1	18	9						
5	2	2	20	10	5					
11	5	1	22	11						
3	1	3	24	12	6	3				
13	6	1	26	13						
7	3	2	28	14	7					
15	7	1	30	15						
1	0	5	32	16	8	4	2	1		
17	8	1	34	17						
9	4	2	36	18	9					
19	9	1	38	19						
5	2	3	40	20	10	5				
21	10	1	42	21						
11	5	2	44	22	11					
23	11	1	46	23						
3	1	4	48	24	12	6	3			
25	12	1	50	25						
13	6	2	52	26	13					
27	13	1	54	27						
7	3	3	56	28	14	7				
29	14	1	58	29						
15	7	2	60	30	15					
31	15	1	62	31						
1	0	6	64	32	16	8	4	2	1	
33	16	1	66	33						
17	8	2	68	34	17					
35	17	1	70	35						
9	4	3	72	36	18	9				
37	18	1	74	37						
19	9	2	76	38	19					
39	19	1	78	39						
5	2	4	80	40	20	10	5			
41	20	1	82	41						
21	10	2	84	42	21					
43	21	1	86	43						
11	5	3	88	44	22	11				
45	22	1	90	45						
23	11	2	92	46	23					
47	23	1	94	47						
3	1	5	96	48	24	12	6	3		
49	24	1	98	49						
25	12	2	100	50	25					
51	25	1	102	51						
13	6	3	104	52	26	13				
53	26	1	106	53						
27	13	2	108	54	27					
55	27	1	110	55						
7	3	4	112	56	28	14	7			
57	28	1	114	57						
29	14	2	116	58	29					
59	29	1	118	59						
15	7	3	120	60	30	15				
61	30	1	122	61						
31	15	2	124	62	31					
63	31	1	126	63						
1	0	7	128	64	32	16	8	4	2	1

$$2+4n = (1+2n)*2^t,\ n\in\mathbb{N},\ t=1$$

$$4+4n = \begin{cases} (1+n)*2^2, & n\in\mathbb{N} \\ 4p, & n\in\mathbb{N},\ p\in\mathbb{N}_{>0} \\ (1+2m)*2^t, & m\in\mathbb{N},\ t>1\in\mathbb{N} \end{cases}$$

m+1=ordinal of the sequence that shares the same t with $t\in\mathbb{N}$

e.g. $m=4, t=3 \Rightarrow 72=8+16*4 \Rightarrow (1+2*4)*2^3 \Rightarrow 9*2^3=72$, $m+1=5$, therefore the fifth ordinal of the sequence 8+16m and the fifth ordinal of the sequence 1+2m=9 with t=3.

e.g. $m=12, t=0 \Rightarrow 25=1+2*12$, $m+1=13$

therefore the thirteenth ordinal of the sequence 1+2m with t=0

We highlight the regularity of the sequences that repeat themselves at deterministic intervals:

Table 2.3.2.3.2.

ODD = 1+2m	m	t	EVEN = $(1+2m)*2^t$	2+4m (t=1)	4+8m (t=2)	8+16m (t=3)	16+32m (t=4)	32+64m (t=5)	64+128m (t=6)	128+256m (t=7)
				$\frac{EVEN}{2^t}$ →						
1	0	1	2	1						
3	1	1	6	3						
5	2	1	10	5						
7	3	1	14	7						
9	4	1	18	9						
11	5	1	22	11						
13	6	1	26	13						
15	7	1	30	15						
17	8	1	34	17						
19	9	1	38	19						
1	0	2	4	2	1					
3	1	2	12	6	3					
5	2	2	20	10	5					
7	3	2	28	14	7					
9	4	2	36	18	9					
11	5	2	44	22	11					
13	6	2	52	26	13					
15	7	2	60	30	15					
17	8	2	68	34	17					
19	9	2	76	38	19					
21	10	2	84	42	21					
1	0	3	8	4	2	1				
3	1	3	24	12	6	3				
5	2	3	40	20	10	5				
7	3	3	56	28	14	7				
9	4	3	72	36	18	9				
11	5	3	88	44	22	11				
13	6	3	104	52	26	13				
15	7	3	120	60	30	15				
17	8	3	136	68	34	17				
19	9	3	152	76	38	19				
1	0	4	16	8	4	2	1			
3	1	4	48	24	12	6	3			
5	2	4	80	40	20	10	5			
7	3	4	112	56	28	14	7			
9	4	4	144	72	36	18	9			
11	5	4	176	88	44	22	11			

→ $2^t+2^{t+1}m$

In the table on the left we report the same numbers as the previous one, ordering them according to t.

Table 2.3.2.3.3.

	$(1+2m)*2^t$		
1+2m	2+2m	t=tmax	m
1	2	1	0
1	4	2	0
3	6	1	1
1	8	3	0
5	10	1	2
3	12	2	1
7	14	1	3
1	16	4	0
9	18	1	4
5	20	2	2
11	22	1	5
3	24	3	1
13	26	1	6
7	28	2	3
15	30	1	7
1	32	5	0
17	34	1	8
9	36	2	4
19	38	1	9
5	40	3	2
21	42	1	10
11	44	2	5
23	46	1	11
3	48	4	1
25	50	1	12
13	52	2	6
27	54	1	13
7	56	3	3
29	58	1	14
15	60	2	7
31	62	1	15
1	64	6	0
33	66	1	16
17	68	2	8
35	70	1	17
9	72	3	4
37	74	1	18
19	76	2	9
39	78	1	19
5	80	4	2
41	82	1	20
21	84	2	10
43	86	1	21
11	88	3	5
45	90	1	22
23	92	2	11
47	94	1	23
3	96	5	1

In Table 2.3.2.3.3. we highlight cycle 16 of tmax. The sequences $(1+2m)*2^t$ will have a root of 2^t and a module 2^{t+1}, therefore they will begin with a phase shift of 2^t from 0, as can be seen from the tables. The first 15 values of t=tmax are repeated infinitely maintaining the same position in the cycle, while the sixteenth will alternate 5 with >5 following its own cadence $2^{t+1}*m$, $m \in \mathbb{N}$, as can be deduced from Direct Graph 2.3.2.3. and from the following table:

Table 2.3.2.3.4.

	$(1+2m)*2^{tmax}$		
1+2m	2+2m	tmax	m
1	64	6	0
3	192	6	1
5	320	6	2
7	448	6	3

We can therefore state: $\{2n\}=\{2^t+2^{t+1}*m\}$, $n,t \in \mathbb{N}_{>0}$, $m \in \mathbb{N}$

The set of positive integers is: $\{\mathbb{N}_{>0}\} = \{2^t+2^{t+1}m\}$, $t,m \in \mathbb{N}$ □

Definition 2.4.

Module 3: highlights multiples of 3.

The first 3 numbers of each column can be represented geometrically as the vertices of 3 equilateral triangles inscribed in a circle, which is equivalent to saying:

Multiples of 3

$1+3x \equiv \{1,4,7\} \pmod 9$; $2+3x \equiv \{2,5,8\} \pmod 9$; $3x \equiv \{0,3,6\} \pmod 9$

The 3x+1 algorithm seen in optics (mod 3):

1 + 3x

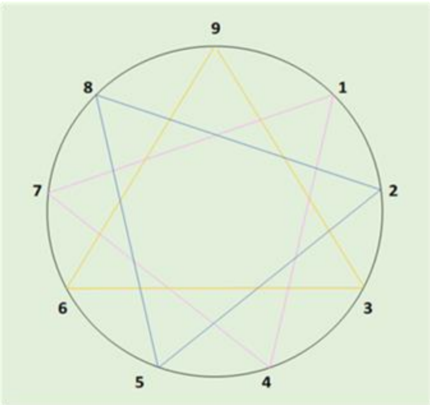
1+3x	2+3x	3x
residue = 1	residue = 2	residue = 0
1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
19	20	21
22	23	24
25	26	27
28	29	30

3 inscribed triangles:

1-4-7

2-5-8

3-6-9



Definition 2.5.

Module 6: distinguish between EVEN and ODD, multiples of 3 ODD and EVEN

		multiples of 3			multiples of 3
ODD	EVEN	ODD	EVEN	ODD	EVEN
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60

The EVEN numbers root 4 are $\equiv 4 \pmod 6$ then the inverse function applies:

$$2n, \frac{n-1}{3}$$

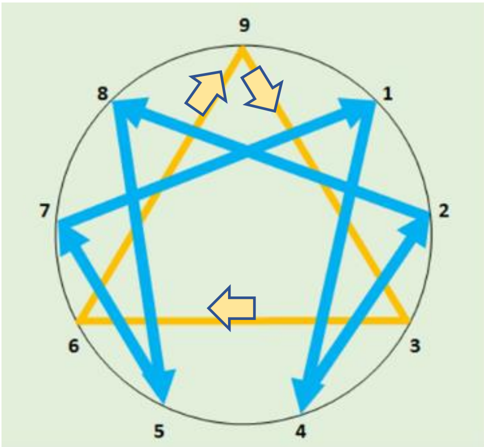
$$\frac{4-1}{3}=1, \ \frac{10-1}{3}=3, \ \frac{16-1}{3}=5\ldots \Rightarrow (1+2n)*3+1=\{EVEN \equiv 4 (mod6)\},\ n \in \mathbb{N}$$

Definition 2.6. **Module 9**: EVEN numbers that have roots 1-4-7 are ≡ 4 (mod 6), EVEN numbers that have roots 2-5-8 are not. Multiples of 3 have roots 0-3-6. All columns alternate EVEN and ODD numbers.

EVEN≡4 (mod6)		multiples di 3	EVEN≡4 (mod6)		multiples di 3	EVEN≡4 (mod6)		multiples di 3
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63
64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81
82	83	84	85	86	87	88	89	90

ENNEAGRAM: the ancient symbol is a graphic and geometric representation of arithmetic modulo 9, 6, 3,2 and theosophical reduction: it is known how the iterative reduction of the result of the sums of the individual digits of a positive integer up to its numerical root leads to a number between 1 and 9, which corresponds to the residue {1,2,3,4,5,6,7,8,0}(mod9).

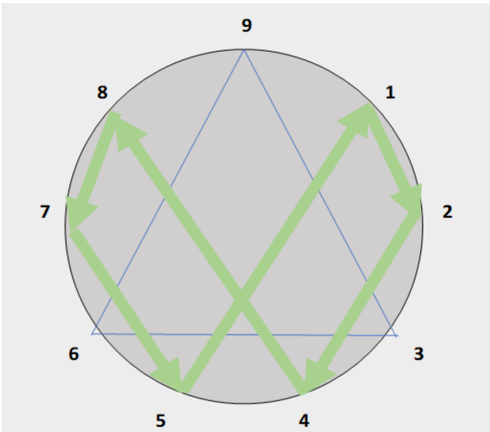
Directed graph 2.6.1.
The triangle 3-6-9 which expresses the multiples of 3 which are ≡ 0,3,6 (mod9) is excluded from the cycle of residue class {[1],[4],[2],[8],[5],[7]}mod9,



digits that correspond to the periodic decimals obtained by dividing the unit by 7:

$$\frac{1}{7} = 0,\overline{142857}$$

By modifying the flow of the latter we obtain the
Directed graph 2.6.2.
We get the cycle of the residue class:
{[1],[2],[4],[8],[7],[5]} mod9

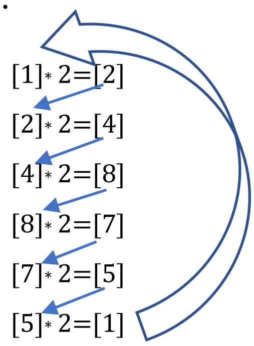


Lemma 2.7.

Let's consider EVEN numbers and residue class (mod9):

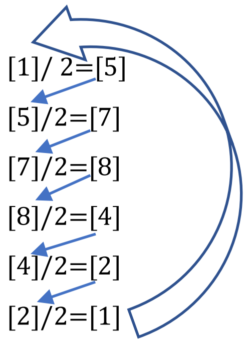
Residue class cycle {[1],[2],[4],[8],[7],[5]} : $x*2=y$

$x \equiv 1 (mod9) * 2 \Rightarrow y \equiv 2 \ (mod9) \Rightarrow$
$x \equiv 2 (mod9) * 2 \Rightarrow y \equiv 4 \ (mod9) \Rightarrow$
$x \equiv 4 (mod9) * 2 \Rightarrow y \equiv 8 \ (mod9) \Rightarrow$
$x \equiv 8 (mod9) * 2 \Rightarrow y \equiv 7 \ (mod9) \Rightarrow$
$x \equiv 7 (mod9) * 2 \Rightarrow y \equiv 5 \ (mod9) \Rightarrow$
$x \equiv 5 (mod9) * 2 \Rightarrow y \equiv 1 \ (mod9) \Rightarrow$



Now let's divide /2 instead of multiplying *2: $\frac{x}{2} = y$

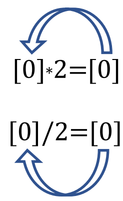
$x \equiv 1 (mod9) \ / \ 2 \Rightarrow y \equiv 5 \ (mod9) \Rightarrow$
$x \equiv 5 (mod9) \ / \ 2 \Rightarrow y \equiv 7 \ (mod9) \Rightarrow$
$x \equiv 7 (mod9) \ / \ 2 \Rightarrow y \equiv 8 \ (mod9) \Rightarrow$
$x \equiv 8 (mod9) \ / \ 2 \Rightarrow y \equiv 4 \ (mod9) \Rightarrow$
$x \equiv 4 (mod9) \ / \ 2 \Rightarrow y \equiv 2 \ (mod9) \Rightarrow$
$x \equiv 2 (mod9) \ / \ 2 \Rightarrow y \equiv 1 \ (mod9) \Rightarrow$



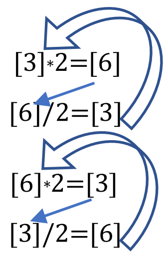
Both operations take us back to the starting residue class (mod9).
The multiplication operation can become division since MCD (2,9)=1 therefore we respect the invariance of the 2 arithmetic operations.

Residue class cycle {[0],[3],[6]}: $x*2=y$, $\frac{x}{2} = y$

$x \equiv 0 (mod9) * 2 \Rightarrow y \equiv 0 \ (mod9) \Rightarrow$
$x \equiv 0 (mod9) \ / \ 2 \Rightarrow y \equiv 0 \ (mod9) \Rightarrow$



$x \equiv 3 (mod9) * 2 \Rightarrow y \equiv 6 \ (mod9) \Rightarrow$
$x \equiv 6 (mod9) \ / \ 2 \Rightarrow y \equiv 3 \ (mod9) \Rightarrow$
$x \equiv 6 (mod9) * 2 \Rightarrow y \equiv 3 \ (mod9) \Rightarrow$
$x \equiv 3 (mod9) \ / \ 2 \Rightarrow y \equiv 6 \ (mod9) \Rightarrow$



Thanks to the theosophical reduction and the cyclical nature of modular arithmetic, which is revealed once the modulus is reached and is manifested for the infinity of whole numbers, we can by induction extend what has been stated to all numbers EVEN>0. The inductive basis and the subsequent inductive step can be trivially verified by choosing any EVEN number from the following table:

Table 2.8.

Matrix (mod9)

<u>The matrix modulo 9 aligns multiples of 3 in 3 columns: 3-6-9</u>

Let's change the order of the 9 columns to better appreciate the triangle of roots 3-6-0. We highlight with colours some sequences of integers generated by the *2 condition:

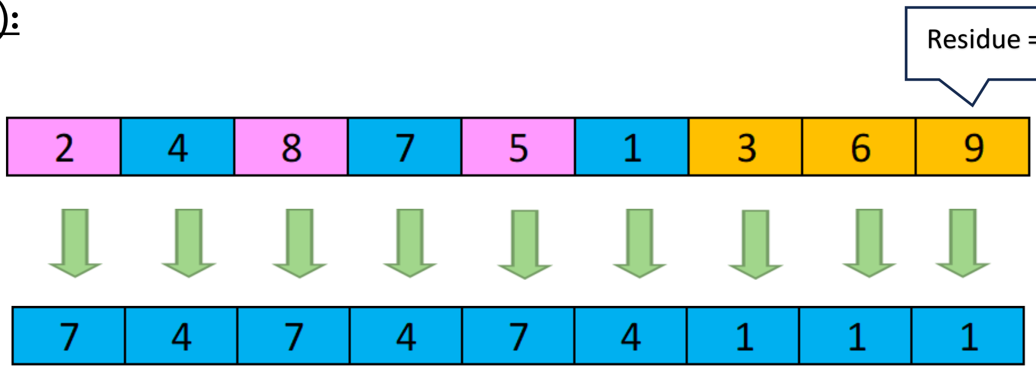
*2

2	4	8	7	5	1	3	6	9
11	13	17	16	14	10	12	15	18
20	22	26	25	23	19	21	24	27
29	31	35	34	32	28	30	33	36
38	40	44	43	41	37	39	42	45
47	49	53	52	50	46	48	51	54
56	58	62	61	59	55	57	60	63
65	67	71	70	68	64	66	69	72
74	76	80	79	77	73	75	78	81
83	85	89	88	86	82	84	87	90
92	94	98	97	95	91	93	96	99
101	103	107	106	104	100	102	105	108
110	112	116	115	113	109	111	114	117
119	121	125	124	122	118	120	123	126
128	130	134	133	131	127	129	132	135
137	139	143	142	140	136	138	141	144
146	148	152	151	149	145	147	150	153
155	157	161	160	158	154	156	159	162
164	166	170	169	167	163	165	168	171
173	175	179	178	176	172	174	177	180
182	184	188	187	185	181	183	186	189
191	193	197	196	194	190	192	195	198
200	202	206	205	203	199	201	204	207
209	211	215	214	212	208	210	213	216
218	220	224	223	221	217	219	222	225
227	229	233	232	230	226	228	231	234
236	238	242	241	239	235	237	240	243
245	247	251	250	248	244	246	249	252
254	256	260	259	257	253	255	258	261

The matrix continues to infinity and contains all positive numbers. □

Lemma 2.9.

<u>We apply the 3x+1 condition to all the ODD numbers of the matrix (mod9):</u>



The $3x+1$ condition has 3 effects:

2.9.1. Converts ODD numbers x to multiples of 3 = 3x and thanks to the sum returns EVEN numbers.

2.9.2. The EVEN numbers generated by the condition, which are ≡ 4 (mod6), pour into the root triangle {1, 4, 7} (mod9) only.

Generalizing:

$$3*(9k+0)+1=27k+1 \equiv 1 \pmod 9$$
$$3*(9k+1)+1=27k+4 \equiv 4 \pmod 9$$
$$3*(9k+2)+1=27k+7 \equiv 7 \pmod 9$$
$$3*(9k+3)+1=27k+10 \equiv 1 \pmod 9 \quad ; \quad 10 \equiv 1 \pmod 9$$
$$3*(9k+4)+1=27k+13 \equiv 4 \pmod 9 \quad ; \quad 13 \equiv 4 \pmod 9$$
$$3*(9k+5)+1=27k+16 \equiv 7 \pmod 9 \quad ; \quad 16 \equiv 7 \pmod 9$$
$$3*(9k+6)+1=27k+19 \equiv 1 \pmod 9 \quad ; \quad 19 \equiv 1 \pmod 9$$
$$3*(9k+7)+1=27k+22 \equiv 4 \pmod 9 \quad ; \quad 22 \equiv 4 \pmod 9$$
$$3*(9k+8)+1=27k+25 \equiv 7 \pmod 9 \quad ; \quad 25 \equiv 7 \pmod 9$$

Proof

$$3*(9k+n)+1\equiv\{1,4,7\}(\text{mod}9) \Rightarrow 27k+3n+1\equiv\{1,4,7\}(\text{mod}9)$$

since $27k \equiv 0 \pmod 9$ and $\mathbf{3n+1\equiv\{1,4,7\}(mod9)}$, $k,n\in\mathbb{N}$

$$n=2m+1 \Rightarrow \mathbf{3*(2m+1)+1=6m+4} \Rightarrow$$

$$\begin{cases} 6m+4 \equiv 1 \pmod 9 \text{ if } m\equiv 1 \pmod 3 \Rightarrow m=3p+1 \\ 6m+4 \equiv 4 \pmod 9 \text{ if } m\equiv 0 \pmod 3 \Rightarrow m=3p+0 \\ 6m+4 \equiv 7 \pmod 9 \text{ if } m\equiv 2 \pmod 3 \Rightarrow m=3p+2 \end{cases}$$

$$[6]*[3]=[18], \quad 18p\equiv 0 \pmod 9$$

$m\equiv\{0,1,2\}(\text{mod}3)$

$m,p\in\mathbb{N}$

$$6*(3p+1)+4 \Rightarrow 18p+10 \equiv 1 \pmod 9$$
$$6*(3p+0)+4 \Rightarrow 18p+4 \equiv 4 \pmod 9$$
$$6*(3p+2)+4 \Rightarrow 18p+16 \equiv 7 \pmod 9$$

which is equivalent to multiplying $*3+1$ the numbers ODD≡{1,3,5}(mod6):

$$\begin{cases} 3*(6p+3)+1 \Rightarrow 18p+10 \equiv 1 \pmod 9 \\ 3*(6p+1)+1 \Rightarrow 18p+4 \equiv 4 \pmod 9 \\ 3*(6p+5)+1 \Rightarrow 18p+16 \equiv 7 \pmod 9 \end{cases}$$

2.9.3. Merges into the column with root 1 multiples of 3 ODD:

$$(3k) * 3+1 = 9k+1$$

which are ≡ 3 (mod6) and ≡ 0,3,6 (mod9) and we can write as:

$$(2m+1) * 3 = 6m+3$$

summing 1 we get: **6m+3+1 = 6m+4**

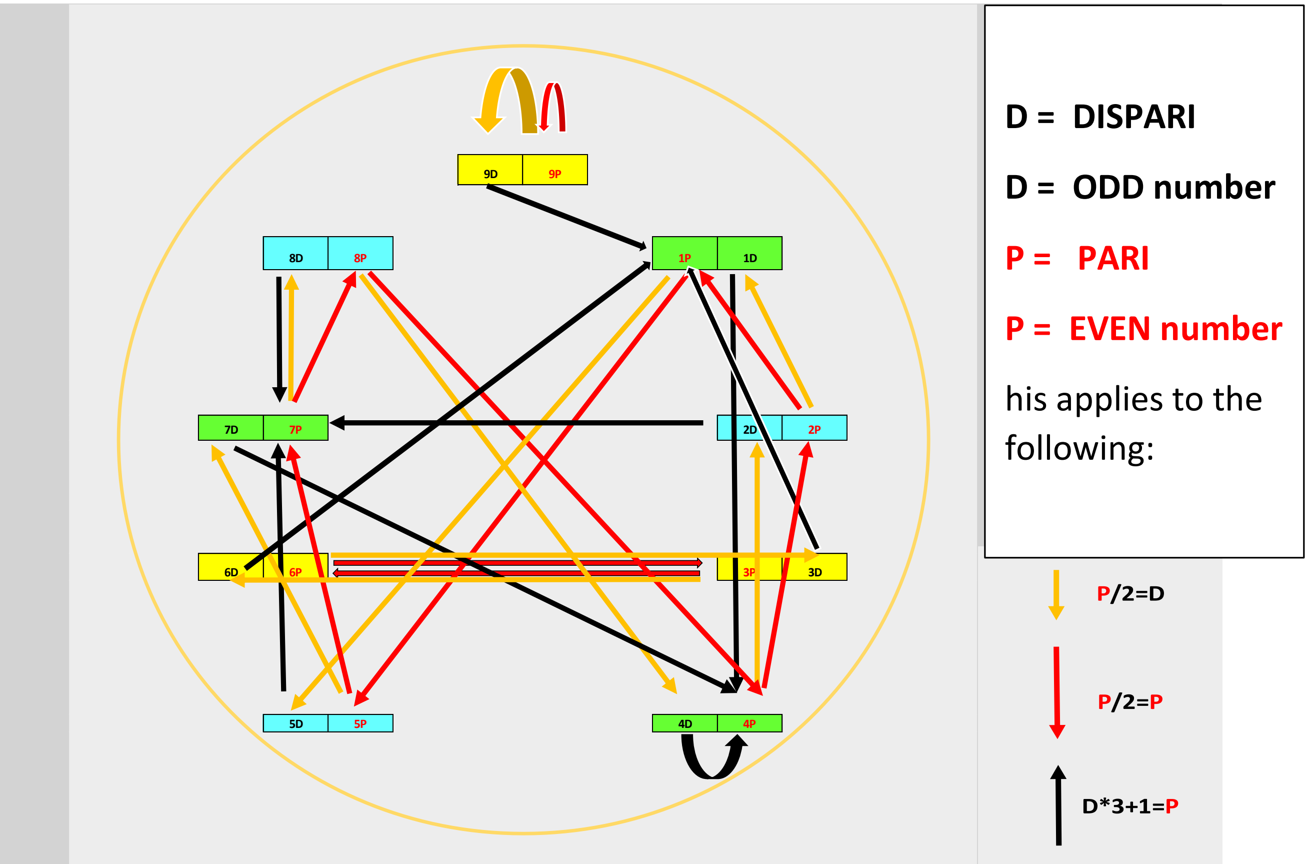
Thanks to modular arithmetic we can use the inductive method and extend what is stated to all natural numbers.

It is important to understand how the condition connects the residue class cycle {[0],[3],[6]} with the residue class cycle {[1],[2],[4],[8],[7],[5]}, which, thanks to the /2 condition, leads to 1. In fact, it eliminates multiples of 3 from subsequent counts, thus reconciling the numbers 2 and 3 which are notoriously co-prime. □

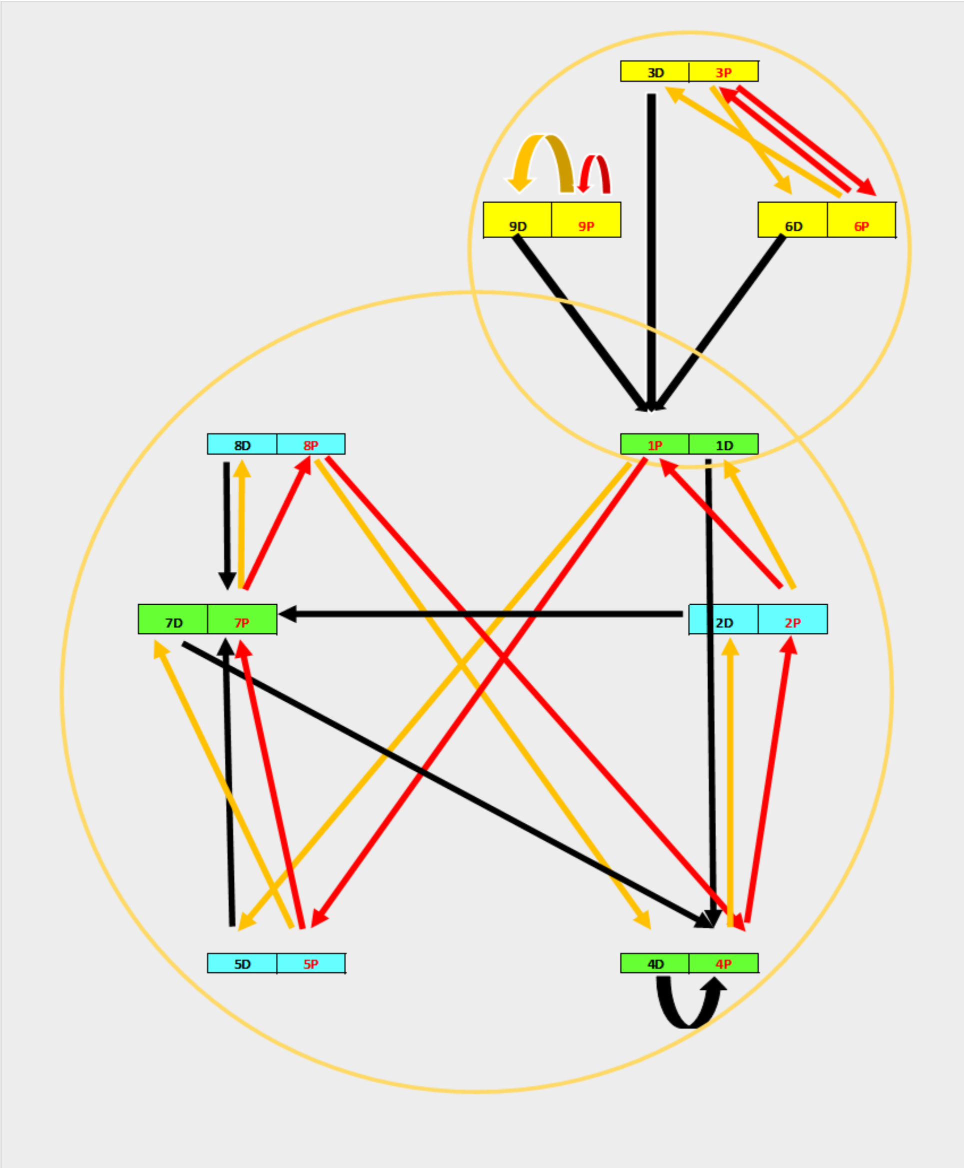
We express what has been stated through 3 flow diagrams (mod9) which are the same but highlight the circulation, let's reverse the direction of the arrows in the fourth:

Theorem 2.10. Directed graph All positive integers, represented by the following directed graphs, can be reached by applying the 2 conditions and the inverse function.

Flowchart 1: distinguishing between roots and EVEN and ODD numbers:



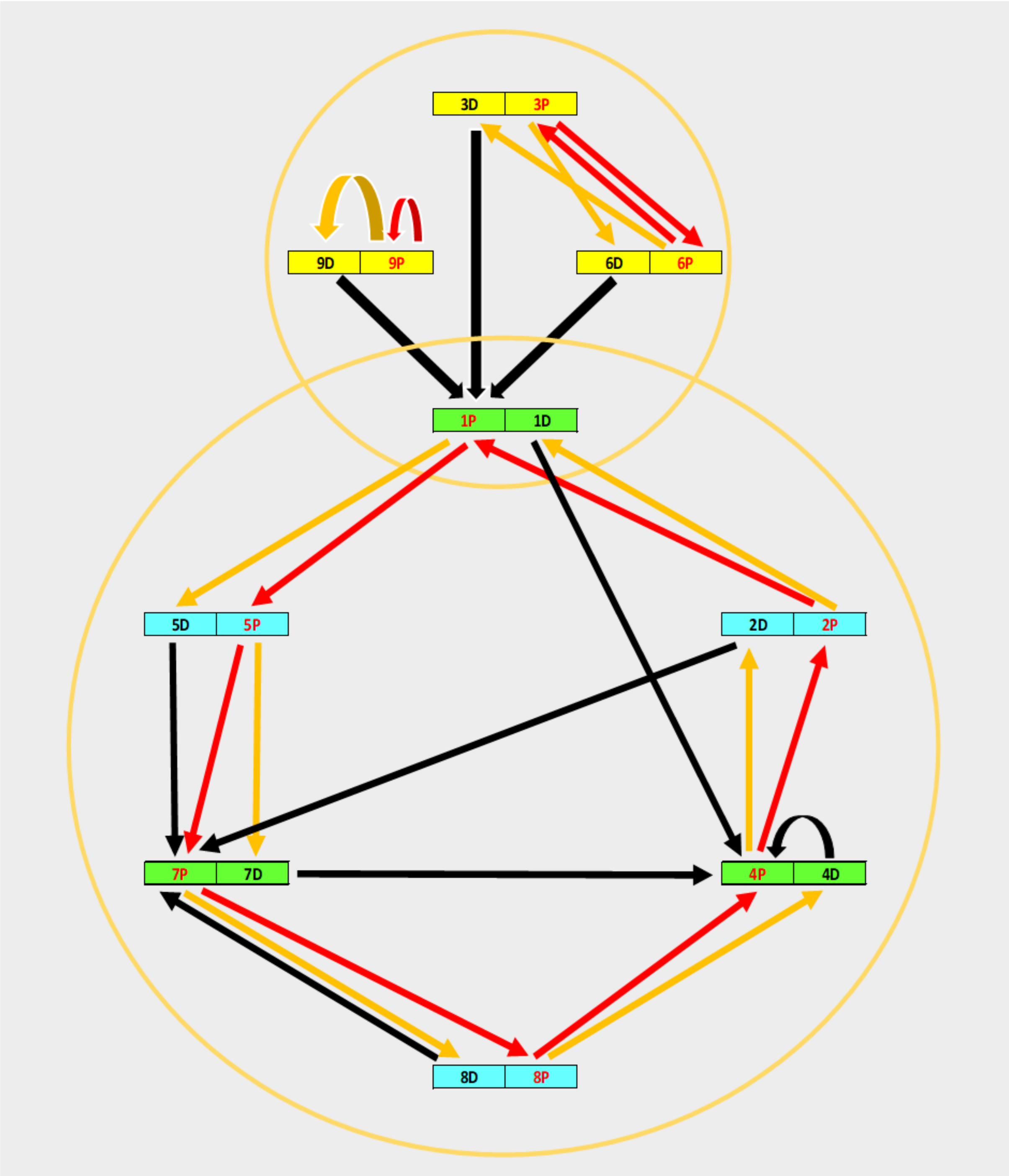
Flowchart 2 where we highlight the transition from **Enneagram to Hexagram** in the form of ∞ or form similar to the Lorenz attractor ∞ (red arrows inside the large circle):



All numbers ≡ 0,3,6 (mod9) are multiples of 3 and enter the Hexagram cycle and are connected to the condition / 2.

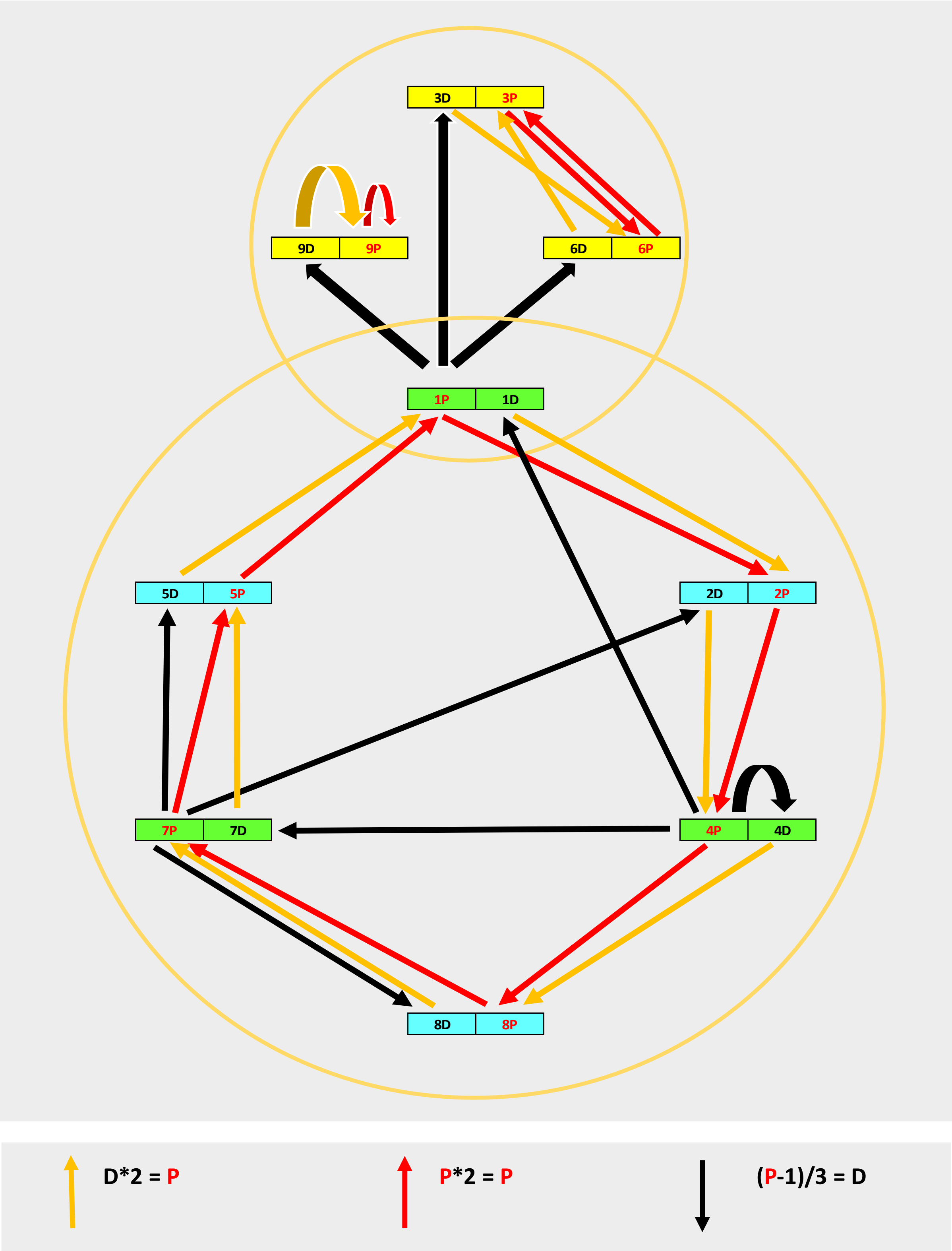
Multiples of 3, after applying the 2 conditions, become: EVEN numbers **≡ 1 (mod9)** and are divided by 2^{tmax} generating an ODD number ≡ {1,4,7} (mod9) which is ≡ 1 (mod6) and generating an ODD number ≡ {2,5,8} (mod9) which is ≡ 5 (mod6).

Flowchart 3: we arrange the Hexagram in the form of zero (red arrows inside the large circle):



The numbers powers of 2 follow the red path /2 until they deviate (orange arrow) and reach 1. The EVEN numbers, multiples of 2, follow the red path until they deviate (orange arrow) on an ODD number which will become, after application of the condition 3x+1, EVEN ≡ 4,7 (mod9).

Flowchart 4. Having seen Lemma 2.7. we reverse the direction of the arrows and replace the conditions with those of the inverse function, **the *3* patterns become the Collatz graph:**



Proof Let's analyze the EVEN nodes {1,4,7} and check how the numbers belonging to this set, inserted into the inverse function, reach all the ODD numbers:
Looking at Table 2.8.1. it is trivial to point out that:

1P={10+18m}, 4P={4+18m}, 7P={16+18m}.

The modulus is 2∗9=18 because the (mod9) alternates EVEN and ODD.

We insert 1P into the inverse formula and obtain the sequence representing multiples of 3 ODD ={3D,6D,9D}:

$$\frac{10+18m-1}{3} = 3 + 6m \quad , m \in \mathbb{N}$$

{3n,1+3n,2+3n}={ℕ}, n∈ℕ

m=3n ⇨ 3+6∗3n=3+18n ⇨ 3+18n=3D, n∈ℕ

m=1+3n ⇨ 3+6∗(1+3n)=9+18n ⇨ 9+18n=9D, n∈ℕ

m=2+3n ⇨ 3+6∗(2+3n)=15+18n ⇨ 15+18n=6D, n∈ℕ

We insert 4P into the inverse formula and obtain the sequence:

$$\frac{4+18m-1}{3} = 1 + 6m \quad , m \in \mathbb{N}$$

m=3n ⇨ 1+6∗3n=1+18n ⇨ 1+18n=1D, n∈ℕ, n=0 ⇨ 1D=1

m=1+3n ⇨ 1+6∗(1+3n)=7+18n ⇨ 7+18n=7D, n∈ℕ

m=2+3n ⇨ 1+6∗(2+3n)=13+18n ⇨ 13+18n=4D, n∈ℕ

We insert 7P into the inverse formula and obtain the sequence:

$$\frac{16+18m-1}{3} = 5 + 6m \quad , m \in \mathbb{N}$$

m=3n ⇨ 5+6∗3n=5+18n ⇨ 5+18n=5D, n∈ℕ

m=1+3n ⇨ 5+6∗(1+3n)=11+18n ⇨ 11+18n=2D, n∈ℕ

m=2+3n ⇨ 5+6∗(2+3n)=17+18n ⇨ 17+18n=8D, n∈ℕ

Since the numerical set {1+6m,3+6m,5+6m}={1+2n}, Definition 2.5., and {1+2n}={1D,2D,3D,4D,5D,6D,7D,8D,9D} we have shown that all ODD numbers are reached by the inverse formula and are present in Flowchart 4. With Theorem 2.3. we have proof that by multiplying the ODD numbers iteratively by 2 we obtain the EVEN numbers. □

Proof the conjecture 2.11.

In flowchart n°4 all positive integers are represented using modulo 9, distinguishing between EVEN and ODD integers. The same represents the tree and runs through the Collatz graph respecting the inverse function, as highlighted by Theorems 2.3.-2.10 and from Lemmas 2.7.-2.9. Therefore, the conjecture is true. □

Proof of the conjecture 2.12.

The conjecture is true $\forall n \equiv 1(\text{mod}2) \Leftrightarrow 3n+1 \equiv 4(\text{mod}6)$

The conjecture imposes 3n+1 if n is ODD: $n\equiv 1(\text{mod}2) \Rightarrow n=2m+1$

hence $3*(2m+1)+1= 6m+4$ and $6m+4 \equiv 4 \ (\text{mod}6)$

therefore $3*(2m+1)+1\equiv 4 \ (\text{mod}6)$, $m\in\mathbb{N}$

Proof of the necessary condition. The direct implication is true since $\forall n\equiv 1(\text{mod}2) \Rightarrow 3n+1=6m+4$ and by the reflexive property of congruences 2 equal numbers are congruent, Statement 2.1.2., so the product of the ODD numbers $\in\mathbb{N} * 3+1$ is $\equiv 4$ (mod6).

Proof of the sufficient condition. The inverse implication is true since the negation of the direct is true:

$3n+1\equiv 4(\text{mod}6)$ is equivalent to n not EVEN.

The conjecture does not involve the multiplication of EVEN numbers $\in\mathbb{N}_{>0} * 3+1$ and in any case it would not be $\equiv 4$ (mod6):

$n\equiv 0(\text{mod}2)\Rightarrow n=2k$ so $3*2k+1=6k+1$ and $6k+1$ it is not $\equiv 4(\text{mod}6)$ and there are no positive integers $\neq$ from $\{2k,1+2m\}$, $k\in\mathbb{N}_{>0}$, $m\in\mathbb{N}$.

Seen in another light, stating that 2 numerical expressions $:=\{a,b\}\in\mathbb{N}$ are congruent modul p $:=p\in\mathbb{N}_{>0}$, means that there is at least one natural number $m\in\mathbb{N}$ that satisfies the equality: $a-b=p*m \Rightarrow a=p*m+b \Rightarrow m=\frac{a-b}{p}$

e.g. $a=27, p=9, b=0 \Rightarrow 27=9m+0 \Rightarrow m=\frac{27-0}{9} \Rightarrow m=3$, m=integer, therefore $27-0=9*3 \Rightarrow 27=27$ and $27 \equiv 0(\text{mod}9)$

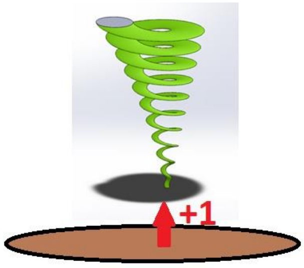
e.g. $a=27k, k\in\mathbb{N}, p=9, b=0 \Rightarrow 27k=9m+0 \Rightarrow m=\frac{27k-0}{9} \Rightarrow m=3k$ 3k=integer, therefore $27k-0=9*3k \Rightarrow 27k=27k$ therefore $27k \equiv 0(\text{mod}9)$

our case $a=3n+1$, $n=2k+1, k\in\mathbb{N}, n\in\mathbb{N}_{>0}$, $b=6k+4$, $p=6$

$m=\frac{3*(2k+1)+1-6k-4}{6} \Rightarrow m=\frac{0}{6} \Rightarrow m=0 \Rightarrow m=$ integer $\Rightarrow a-b=6*0 \Rightarrow 0=6*0 \Rightarrow a=b \Rightarrow 3*(2k+1)+1 \equiv 6k+4(\text{mod}6)\Rightarrow 3*(2k+1)+1\equiv 4(\text{mod}6)$ □

Statement 2.1.3.
The graph tree and flow diagram in 3 dimensions become a 3D roller coaster (flow diagram 2) or a 3D spiral (flow diagram 3). We could allegorically describe them as a huge slide where <u>numerical gravity</u>, represented by divisibility by 2, pushes all positive integers towards the foreground at +1.

The powers of 2 are the fulcrum, the bond, through which the algorithm links all positive integers.



A possible way to prove the conjecture is to proof:

- There are no loops except for 1-4-2-1...
  Given the flow pattern we can state that there are no other loops except 1-4-2-1... and that D1=1 is the only possible solution of this routine:
  D1*3+1=P4 , P4/2=P2 , P2/2= D1 , P4/4= D1
  1*3+1 =4 , 4/2=2 , 2/2 = 1 , 4/4 = 1
  (D1*3+1)/4 = D1 ⇨ 1 = 4D1-3D1 ⇨ 1=D1
  (x*3+1)/4 = x has solution x=1
  A three-dimensional view of Flowchart 3 allows us to visualize that the mechanism $D8_1*3+1=P7$, $P7/2=D8_2$ and $D8_1$ is trivially $\neq D8_2$ so it is not a loop.
- There are no routines that lead to infinity. This is true if all positive numbers are present in the Collatz graph, statement that we have just show.

Below we will formulate the equations that arise from the use of the 2 conditions. We will find the correlations with the binary code and how the possible routines can only decrease.

We highlight the fundamental action of the powers of 2:

3 Operation of the algorithm

Theorem 3.1. Equation 3.1.1. is true.

Equation 3.1.1.

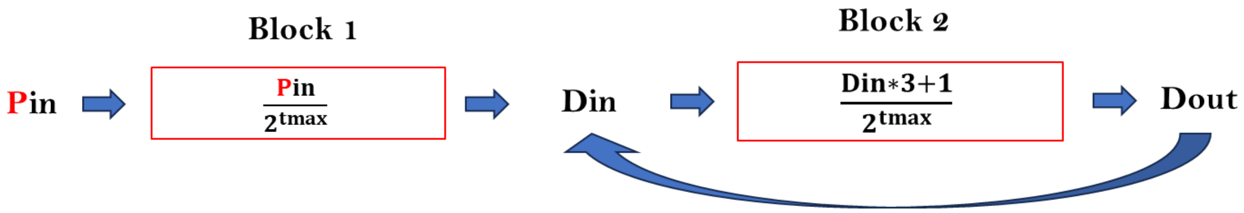
$$\frac{\left(\frac{\left(\frac{\left(\frac{Din*3+1}{2^{tmax}}\right)*3+1}{2^{tmax}}\right)*3+1}{2^{tmax}}\ldots\right)*3+1}{2^{tmax}} = Dout$$

Din determines the number of: $\frac{3x+1}{2^{tmax}}$, number of $\frac{3x+1}{2^{tmax}} \in \mathbb{N}_{>0}$

Proof

Given Theorem 2.3. we can state all positive integers are present in the binary positional numbering system, where the numbers 1 (high level) represent a power of 2 with an exponent, which starts from 0, increasing from right to left. The sum of the values obtained by raising 2 to its exponent gives the base number 10. An EVEN number, expressed with binary numeration, will have one or more less significant digits marked by a 0 (low level). Eliminating the least significant zeros is equivalent to dividing by 2 as many times as there are zeros. The number thus divided is an ODD number. All EVEN numbers subjected to the /2 condition, one or more times, become ODD. The number 1 is an ODD number and is connected to EVEN numbers thanks to the /2 condition.

Lemma 3.2. Positive feedback block diagram



If the chosen number is EVEN it will be processed by block 1 which will give block 2 an ODD number. If the chosen number is ODD it will be processed by block 2 which will apply the 2 conditions: 3x+1 and $/2^{tmax}$ and will return an ODD number which will be processed from block 2:

Proof

Positive feedback, i.e. the application of the 2 conditions, brings Dout back to the input of block 2 and sums it to 0, since the EVEN number is absent or has already been processed and has given up its energy. The subsequent cycles will be exclusively those of block 2. □

Definition 3.2.1.

Module 32: has the peculiarity of line up the powers of 2.
Looking for **regularities in the distribution of powers of 2**, let us observe how the input ODD numbers inserted into a 16-column table behave. The same, applying the 3x+1 condition, generate an EVEN number that is divisible by 2^{tmax}:

Table 3.2.1.1.

Exponent of 2 — Module operation: Din mod9

2	Din mod9	1	Din mod9	4	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	3	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	>=5	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	3	Din mod9	1	Din mod9
1	1	3	3	5	5	7	7	9	0	11	2	13	4	15	6	17	8	19	1	21	3	23	5	25	7	27	0	29	2	31	4
33	6	35	8	37	1	39	3	41	5	43	7	45	0	47	2	49	4	51	6	53	8	55	1	57	3	59	5	61	7	63	0
65	2	67	4	69	6	71	8	73	1	75	3	77	5	79	7	81	0	83	2	85	4	87	6	89	8	91	1	93	3	95	5
97	7	99	0	101	2	103	4	105	6	107	8	109	1	111	3	113	5	115	7	117	0	119	2	121	4	123	6	125	8	127	1
129	3	131	5	133	7	135	0	137	2	139	4	141	6	143	8	145	1	147	3	149	5	151	7	153	0	155	2	157	4	159	6
161	8	163	1	165	3	167	5	169	7	171	0	173	2	175	4	177	6	179	8	181	1	183	3	185	5	187	7	189	0	191	2
193	4	195	6	197	8	199	1	201	3	203	5	205	7	207	0	209	2	211	4	213	6	215	8	217	1	219	3	221	5	223	7
225	0	227	2	229	4	231	6	233	8	235	1	237	3	239	5	241	7	243	0	245	2	247	4	249	6	251	8	253	1	255	3
257	5	259	7	261	0	263	2	265	4	267	6	269	8	271	1	273	3	275	5	277	7	279	0	281	2	283	4	285	6	287	8
289	1	291	3	293	5	295	7	297	0	299	2	301	4	303	6	305	8	307	1	309	3	311	5	313	7	315	0	317	2	319	4
321	6	323	8	325	1	327	3	329	5	331	7	333	0	335	2	337	4	339	6	341	8	343	1	345	3	347	5	349	7	351	0
353	2	355	4	357	6	359	8	361	1	363	3	365	5	367	7	369	0	371	2	373	4	375	6	377	8	379	1	381	3	383	5
385	7	387	0	389	2	391	4	393	6	395	8	397	1	399	3	401	5	403	7	405	0	407	2	409	4	411	6	413	8	415	1
417	3	419	5	421	7	423	0	425	2	427	4	429	6	431	8	433	1	435	3	437	5	439	7	441	0	443	2	445	4	447	6
449	8	451	1	453	3	455	5	457	7	459	0	461	2	463	4	465	6	467	8	469	1	471	3	473	5	475	7	477	0	479	2
481	4	483	6	485	8	487	1	489	3	491	5	493	7	495	0	497	2	499	4	501	6	503	8	505	1	507	3	509	5	511	7
513	0	515	2	517	4	519	6	521	8	523	1	525	3	527	5	529	7	531	0	533	2	535	4	537	6	539	8	541	1	543	3
545	5	547	7	549	0	551	2	553	4	555	6	557	8	559	1	561	3	563	5	565	7	567	0	569	2	571	4	573	6	575	8
577	1	579	3	581	5	583	7	585	0	587	2	589	4	591	6	593	8	595	1	597	3	599	5	601	7	603	0	605	2	607	4
609	6	611	8	613	1	615	3	617	5	619	7	621	0	623	2	625	4	627	6	629	8	631	1	633	3	635	5	637	7	639	0
641	2	643	4	645	6	647	8	649	1	651	3	653	5	655	7	657	0	659	2	661	4	663	6	665	8	667	1	669	3	671	5
673	7	675	0	677	2	679	4	681	6	683	8	685	1	687	3	689	5	691	7	693	0	695	2	697	4	699	6	701	8	703	1
705	3	707	5	709	7	711	0	713	2	715	4	717	6	719	8	721	1	723	3	725	5	727	7	729	0	731	2	733	4	735	6
737	8	739	1	741	3	743	5	745	7	747	0	749	2	751	4	753	6	755	8	757	1	759	3	761	5	763	7	765	0	767	2
769	4	771	6	773	8	775	1	777	3	779	5	781	7	783	0	785	2	787	4	789	6	791	8	793	1	795	3	797	5	799	7
801	0	803	2	805	4	807	6	809	8	811	1	813	3	815	5	817	7	819	0	821	2	823	4	825	6	827	8	829	1	831	3
833	5	835	7	837	0	839	2	841	4	843	6	845	8	847	1	849	3	851	5	853	7	855	0	857	2	859	4	861	6	863	8
865	1	867	3	869	5	871	7	873	0	875	2	877	4	879	6	881	8	883	1	885	3	887	5	889	7	891	0	893	2	895	4
897	6	899	8	901	1	903	3	905	5	907	7	909	0	911	2	913	4	915	6	917	8	919	1	921	3	923	5	925	7	927	0
929	2	931	4	933	6	935	8	937	1	939	3	941	5	943	7	945	0	947	2	949	4	951	6	953	8	955	1	957	3	959	5
961	7	963	0	965	2	967	4	969	6	971	8	973	1	975	3	977	5	979	7	981	0	983	2	985	4	987	6	989	8	991	1
993	3	995	5	997	7	999	0	1001	2	1003	4	1005	6	1007	8	1009	1	1011	3	1013	5	1015	7	1017	0	1019	2	1021	4	1023	6
1025	8	1027	1	1029	3	1031	5	1033	7	1035	0	1037	2	1039	4	1041	6	1043	8	1045	1	1047	3	1049	5	1051	7	1053	0	1055	2
1057	4	1059	6	1061	8	1063	1	1065	3	1067	5	1069	7	1071	0	1073	2	1075	4	1077	6	1079	8	1081	1	1083	3	1085	5	1087	7
1089	0	1091	2	1093	4	1095	6	1097	8	1099	1	1101	3	1103	5	1105	7	1107	0	1109	2	1111	4	1113	6	1115	8	1117	1	1119	3
1121	5	1123	7	1125	0	1127	2	1129	4	1131	6	1133	8	1135	1	1137	3	1139	5	1141	7	1143	0	1145	2	1147	4	1149	6	1151	8
1153	1	1155	3	1157	5	1159	7	1161	0	1163	2	1165	4	1167	6	1169	8	1171	1	1173	3	1175	5	1177	7	1179	0	1181	2	1183	4
1185	6	1187	8	1189	1	1191	3	1193	5	1195	7	1197	0	1199	2	1201	4	1203	6	1205	8	1207	1	1209	3	1211	5	1213	7	1215	0
1217	2	1219	4	1221	6	1223	8	1225	1	1227	3	1229	5	1231	7	1233	0	1235	2	1237	4	1239	6	1241	8	1243	1	1245	3	1247	5
1249	7	1251	0	1253	2	1255	4	1257	6	1259	8	1261	1	1263	3	1265	5	1267	7	1269	0	1271	2	1273	4	1275	6	1277	8	1279	1
1281	3	1283	5	1285	7	1287	0	1289	2	1291	4	1293	6	1295	8	1297	1	1299	3	1301	5	1303	7	1305	0	1307	2	1309	4	1311	6
1313	8	1315	1	1317	3	1319	5	1321	7	1323	0	1325	2	1327	4	1329	6	1331	8	1333	1	1335	3	1337	5	1339	7	1341	0	1343	2

By vertically displacing the columns of the matrix of ODD numbers (mod32), we obtain rows of data that have the same (mod9):

Table 3.2.1.2.

2	Din mod9	1	Din mod9	>=5	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	3	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	4	Din mod9	1	Din mod9	2	Din mod9	1	Din mod9	3	Din mod9	1	Din mod9
																												13	4	31	4
																								9	0	27	0	45	0	63	0
																				5	5	23	5	41	5	59	5	77	5	95	5
																1	1	19	1	37	1	55	1	73	1	91	1	109	1	127	1
														15	6	33	6	51	6	69	6	87	6	105	6	123	6	141	6	159	6
										11	2	29	2	47	2	65	2	83	2	101	2	119	2	137	2	155	2	173	2	191	2
						7	7	25	7	43	7	61	7	79	7	97	7	115	7	133	7	151	7	169	7	187	7	205	7	223	7
		3	3	21	3	39	3	57	3	75	3	93	3	111	3	129	3	147	3	165	3	183	3	201	3	219	3	237	3	255	3
17	8	35	8	53	8	71	8	89	8	107	8	125	8	143	8	161	8	179	8	197	8	215	8	233	8	251	8	269	8	287	8
49	4	67	4	85	4	103	4	121	4	139	4	157	4	175	4	193	4	211	4	229	4	247	4	265	4	283	4	301	4	319	4
81	0	99	0	117	0	135	0	153	0	171	0	189	0	207	0	225	0	243	0	261	0	279	0	297	0	315	0	333	0	351	0
113	5	131	5	149	5	167	5	185	5	203	5	221	5	239	5	257	5	275	5	293	5	311	5	329	5	347	5	365	5	383	5
145	1	163	1	181	1	199	1	217	1	235	1	253	1	271	1	289	1	307	1	325	1	343	1	361	1	379	1	397	1	415	1
177	6	195	6	213	6	231	6	249	6	267	6	285	6	303	6	321	6	339	6	357	6	375	6	393	6	411	6	429	6	447	6
209	2	227	2	245	2	263	2	281	2	299	2	317	2	335	2	353	2	371	2	389	2	407	2	425	2	443	2	461	2	479	2
241	7	259	7	277	7	295	7	313	7	331	7	349	7	367	7	385	7	403	7	421	7	439	7	457	7	475	7	493	7	511	7
273	3	291	3	309	3	327	3	345	3	363	3	381	3	399	3	417	3	435	3	453	3	471	3	489	3	507	3	525	3	543	3
305	8	323	8	341	8	359	8	377	8	395	8	413	8	431	8	449	8	467	8	485	8	503	8	521	8	539	8	557	8	575	8
337	4	355	4	373	4	391	4	409	4	427	4	445	4	463	4	481	4	499	4	517	4	535	4	553	4	571	4	589	4	607	4
369	0	387	0	405	0	423	0	441	0	459	0	477	0	495	0	513	0	531	0	549	0	567	0	585	0	603	0	621	0	639	0
401	5	419	5	437	5	455	5	473	5	491	5	509	5	527	5	545	5	563	5	581	5	599	5	617	5	635	5	653	5	671	5
433	1	451	1	469	1	487	1	505	1	523	1	541	1	559	1	577	1	595	1	613	1	631	1	649	1	667	1	685	1	703	1
465	6	483	6	501	6	519	6	537	6	555	6	573	6	591	6	609	6	627	6	645	6	663	6	681	6	699	6	717	6	735	6
497	2	515	2	533	2	551	2	569	2	587	2	605	2	623	2	641	2	659	2	677	2	695	2	713	2	731	2	749	2	767	2
529	7	547	7	565	7	583	7	601	7	619	7	637	7	655	7	673	7	691	7	709	7	727	7	745	7	763	7	781	7	799	7
561	3	579	3	597	3	615	3	633	3	651	3	669	3	687	3	705	3	723	3	741	3	759	3	777	3	795	3	813	3	831	3
593	8	611	8	629	8	647	8	665	8	683	8	701	8	719	8	737	8	755	8	773	8	791	8	809	8	827	8	845	8	863	8
625	4	643	4	661	4	679	4	697	4	715	4	733	4	751	4	769	4	787	4	805	4	823	4	841	4	859	4	877	4	895	4
657	0	675	0	693	0	711	0	729	0	747	0	765	0	783	0	801	0	819	0	837	0	855	0	873	0	891	0	909	0	927	0
689	5	707	5	725	5	743	5	761	5	779	5	797	5	815	5	833	5	851	5	869	5	887	5	905	5	923	5	941	5	959	5
721	1	739	1	757	1	775	1	793	1	811	1	829	1	847	1	865	1	883	1	901	1	919	1	937	1	955	1	973	1	991	1
753	6	771	6	789	6	807	6	825	6	843	6	861	6	879	6	897	6	915	6	933	6	951	6	969	6	987	6	1005	6	1023	6
785	2	803	2	821	2	839	2	857	2	875	2	893	2	911	2	929	2	947	2	965	2	983	2	1001	2	1019	2	1037	2	1055	2
817	7	835	7	853	7	871	7	889	7	907	7	925	7	943	7	961	7	979	7	997	7	1015	7	1033	7	1051	7	1069	7	1087	7
849	3	867	3	885	3	903	3	921	3	939	3	957	3	975	3	993	3	1011	3	1029	3	1047	3	1065	3	1083	3	1101	3	1119	3
881	8	899	8	917	8	935	8	953	8	971	8	989	8	1007	8	1025	8	1043	8	1061	8	1079	8	1097	8	1115	8	1133	8	1151	8
913	4	931	4	949	4	967	4	985	4	1003	4	1021	4	1039	4	1057	4	1075	4	1093	4	1111	4	1129	4	1147	4	1165	4	1183	4
945	0	963	0	981	0	999	0	1017	0	1035	0	1053	0	1071	0	1089	0	1107	0	1125	0	1143	0	1161	0	1179	0	1197	0	1215	0
977	5	995	5	1013	5	1031	5	1049	5	1067	5	1085	5	1103	5	1121	5	1139	5	1157	5	1175	5	1193	5	1211	5	1229	5	1247	5
1009	1	1027	1	1045	1	1063	1	1081	1	1099	1	1117	1	1135	1	1153	1	1171	1	1189	1	1207	1	1225	1	1243	1	1261	1	1279	1
1041	6	1059	6	1077	6	1095	6	1113	6	1131	6	1149	6	1167	6	1185	6	1203	6	1221	6	1239	6	1257	6	1275	6	1293	6	1311	6
1073	2	1091	2	1109	2	1127	2	1145	2	1163	2	1181	2	1199	2	1217	2	1235	2	1253	2	1271	2	1289	2	1307	2	1325	2	1343	2

We highlight how 9*2 is the horizontal increase of ODD numbers.

Definition 3.3.

We look for cyclicality in the manifestation of the algorithm through powers of 2:

Cycle 16 of tmax applying the algorithm:

$$Din*3+1 = P1\text{-}P4\text{-}P7$$

$$\frac{P1-P4-P7}{2^{tmax}} = \frac{P}{2^{tmax}}$$

$$\frac{P}{2^{tmax}} = Dout$$

D = ODD number
P = EVEN number
Valid for all of the following tables:

Table 3.4.

We find cycle 16 of tmax in Table 2.3.2.3.3. with the variable that alternates {5,>5} appearing at the eleventh ordinal instead of at the sixteenth. The average of the 16 exponents of 2 is minimum: 31/16 = 1.9375 with the eleventh ordinal =5, equal to 2 with exponent =6, or > 2 when the exponent >6.

The table continues infinitely and contains all the Din.

Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
1	1	1	4	4	2	1
3	3	3	10	1	1	5
5	5	5	16	7	4	1
7	7	7	22	4	1	11
0	9	9	28	1	2	7
2	11	11	34	7	1	17
4	13	13	40	4	3	5
6	15	15	46	1	1	23
8	17	17	52	7	2	13
1	19	19	58	4	1	29
3	21	21	64	1	6	1
5	23	23	70	7	1	35
7	25	25	76	4	2	19
0	27	27	82	1	1	41
2	29	29	88	7	3	11
4	31	31	94	4	1	47
6	33	1	100	1	2	25
8	35	3	106	7	1	53
1	37	5	112	4	4	7
3	39	7	118	1	1	59
5	41	9	124	7	2	31
7	43	11	130	4	1	65
0	45	13	136	1	3	17
2	47	15	142	7	1	71
4	49	17	148	4	2	37
6	51	19	154	1	1	77
8	53	21	160	7	5	5
1	55	23	166	4	1	83
3	57	25	172	1	2	43
5	59	27	178	7	1	89
7	61	29	184	4	3	23
0	63	31	190	1	1	95
2	65	1	196	7	2	49
4	67	3	202	4	1	101
6	69	5	208	1	4	13
8	71	7	214	7	1	107
1	73	9	220	4	2	55
3	75	11	226	1	1	113

Definition 3.5. **We filter the powers of 2.**

With the same tmax, the reason for the progression of the ODD input number (Din-Din-1) is an expression of the powers of 2. It is exactly 2^{tmax+1}. If we multiply: $3*2^{tmax+1}=6*2^{tmax}$ we obtain the increase of the numbers ≡ 4 (mod6) expressed in the column P1-P4-P7. <u>While Dout increase is 6</u>.

This applies to all values of tmax with $tmax \in \mathbb{N}_{>0}$

Tabelle 3.6.

Din-Din-1	Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
	3	3	3	10	1	1	5
4	7	7	7	22	4	1	11
4	2	11	11	34	7	1	17
4	6	15	15	46	1	1	23
4	1	19	19	58	4	1	29
4	5	23	23	70	7	1	35
	1	1	1	4	4	2	1
8	0	9	9	28	1	2	7
8	8	17	17	52	7	2	13
8	7	25	25	76	4	2	19
8	6	33	1	100	1	2	25
8	5	41	9	124	7	2	31
	4	13	13	40	4	3	5
16	2	29	29	88	7	3	11
16	0	45	13	136	1	3	17
16	7	61	29	184	4	3	23
16	5	77	13	232	7	3	29
16	3	93	29	280	1	3	35
	5	5	5	16	7	4	1
32	1	37	5	112	4	4	7
32	6	69	5	208	1	4	13
32	2	101	5	304	7	4	19
32	7	133	5	400	4	4	25
32	3	165	5	496	1	4	31

5+6∗n
1+6∗n
5+6∗n
1+6∗n

Din-Din-1	Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
	8	53	21	160	7	5	5
64	0	117	21	352	1	5	11
64	1	181	21	544	4	5	17
64	2	245	21	736	7	5	23
64	3	309	21	928	1	5	29
64	4	373	21	1120	4	5	35
	3	21	21	64	1	6	1
128	5	149	21	448	7	6	7
128	7	277	21	832	4	6	13
128	0	405	21	1216	1	6	19
128	2	533	21	1600	7	6	25
128	4	661	21	1984	4	6	31
	6	213	21	640	1	7	5
256	1	469	21	1408	4	7	11
256	5	725	21	2176	7	7	17
256	0	981	21	2944	1	7	23
256	4	1237	21	3712	4	7	29
256	8	1493	21	4480	7	7	35
	4	85	21	256	4	8	1
512	3	597	21	1792	1	8	7
512	2	1109	21	3328	7	8	13
512	1	1621	21	4864	4	8	19
512	0	2133	21	6400	1	8	25
512	8	2645	21	7936	7	8	31

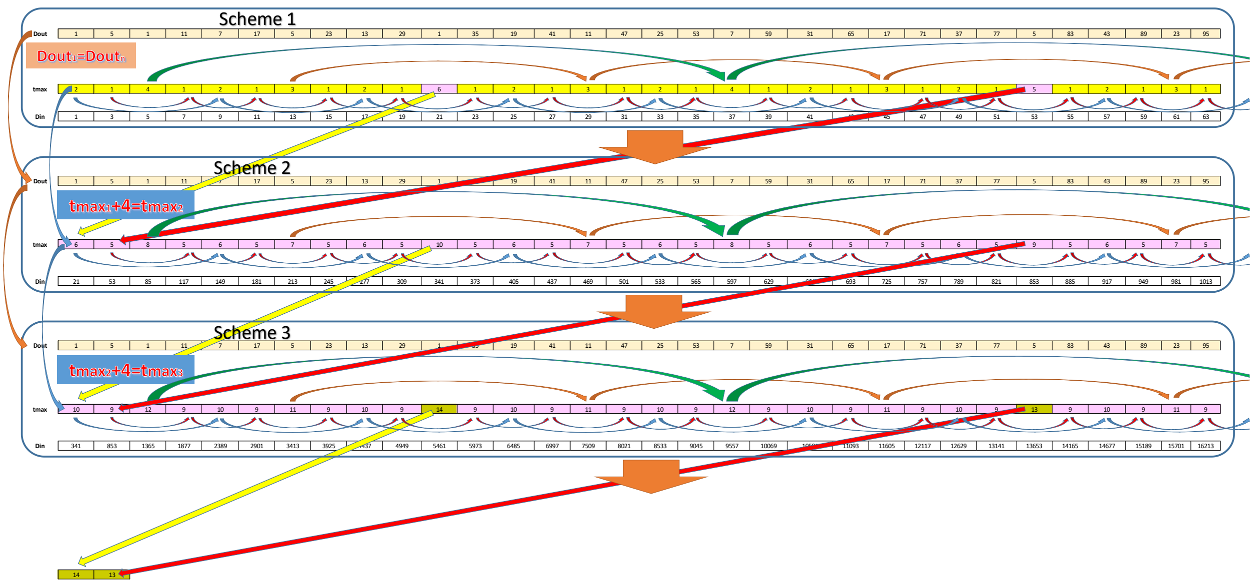
5+6∗n
1+6∗n
5+6∗n
1+6∗n

Din $\equiv$ 21 (mod32).

Equation 3.7.

$Din_{+n}=Din_{start}+2^{tmax+1}*n$, $n\in\mathbb{N}$, $tmax\in\mathbb{N}_{>0}$ **n+1=the nth Din which, multiplied ∗3+1, will be divided by 2 with the same exponent tmax.**
Din_{start}=1th Din with a given tmax.

Directed graph 3.7.1.



Scheme 1: Din_{+1}-Din=2 $\Rightarrow 2^1$
Scheme 2: Din_{+1}-Din=32 $\Rightarrow 2^5$ $2^1*2^4=2^5$, scheme exponent 1+4 = scheme exponent 2
Scheme 3: Din_{+1}-Din=512 $\Rightarrow 2^9$ $2^5*2^4=2^9$, scheme exponent 2+4 = scheme exponent 3
…

We highlight how the tmax of the previous scheme +4 become the tmax of the following one. All this repeats infinitely and allows us a "logarithmic" vision of the graph itself. The Directed graph shows how Din≡21(mod32) are not reached by Equation 3.7. with tmax <5. The same become Din_{start}

and all the other Din of the sequences that share tmax>4. We derive Din from the equation of block 2 and obtain the inverse formula:

$$\text{Din} = \frac{\text{Dout} * 2^{\text{tmax}} - 1}{3}$$

We will prove with Lemma 3.16. that $\text{Dout}*2^{\text{tmax}} \equiv 4 \pmod 6$.

Dout schema 1 = 1 ⇨ $1*2^2 = 4$ ⇨ $\frac{4-1}{3} = 1$ ⇨ Din= 1

Dout scheme 2 = 1 ⇨ $1*2^6 = 64$ ⇨ $\frac{64-1}{3} = 21$ ⇨ Din= 21

Dout scheme 3 = 1 ⇨ $1*2^{10}=1024$ ⇨ $\frac{1024-1}{3} = 341$ ⇨ Din= 341

Dout scheme 4 = 1 ⇨ $1*2^{14}=16384$ ⇨ $\frac{16384-1}{3} = 5461$ ⇨ Din= 5461

Dout scheme 1 = 5 ⇨ $5*2^1 = 10$ ⇨ $\frac{10-1}{3} = 3$ ⇨ Din= 3

Dout scheme 2 = 5 ⇨ $5*2^5 = 160$ ⇨ $\frac{160-1}{3} = 53$ ⇨ Din= 53

Dout scheme 3 = 5 ⇨ $5*2^9= 2560$ ⇨ $\frac{2560-1}{3} = 853$ ⇨ Din= 853

Dout scheme 4 = 5 ⇨ $5*2^{13}=40960$ ⇨ $\frac{40960-1}{3} = 13653$ ⇨ Din=13653

The Directed Graph 3.7.1. shows how scheme 1 repeats infinitely and allows all Din∈ℕ to reach the possible Dout thanks to the powers of 2. The eleventh ordinal of each cycle 16 of tmax is connected to a Din≡ 21 (mod32).

Equation 3.7.2.

Having seen the Direct Graph 3.7.1. we can write:

$\text{Din}_{start}=\frac{1*2^{\text{tmax}}-1}{3}$ with tmax EVEN, $\text{Din}_{start} =\frac{5*2^{\text{tmax}}-1}{3}$ with tmax ODD

Equation 3.7.2.1. Equation 3.7. becomes:

$\text{Din}_{+n}=\frac{1*2^{\text{tmax}}-1}{3}+2^{\text{tmax}+1}*n$, n∈ℕ, Din who share the same tmax EVEN>0

$\text{Din}_{+n}=\frac{5*2^{\text{tmax}}-1}{3}+2^{\text{tmax}+1}*n$, n∈ℕ, Din who share the same tmax ODD

Definition 3.7.2.2.

$$j = \text{tmax mod}2 *4+1 \text{ , } j \in \{1,5\}$$

definition assumed 3.7.2.2., the Equations 3.7.2.1. they become:

Equation 3.7.2.3.

$$\text{Din} = \frac{j*2^{tmax}-1}{3} + 2^{tmax+1}*n, \quad n \in \mathbb{N}$$

Equation 3.7.2.4.

Let's define: $v = \frac{\text{Din} - \text{Din mod}32}{32} \Rightarrow$

v+1= ordinal number of cycle 16 of tmax.

e.g. 5273≡25(mod32) ⇨ $v+1 = \frac{5273-25}{32} + 1 =$ 165th ⇨ the number 5273 is included in the 165th cycle 16 of tmax.

Equation 3.7.2.5.

d+1=(v+1)*a-b, a= number of Din present in each cycle 16 that shares the same tmax.

d+1=ordinal of Din which shares the same tmax:={1,2,3,4,>4}:

Din ≡ {3,7,11,15,19,23,27,31}(mod32) ⇨ tmax= 1 ⇨ a=8

Din mod32=31 ⇨ b=1, Din mod32=27 ⇨ b=2
Din mod32=23 ⇨ b=3, Din mod32=19 ⇨ b=4
Din mod32=15 ⇨ b=5, Din mod32=11 ⇨ b=6
Din mod32= 7 ⇨ b=7, Din mod32= 3 ⇨ b=8

Din ≡ {1,9,17,25}(mod32) ⇨ tmax= 2 ⇨ a=4

Din mod32=25 ⇨ b=1, Din mod32=17 ⇨ b=2
Din mod32= 9 ⇨ b=3, Din mod32= 1 ⇨ b=4

Din ≡ {13,29}(mod32) ⇨ tmax= 3 ⇨ a=2

Din mod32=29 ⇨ b=1, Din mod32=13 ⇨ b=2

Din ≡ {5} (mod32) ⇨ tmax= 4 ⇨ a=1

Din ≡ {21} (mod32) ⇨ tmax> 4 ⇨ a=1

e.g. Din=5265 ⇨ tmax=2, a=4, b=2 ⇨ 165*4-2+1=659° Din which shares tmax=2

e.g. Din=1493 ⇨ tmax>4, a=1, b=1 ⇨ 47*1-1+1= 47° Din which shares tmax>4

with tmax:={1,2,3,4} we will have d=n of the Equation 3.7.2.3. ⇨

$$n=\left(\frac{Din-Din\bmod 32}{32}+1\right)*a-b$$

we will then have:

Equation 3.7.2.6. $tmax:=\{1,2,3,4\} \Rightarrow$

$$Din=\frac{j*2^{tmax}-1}{3}+2^{tmax+1}*\left(\left(\frac{Din-Din\bmod 32}{32}+1\right)*a-b\right)$$

Definition 3.8. $tmax>3 \Rightarrow$

$tmax=4$ manifests itself every $2=2^0$ cycle 16 of tmax $\Rightarrow 2^{tmax}-2^0=2^4$
$tmax=5$ manifests itself every $2=2^1$ cycle 16 of tmax $\Rightarrow 2^{tmax}-2^1=2^4$
$tmax=6$ manifests itself every $4=2^2$ cycle 16 of tmax $\Rightarrow 2^{tmax}-2^2=2^4$
$tmax=7$ manifests itself every $8=2^3$ cycle 16 of tmax $\Rightarrow 2^{tmax}-2^3=2^4$
$tmax=8$ manifests itself every $16=2^4$ cycle 16 of tmax $\Rightarrow 2^{tmax}-2^4=2^4$
...

Equation 3.8.1.
To obtain d=n with $tmax>3$ we must divide v by 2^{tmax-4} and subtract the mantissa function:

$$n=\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}-\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}\bmod 1, \Rightarrow$$

$$Din=\frac{j*2^{tmax}-1}{3}+2^{tmax+1}*\left(\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}-\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}\bmod 1\right)$$

Let us equate n obtained from Equation 3.7.2.3. a n obtained from 3.8.1. and we obtain:

Equation 3.8.2. $n=n \Rightarrow$

$$\frac{Din-\frac{j*2^{tmax}-1}{3}}{2^{tmax+1}}=\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}-\frac{\frac{Din-Din\bmod 32}{32}}{2^{tmax-4}}\bmod 1, \quad tmax>3,\ tmax\in\mathbb{N}$$

Equation 3.8.3. From the 3 equations:

$$\begin{cases}\frac{Din*3+1}{2^{tmax}}=Dout, & \text{Equation in block 2}\\ Din=\frac{j*2^{tmax}-1}{3}+2^{tmax+1}*n, & \text{Equation 3.7.2.3.}\\ Dout=j+6n, & \text{Equation obtained from Direct Graph 3.71. and from Tables 3.6.}\end{cases}$$

we obtain: $\dfrac{\left(\frac{j*2^{tmax}-1}{3}+2^{tmax+1}*n\right)*3+1}{2^{tmax}} = j+6n \Rightarrow$

$\dfrac{j*2^{tmax}-1+6*2^{tmax}*n+1}{2^{tmax}} = j+6n \Rightarrow \quad j+6n = j+6n \Rightarrow$

n=n, $n \in \mathbb{N}$, $tmax \in \mathbb{N}_{>0}$,
equality showing the connection between Din and Dout made by the 2 conditions and tmax, i.e. we can write Din as a function of the same independent variable n that determines Dout together with j. We have previously formalized the "independent" variable n as a function of Din and the internal parameters: tmax, a, b deriving from the algorithm seen through module 32.

Table 3.9. We report the values of Direct Graph 3.7.1 in the following table. The reasons for the progressions will be: $2^{1+4\theta}$, $\theta \in \mathbb{N} \Rightarrow$
$\theta=0 \Rightarrow 2^{1}$, $\theta=1 \Rightarrow 2^{5}$, $\theta=2 \Rightarrow 2^{9}$, $\theta=3 \Rightarrow 2^{13}$, $\theta=4 \Rightarrow 2^{17}$,…

Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
1	1	4	4	2	1
3	3	10	1	1	5
5	5	16	7	4	1
7	7	22	4	1	11
9	9	28	1	2	7
11	11	34	7	1	17
13	13	40	4	3	5
15	15	46	1	1	23
17	17	52	7	2	13
19	19	58	4	1	29
21	21	64	1	6	1
23	23	70	7	1	35
25	25	76	4	2	19
27	27	82	1	1	41
29	29	88	7	3	11
31	31	94	4	1	47
33	1	100	1	2	25
35	3	106	7	1	53
37	5	112	4	4	7
39	7	118	1	1	59
41	9	124	7	2	31
43	11	130	4	1	65
45	13	136	1	3	17
47	15	142	7	1	71
49	17	148	4	2	37
51	19	154	1	1	77
53	21	160	7	5	5
55	23	166	4	1	83
57	25	172	1	2	43
59	27	178	7	1	89
61	29	184	4	3	23
63	31	190	1	1	95

…

Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
21	21	64	1	6	1
53	21	160	7	5	5
85	21	256	4	8	1
117	21	352	1	5	11
149	21	448	7	6	7
181	21	544	4	5	17
213	21	640	1	7	5
245	21	736	7	5	23
277	21	832	4	6	13
309	21	928	1	5	29
341	21	1024	7	10	1
373	21	1120	4	5	35
405	21	1216	1	6	19
437	21	1312	7	5	41
469	21	1408	4	7	11
501	21	1504	1	5	47
533	21	1600	7	6	25
565	21	1696	4	5	53
597	21	1792	1	8	7
629	21	1888	7	5	59
661	21	1984	4	6	31
693	21	2080	1	5	65
725	21	2176	7	7	17
757	21	2272	4	5	71
789	21	2368	1	6	37
821	21	2464	7	5	77
853	21	2560	4	9	5
885	21	2656	1	5	83
917	21	2752	7	6	43
949	21	2848	4	5	89
981	21	2944	1	7	23
1013	21	3040	7	5	95

…

tmax+4

Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
341	21	1024	7	10	1
853	21	2560	4	9	5
1365	21	4096	1	12	1
1877	21	5632	7	9	11
2389	21	7168	4	10	7
2901	21	8704	1	9	17
3413	21	10240	7	11	5
3925	21	11776	4	9	23
4437	21	13312	1	10	13
4949	21	14848	7	9	29
5461	21	16384	4	14	1
5973	21	17920	1	9	35
6485	21	19456	7	10	19
6997	21	20992	4	9	41
7509	21	22528	1	11	11
8021	21	24064	7	9	47
8533	21	25600	4	10	25
9045	21	27136	1	9	53
9557	21	28672	7	12	7
10069	21	30208	4	9	59
10581	21	31744	1	10	31
11093	21	33280	7	9	65
11605	21	34816	4	11	17
12117	21	36352	1	9	71
12629	21	37888	7	10	37
13141	21	39424	4	9	77
13653	21	40960	1	13	5
14165	21	42496	7	9	83
14677	21	44032	4	10	43
15189	21	45568	1	9	89
15701	21	47104	7	11	23
16213	21	48640	4	9	95

…

Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
5461	21	16384	1	14	1
13653	21	40960	7	13	5
21845	21	65536	4	16	1
30037	21	90112	1	13	11
38229	21	114688	7	14	7
46421	21	139264	4	13	17
54613	21	163840	1	15	5
62805	21	188416	7	13	23
70997	21	212992	4	14	13
79189	21	237568	1	13	29
87381	21	262144	7	18	1
95573	21	286720	4	13	35
103765	21	311296	1	14	19
111957	21	335872	7	13	41
120149	21	360448	4	15	11
128341	21	385024	1	13	47
136533	21	409600	7	14	25
144725	21	434176	4	13	53
152917	21	458752	1	16	7
161109	21	483328	7	13	59
169301	21	507904	4	14	31
177493	21	532480	1	13	65
185685	21	557056	7	15	17
193877	21	581632	4	13	71
202069	21	606208	1	14	37
210261	21	630784	7	13	77
218453	21	655360	4	17	5
226645	21	679936	1	13	83
234837	21	704512	7	14	43
243029	21	729088	4	13	89
251221	21	753664	1	15	23
259413	21	778240	7	13	95

…

Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
87381	21	262144	7	18	1
218453	21	655360	4	17	5
349525	21	1048576	1	20	1
480597	21	1441792	7	17	11
611669	21	1835008	4	18	7
742741	21	2228224	1	17	17
873813	21	2621440	7	19	5
1004885	21	3014656	4	17	23
1135957	21	3407872	1	18	13
1267029	21	3801088	7	17	29
1398101	21	4194304	4	22	1
1529173	21	4587520	1	17	35
1660245	21	4980736	7	18	19
1791317	21	5373952	4	17	41
1922389	21	5767168	1	19	11
2053461	21	6160384	7	17	47
2184533	21	6553600	7	18	25
2315605	21	6946816	4	17	53
2446677	21	7340032	1	20	7
2577749	21	7733248	7	17	59
2708821	21	8126464	4	18	31
2839893	21	8519680	1	17	65
2970965	21	8912896	7	19	17
3102037	21	9306112	4	17	71
3233109	21	9699328	1	18	37
3364181	21	10092544	7	17	77
3495253	21	10485760	4	21	5
3626325	21	10878976	1	17	83
3757397	21	11272192	7	18	43
3888469	21	11665408	4	17	89
4019541	21	12058624	1	19	23
4150613	21	12451840	7	17	95

…

Equation 3.10.
We can obtain the Din of the table with Equation 3.7.2.3., or with the following equation which produces the sequences of the Din of all the schemes of the Directed Graph 3.7.1., where scheme 1 is repeated infinite times by varying $\theta \in \mathbb{N}$:

$Din = \dfrac{2^{2+4\theta}-1}{3} + 2^{tmax-1}*m$, $\theta, m \in \mathbb{N}$, $tmax \in \mathbb{N}_{>0}$, $tmax = 2+4\theta \Rightarrow$

$tmax-1 = 1+4\theta \Rightarrow$

$Din = \dfrac{2^{2+4\theta}-1}{3} + 2^{1+4\theta}*m$, $\theta, m \in \mathbb{N} \Rightarrow \quad Din = \dfrac{2^{1+4\theta}*(2+3m)-1}{3} \Rightarrow$

$$2+3m = \frac{3*Din+1}{2^{1+4\theta}} \Rightarrow 4+6m = \frac{3*Din+1}{2^{4\theta}}$$

$\theta=0 \Rightarrow Din=\frac{2^2-1}{3}+2^1*m \Rightarrow Din=\frac{2^1*(2+3m)-1}{3} \Rightarrow Din=1+2*m$

$\theta=1 \Rightarrow Din=\frac{2^6-1}{3}+2^5*m \Rightarrow Din=\frac{2^5*(2+3m)-1}{3} \Rightarrow Din=21+32*m$

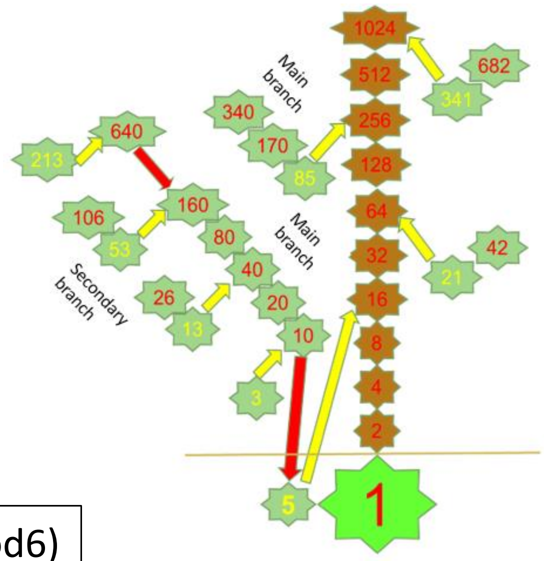
which expresses the succession of the Din≡21(mod32)

$\theta=2 \Rightarrow Din=\frac{2^{10}-1}{3}+2^9*m \Rightarrow Din=\frac{2^9*(2+3m)-1}{3} \Rightarrow Din=341+512*m$

…

Definition 3.11.

The tree of the Collatz graph: the EVEN numbers that cannot be reached using the 3x+1 condition (e.g. 26) after division by 2^{tmax} become ODD and enter Block 2 and the graph.



5 and 1 = roots (mod6)

Lemma 3.12.

We derive the equations in block 2 by observing the data in the tables that highlight the powers of 2. Once we have found the equations we will demonstrate the equality of 2 ways of expressing the input ODD number. We can write for the exponents of 2 ODD:

First ODD number input where the exponent of 2 ODD t = 1 appears for the first time

Sequence: 3,7,11,15… ; (3+4n)= 3 with n=0
(3+4n)= 7 with n=1 ; (3+4n)= 11 with n=2

$$\frac{(3+4n)*3+1}{2^1}$$

We can write 2^{t+1}

$$\frac{(13+16n)*3+1}{2^3}$$

$$\frac{(53+64n)*3+1}{2^5}$$

We are looking for an increase:

$213-53=160 \Rightarrow 10*16 = 10*2^4 \Rightarrow 160=10*2^{t-1}$

$53-13= 40 \Rightarrow 10*4 = 10*2^2 \Rightarrow 40=10*2^{t-1}$

$13-3 = 10 \Rightarrow 10*1 = 10*2^0 \Rightarrow 10=10*2^{t-1}$

$$\frac{(213+256n)*3+1}{2^7}$$

t= exponent of 2

Equation 3.12.1.

in block 2 for ODD powers of 2 with $n \in \mathbb{N}$ and $t \in \mathbb{N}_{>0}$:

$$\frac{\left(\frac{5*2^{t}-1}{3}+2^{t+1}*n\right)*3+1}{2t}=5+6n$$

$$\frac{5*2^{t}-1}{3}+2^{t+1}*n=Din$$

$$Din*3+1=P$$

$$\frac{P}{2^{t}}=Dout$$

Root $=\frac{10*2^{t-1}-1}{3}=\frac{5*2^{t}-1}{3}$ obtained by the inverse function becomes the secondary branch of the tree

Module $=2^{t+1}$

$n = 0$

t=D	inverse formula $\frac{P-1}{3}=Din$	$Din*3+1=P$	$\frac{Din*3+1}{2^{t}}=Dout$
1	3	10	5
3	13	40	5
5	53	160	5
7	213	640	5
9	853	2560	5
11	3413	10240	5
13	13653	40960	5
15	54613	163840	5
17	218453	655360	5
19	873813	2621440	5
21	3495253	10485760	5
23	13981013	41943040	5
25	55924053	167772160	5
27	223696213	671088640	5
29	894784853	2684354560	5
31	3579139413	10737418240	5

synthetic way to generate the Din numbers of the table starting from 13,

$$3+\sum_{t:0}^{\infty}2^{2t}*10$$

or the sequence can be generated recursively:

$a_n = a_{n-1}*q+1$ with $q = 4$ and $a_0 = 3$, $n \in \mathbb{N}_{>0}$

We can write for the exponents of 2 EVEN:

First ODD number input where the exponent of 2 EVEN t = 2 appears for the first time

We can write as 2^{t+1}

We are looking for an increase:

$85 - 21 = 64 \Rightarrow 64 = 2^6$

$21 - 5 = 16 \Rightarrow 16 = 2^4$

$5 - 1 = 4 \Rightarrow 4 = 2^2$

$$\frac{(1 + 8n) * 3 + 1}{2^2}$$

$$\frac{(5 + 32n) * 3 + 1}{2^4}$$

$$\frac{(21 + 128n) * 3 + 1}{2^6}$$

$$\frac{(85 + 512n) * 3 + 1}{2^8}$$

Equation 3.12.1.1.

in block 2 for EVEN powers of 2 with n∈ℕ and t∈$\mathbb{N}_{>0}$:

$$\frac{\left(\frac{2^t - 1}{3} + 2^{t+1} * n\right) * 3 + 1}{2^t} = 1 + 6n$$

$$\frac{2^t - 1}{3} + 2^{t+1} * n = \text{Din}$$

$$\text{Din} * 3 + 1 = \text{P}$$

$$\frac{\text{P}}{2^t} = \text{Dout}$$

Root $= \frac{2^t - 1}{3}$ obtained by the inverse function becomes the main branch of the tree

Module $= 2^{t+1}$

n=0

t=P	inverse formula $\frac{P-1}{3} = \text{Din}$	$\text{Din} * 3 + 1 = \text{P}$ powers of 2, n=0	$\frac{\text{Din} * 3 + 1}{2^t} = \text{Dout}$
2	1	4	1
4	5	16	1
6	21	64	1
8	85	256	1
10	341	1024	1
12	1365	4096	1
14	5461	16384	1
16	21845	65536	1
18	87381	262144	1
20	349525	1048576	1
22	1398101	4194304	1
24	5592405	16777216	1
26	22369621	67108864	1
28	89478485	268435456	1
30	357913941	1073741824	1
32	1431655765	4294967296	1

synthetic way to generate the Din numbers from the table,

$$\sum_{t:0}^{\infty} 2^{2t}$$

or the sequence can be generated recursively:

$$a_n = a_{n-1} * q + 1 \quad \text{with} \quad q = 4 \quad \text{and} \quad a_0 = 1, \quad n \in \mathbb{N}_{>0}$$

Both equations determining the number Din have as modulo 2^{t+1}

The 2 unfolding equations can be written as follows:

$$\frac{Din*3+1}{2^t} = 1 + 6n \qquad \text{with} \qquad t = \text{EVEN}$$

$$Din = \frac{2^t*(1+6n)-1}{3}$$

where $1+6n$ = ODD output

and $2^t*(1+6n)$= EVEN inserted in the inverse formula

$$\frac{Din\ *3+1}{2^t} = 5 + 6n \qquad \text{with} \qquad t = \text{ODD}$$

$$Din = \frac{2^t*(5+6n)-1}{3}$$

where $5+6n$ = ODD output

and $2^t*(5+6n)$= EVEN inserted in the inverse formula

Proof of the equivalence of the 2 ways of expressing Din:

$$Din = Din \quad \Rightarrow \quad \frac{2^t-1}{3} + 2^{t+1} * n = \frac{2^t*(1+6n)-1}{3}$$

$$2^t+2^{t+1}*3n-1 = 2^t+2^t*6n-1$$

$$2^t+2^{t+1}*3n-1 = 2^t+2^{t+1}*3n-1$$

$$Din = Din \quad \Rightarrow \quad \frac{5*2^t-1}{3} + 2^{t+1} * n = \frac{2^t(5+6n)-1}{3}$$

$$5*2^t+2^{t+1}*3n-1 = 5*2^t+2^t*6n -1$$

$$5*2^t+2^{t+1}*3n-1 = 5*2^t+2^{t+1}*3n-1$$

Using Definition 3.7.2.2. we can write:

Equation 3.12.1.2.

$$Din = Din \;\Rightarrow\; \frac{j*2^t-1}{3} + 2^{t+1} * n = \frac{2^t*(j+6n)-1}{3}, \quad :\forall n\in\mathbb{N},\ t\in\mathbb{N}_{>0},\ t=tmax$$

where the first member of the equation is the second of Equation 3.7.2.3. ☐

Lemma 3.12.2. We assume the inverse function and apply the 2 conditions. We multiply the numbers ODD $*2^t$ and insert the product into the inverse formula: $\frac{2^t*(1+2m)-1}{3}$, **varying t and m we obtain Din={1+2n}, n∈ℕ:**

Proof

$t = 1 \quad \frac{1+4m}{3}, \quad m=2+3n \quad \Rightarrow \quad \frac{9+12n}{3} = 3 + 4n,$

$t = 2 \quad \frac{3+8m}{3}, \quad m=3n \quad \Rightarrow \quad \frac{3+24n}{3} = 1 + 8n,$

$t = 3 \quad \frac{7+16m}{3}, \quad m=2+3n \quad \Rightarrow \quad \frac{39+48n}{3} = 13 + 16n,$

$t = 4 \quad \frac{15+32m}{3}, \quad m=3n \quad \Rightarrow \quad \frac{15+96n}{3} = 5 + 32n,$

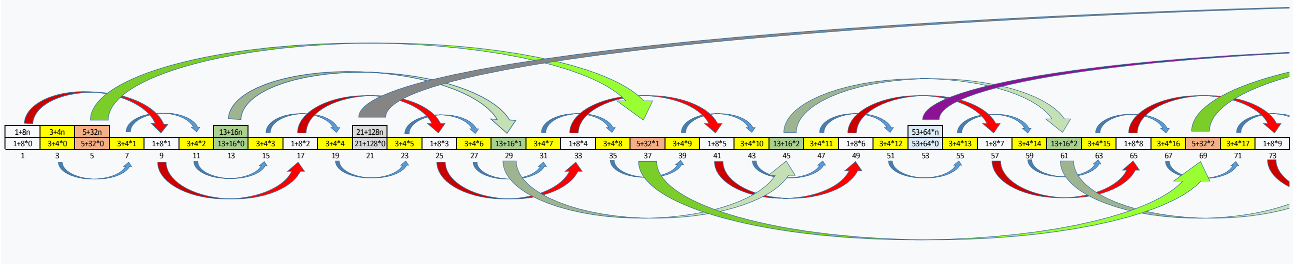
$t = 5 \quad \frac{31+64m}{3}, \quad m=2+3n \quad \Rightarrow \quad \frac{159+192n}{3} = 53 + 64n,$

$t = 6 \quad \frac{63+128m}{3}, \quad m=3n \quad \Rightarrow \quad \frac{63+384n}{3} = 21 + 128n,$

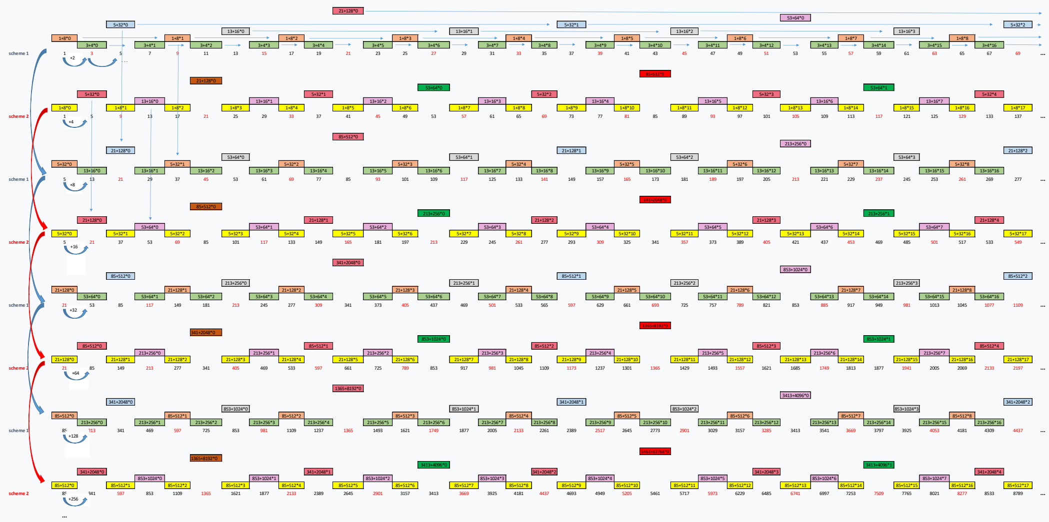
…

if $m\neq 2+3n$ with $t=1+2p$ and if $m\neq 3n$ with $t=2k$ the formula does not generate an integer since $2^t*(1+2m)$ is not $\equiv 4 \pmod 6$. The roots of the sequences obtained are $\frac{j*2^t-1}{3}$ and the modulus 2^{t+1}. All this returns the first member of equation 3.12.1.2.

Directed graph 3.12.2.1. Let's observe how the sequences with a higher modulus generated, fill the gaps left by the previous ones, allowing the algorithm to reach all the ODD numbers:



Directed graph 3.12.2.2.



The two patterns are repeated alternately endlessly. The reasons for Din progressions are the powers of 2 excluding 2^0. By multiplying the modules of any node by 2^2 we obtain the module of the corresponding following node, relating to the same scheme. □

Equation 3.12.3. The Directed Graph 3.12.2.2. proves that the first member of Equation 3.12.1.2. is true, therefore the second is also true, since we have shown the equivalence of the 2 expressions, we therefore write:

$$\frac{2^t*(j+6n)-1}{3} = \frac{2^t*(1+2*(h+3n))-1}{3}, \Rightarrow j+6n=1+2*(h+3n)$$

with h=0 if t=2q, h=2 if t=1+2p, $h\in\{0,2\}$:$\forall t,q\in\mathbb{N}_{>0}$, $n,p\in\mathbb{N}$, $t=t_{max}$

j+6n=1+2m if m=h+3n

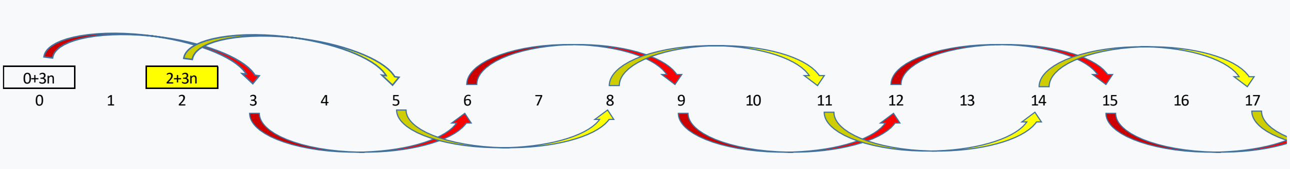
$t=2q \Rightarrow j+6n=1+6n$

$t=2q \Rightarrow m=0+3n \Rightarrow 1+2m=1+2*3n \Rightarrow 1+2m=1+6n$

$t=1+2p \Rightarrow j+6n=5+6n$

$t=1+2p \Rightarrow m=2+3n \Rightarrow 1+2m=1+4+6n \Rightarrow 1+2m=5+6n$

Directed graph 3.12.4. of $m=h+3n \Rightarrow m=\{0+3n, 2+3n\}, n\in\mathbb{N}$



We highlight that in Directed graph 3.12.4. the sequence m=1+3n and

$1+3n\equiv\{1,4,7\}(\bmod 9)$ is missing. This does not surprise us given what was stated in Lemma 2.9.

$1+2*(1+3n)=3+6n$ and $\{3+6n\}$ represents multiples of 3 ODD which when inserted into the inverse formula do not generate integers.

So $\forall m,n \in \mathbb{N}$, $t \in \mathbb{N}_{>0} = t_{max}$

$$\begin{cases} 1+2m = \dfrac{2^t*(1+2*(h+3n))-1}{3} \Rightarrow 4+6m = 2^t*(1+2h+6n) \\ 1+2m = \dfrac{2^t*(j+6n)-1}{3} \Rightarrow m = \dfrac{2^t*(j+6n)-4}{6} \Rightarrow \end{cases}$$

$4+6*\frac{2^t*(j+6n)-4}{6} = 2^t*(1+2h+6n) \Rightarrow j+6n = 1+2h+6n$, **with** $\mathbf{j = 1+2h} \Rightarrow$

if $t=2q \Rightarrow j=1, h=0 \Rightarrow 1=1+2*0$, if $t=1+2p \Rightarrow j=5, h=2 \Rightarrow 5=1+2*2$

$h=\frac{j-1}{2} \Rightarrow 1+2m=1+2*(h+3n) \Rightarrow 1+2m=1+2*\left(\frac{j-1}{2}+3n\right) \Rightarrow 1+2m=j+6n$

so $4+6m = 2^t*(1+2h+6n) \Rightarrow 4+6m = 2^t*(j+6n) \Rightarrow$

Equation 3.12.5. Equation in block 2

$$\frac{\mathbf{(1+2m)*3+1}}{\mathbf{2^{tmax}}} = j+6n, \quad :\forall m,n \in \mathbb{N},\ tmax \in \mathbb{N}_{>0}$$

$j = t \bmod 2 * 4 + 1$, $j \in \{1,5\}$, tmax of $(1+2m)*3+1$

Lemmas 2.7 and 2.9 show that the ODD numbers subjected to the 3x+1 condition and the /2 condition reiterated tmax times generate $Dout\equiv\{1,5\}(\bmod 6)$.

Table 3.12.5.1. Theorem 2.3. it shows that: $\{2n\}=\{2^t*(1+2p)\}$, $n,t \in \mathbb{N}_{>0}$, $m,p \in \mathbb{N} \Rightarrow \frac{2^t*(1+2p)-1}{3}=1+2m \Rightarrow \mathbf{\frac{2n-1}{3} =1+2m}$ **se** $\mathbf{n=2+3m}$ $\frac{2*(2+3m)-1}{3}=1+2m$ $\quad \frac{4+6m-1}{3}=1+2m \quad 4+6m=4+6m$

	4+6m	1+2m
	n=2+3m	
m	2n	(2n-1)/3
	2	0,333333
0	4	1
	6	1,666667
	8	2,333333
1	10	3
	12	3,666667
	14	4,333333
2	16	5
	18	5,666667
	20	6,333333
3	22	7
	24	7,666667
	26	8,333333
4	28	9
	30	9,666667
	32	10,33333
5	34	11
	36	11,66667
	38	12,33333
6	40	13

Equation 3.12.5.2. $\mathbf{\frac{2n-1}{3} =1+2m}$ if $n=2+3m$, $m \in \mathbb{N}$ $n \in \mathbb{N}_{>0}$

The sequence $2+3m$ generates numbers $\equiv\{2,5,8\}(\bmod 9)$,

$2*(2+3m)=4+6m \Rightarrow 4+6m\equiv\{1,4,7\}(\bmod 9)$

Expressing 1+2m as a function of n and t, the variable t will automatically be equal to tmax because it itself determines Din and therefore we can write Equation 3.8.3. :

Equation 3.12.6.

$$\frac{\left(j\frac{2^t-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=\mathbf{j+6n},\ :\forall n\in\mathbb{N},\ t\in\mathbb{N}_{>0}$$

Equation 3.12.7.

Having seen Equation 3.7.2.3. we can write: $n=\frac{Din-Dinstart}{2^{tmax+1}}\ \Rightarrow$

$$n=\frac{Din-\frac{j*2^{tmax}-1}{3}}{2^{tmax+1}}\quad\Rightarrow\quad Dout=\left(\frac{Din-\frac{j*2^{tmax}-1}{3}}{2^{tmax+1}}\right)*6+j\quad\Rightarrow$$

$$\mathbf{Dout}=\left(\frac{Din-\frac{j*2^{tmax}-1}{3}}{2^{tmax}}\right)*3+j\quad\Rightarrow$$

$Dout=\frac{3*Din-j*2^{tmax}+1}{2^{tmax}}+j\ \Rightarrow\ Dout=\frac{Din*3+1}{2^{tmax}}$, which shows the equality of equations 3.12.5. and 3.12.6.

e.g. : $$31=\frac{165-\frac{1*2^4-1}{3}}{2^4}*3+1,\ \Rightarrow\ 31=\frac{165*3+1}{2^4}$$

e.g. : $$35=\frac{1493-\frac{5*2^7-1}{3}}{2^7}*3+5,\ \Rightarrow\ 35=\frac{1493*3+1}{2^7}$$ □

We generate the following tables using the equation 3.12.6.:

Tables 3.13. Let's observe how by varying Din we obtain the same Dout due to the variable t=tmax:

equation for exponents of 2 EVEN
equazione per esponenti di 2 PARI

$$\frac{\left(\frac{2^t-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=1+6n$$

t=>2 n= 0

(mod6)	Din	P = Din*3+1	2^t	Dout = $P/2^t$		Dout = 1+6*n	(mod6)	t
1	1	4	4	1	=	1	1	2
5	5	16	16	1	=	1	1	4
3	21	64	64	1	=	1	1	6
1	85	256	256	1	=	1	1	8
5	341	1024	1024	1	=	1	1	10
3	1365	4096	4096	1	=	1	1	12
1	5461	16384	16384	1	=	1	1	14
5	21845	65536	65536	1	=	1	1	16
3	87381	262144	262144	1	=	1	1	18
1	349525	1048576	1048576	1	=	1	1	20
5	1398101	4194304	4194304	1	=	1	1	22
3	5592405	16777216	16777216	1	=	1	1	24
1	22369621	67108864	67108864	1	=	1	1	26
5	89478485	268435456	268435456	1	=	1	1	28
3	357913941	1073741824	1073741824	1	=	1	1	30
1	1431655765	4294967296	4294967296	1	=	1	1	32
5	5726623061	17179869184	17179869184	1	=	1	1	34
3	22906492245	68719476736	68719476736	1	=	1	1	36
1	91625968981	274877906944	274877906944	1	=	1	1	38
5	366503875925	1099511627776	1099511627776	1	=	1	1	40
1	1466015503701	4398046511104	4398046511104	1	=	1	1	42
5	5864062014805	17592186044416	17592186044416	1	=	1	1	44
3	23456248059221	70368744177664	70368744177664	1	=	1	1	46
1	93824992236885	281474976710656	281474976710656	1	=	1	1	48
5	375299968947541	1125899906842620	1125899906842620	1	=	1	1	50

equation for exponents of 2 ODD
equazione per esponenti di 2 DISPARI

$$\frac{\left(\frac{10*2^{t-1}-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=5+6n$$

t=>1 n= 0

(mod6)	Din	P = Din*3+1	2^t	Dout = $P/2^t$		Dout = 5+6*n	(mod6)	t
3	3	10	2	5	=	5	5	1
1	13	40	8	5	=	5	5	3
5	53	160	32	5	=	5	5	5
3	213	640	128	5	=	5	5	7
1	853	2560	512	5	=	5	5	9
5	3413	10240	2048	5	=	5	5	11
3	13653	40960	8192	5	=	5	5	13
1	54613	163840	32768	5	=	5	5	15
5	218453	655360	131072	5	=	5	5	17
3	873813	2621440	524288	5	=	5	5	19
1	3495253	10485760	2097152	5	=	5	5	21
5	13981013	41943040	8388608	5	=	5	5	23
3	55924053	167772160	33554432	5	=	5	5	25
1	223696213	671088640	134217728	5	=	5	5	27
5	894784853	2684354560	536870912	5	=	5	5	29
3	3579139413	10737418240	2147483648	5	=	5	5	31
1	14316557653	42949672960	8589934592	5	=	5	5	33
5	57266230613	171798691840	34359738368	5	=	5	5	35
3	229064922453	687194767360	137438953472	5	=	5	5	37
3	916259689813	2748779069440	549755813888	5	=	5	5	39
1	3665038759253	10995116277760	2199023255552	5	=	5	5	41
5	14660155037013	43980465111040	8796093022208	5	=	5	5	43
3	58640620148053	175921860444160	35184372088832	5	=	5	5	45
1	234562480592213	703687441776640	140737488355328	5	=	5	5	47
5	938249922368853	2814749767106560	562949953421312	5	=	5	5	49

Tables 3.14 It is trivial to observe how Dout is a function of n.

equation for exponents of 2 EVEN / equazione per esponenti di 2 PARI	t=	t=>2: 2

$$\frac{\left(\frac{2^t-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=1+6n$$

(mod6)	Din	P = Din*3+1	2^t	Dout = P/2^t		Dout = 1+6*n	(mod6)	n
1	1	4	4	1	=	1	1	0
3	9	28	4	7	=	7	1	1
5	17	52	4	13	=	13	1	2
1	25	76	4	19	=	19	1	3
3	33	100	4	25	=	25	1	4
5	41	124	4	31	=	31	1	5
1	49	148	4	37	=	37	1	6
3	57	172	4	43	=	43	1	7
5	65	196	4	49	=	49	1	8
1	73	220	4	55	=	55	1	9
3	81	244	4	61	=	61	1	10
5	89	268	4	67	=	67	1	11
1	97	292	4	73	=	73	1	12

equation for exponents of 2 EVEN / equazione per esponenti di 2 PARI	t=	t=>2: 18

$$\frac{\left(\frac{2^t-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=1+6n$$

(mod6)	Din	P = Din*3+1	2^t	Dout = P/2^t		Dout = 1+6*n	(mod6)	n
3	87381	262144	262144	1	=	1	1	0
5	611669	1835008	262144	7	=	7	1	1
1	1135957	3407872	262144	13	=	13	1	2
3	1660245	4980736	262144	19	=	19	1	3
5	2184533	6553600	262144	25	=	25	1	4
1	2708821	8126464	262144	31	=	31	1	5
3	3233109	9699328	262144	37	=	37	1	6
5	3757397	11272192	262144	43	=	43	1	7
1	4281685	12845056	262144	49	=	49	1	8
3	4805973	14417920	262144	55	=	55	1	9
5	5330261	15990784	262144	61	=	61	1	10
1	5854549	17563648	262144	67	=	67	1	11
3	6378837	19136512	262144	73	=	73	1	12

equation for exponents of 2 ODD / equazione per esponenti di 2 DISPARI	t=	t=>1: 3

$$\frac{\left(\frac{10*2^{t-1}-1}{3}+2^{t+1}*n\right)*3+1}{2^t}=5+6n$$

(mod6)	Din	P = Din*3+1	2^t	Dout = P/2^t		Dout = 5+6*n	(mod6)	n
1	13	40	8	5	=	5	5	0
5	29	88	8	11	=	11	5	1
3	45	136	8	17	=	17	5	2
1	61	184	8	23	=	23	5	3
5	77	232	8	29	=	29	5	4
3	93	280	8	35	=	35	5	5
1	109	328	8	41	=	41	5	6
5	125	376	8	47	=	47	5	7
3	141	424	8	53	=	53	5	8
1	157	472	8	59	=	59	5	9
5	173	520	8	65	=	65	5	10
3	189	568	8	71	=	71	5	11
1	205	616	8	77	=	77	5	12

equation for exponents of 2 ODD / equazione per esponenti di 2 DISPARI	t=	t=>1: 11

$$\frac{\left(\frac{10*2^{t-1}-1}{3}+2^{t+1}*n\right)*3+1}{2^t}\quad 5+6n$$

(mod6)	Din	P = Din*3+1	2^t	Dout = P/2^t		Dout = 5+6*n	(mod6)	n
5	3413	10240	2048	5	=	5	5	0
3	7509	22528	2048	11	=	11	5	1
1	11605	34816	2048	17	=	17	5	2
5	15701	47104	2048	23	=	23	5	3
3	19797	59392	2048	29	=	29	5	4
1	23893	71680	2048	35	=	35	5	5
5	27989	83968	2048	41	=	41	5	6
3	32085	96256	2048	47	=	47	5	7
1	36181	108544	2048	53	=	53	5	8
5	40277	120832	2048	59	=	59	5	9
3	44373	133120	2048	65	=	65	5	10
1	48469	145408	2048	71	=	71	5	11
5	52565	157696	2048	77	=	77	5	12

Remark 3.15.

Numbers in and out of Block 2 ODD-ODD.

The algorithm using the formula: $\frac{3x+1}{2^{tmax}}$ "eliminates" the ODD numbers ≡ 3 (mod6) (multiples of 3 highlighted in yellow), which will not be repeated as input in the next cycle.
The same as can be seen from the table below are ODD ≡ 0,3,6 (mod9) which become EVEN ≡ 1 (mod9) (highlighted in red) after applying the condition 3x+1.

Din	(mod6)	(mod9)	P= Din*3+1	(mod6)	(mod9)	Dout	(mod6)	(mod9)	tmax
1	1		4	4	4	1	1	1	2
3	3	3	10	4	1	5	5	5	1
5	5	5	16	4	7	1	1	1	4
7	1	7	22	4	4	11	5	2	1
9	3	0	28	4	1	7	1	7	2
11	5	2	34	4	7	17	5	8	1
13	1	4	40	4	4	5	5	5	3
15	3	6	46	4	1	23	5	5	1
17	5	8	52	4	7	13	1	4	2
19	1	1	58	4	4	29	5	2	1
21	3	3	64	4	1	1	1	1	6
23	5	5	70	4	7	35	5	8	1
25	1	7	76	4	4	19	1	1	2
27	3	0	82	4	1	41	5	5	1
29	5	2	88	4	7	11	5	2	3
31	1	4	94	4	4	47	5	2	1
33	3	6	100	4	1	25	1	7	2
35	5	8	106	4	7	53	5	8	1
37	1	1	112	4	4	7	1	7	4
39	3	3	118	4	1	59	5	5	1
41	5	5	124	4	7	31	1	4	2
43	1	7	130	4	4	65	5	2	1
45	3	0	136	4	1	17	5	8	3
47	5	2	142	4	7	71	5	8	1
49	1	4	148	4	4	37	1	1	2
51	3	6	154	4	1	77	5	5	1
53	5	8	160	4	7	5	5	5	5
55	1	1	166	4	4	83	5	2	1
57	3	3	172	4	1	43	1	7	2
59	5	5	178	4	7	89	5	8	1
61	1	7	184	4	4	23	5	5	3
63	3	0	190	4	1	95	5	5	1

Lemma 3.16.

We will have 2 possible output roots: {1,5} (mod6)

Proof

According to the properties of congruences modulus n:

$$[a]+[b]=[a+b]$$

$$x \equiv 4(\text{mod}6)+y \equiv 0(\text{mod}6)= z \Rightarrow z \equiv 4(\text{mod}6)$$

$$4+6m \equiv 4(\text{mod}6)$$

$$\left.\begin{array}{l} 2^t*(5+6n) \equiv 4(\text{mod}6) \\ 2^{1+2s}*6n \equiv 0(\text{mod}6) \\ 2^{2s}*10 \equiv 4(\text{mod}6) \end{array}\right\} \begin{array}{l} t=1+2s,\ t\in\mathbb{N}_{>0} \\ n\in\mathbb{N} \\ s\in\mathbb{N} \end{array}$$

s	2^{2s}	2^{2s} mod6	$2^{2s}*10$	$2^{2s}*10$ mod6
0			10	4
1	4	4	40	4
2	16	4	160	4
3	64	4	640	4
4	256	4	2560	4
5	1024	4	10240	4
6	4096	4	40960	4
7	16384	4	163840	4
8	65536	4	655360	4
9	262144	4	2621440	4
10	1048576	4	10485760	4
11	4194304	4	41943040	4
12	16777216	4	167772160	4
13	67108864	4	671088640	4
14	268435456	4	2684354560	4
15	1073741824	4	10737418240	4
16	4294967296	4	42949672960	4
17	17179869184	4	171798691840	4
18	68719476736	4	687194767360	4
19	274877906944	4	2748779069440	4

$$\left.\begin{array}{l} 2^t*(1+6n) \equiv 4(\text{mod}6) \\ 2^{2s}*6n \equiv 0(\text{mod}6) \\ 2^{2s} \equiv 4(\text{mod}6) \end{array}\right\} \begin{array}{l} t=2s,\quad t\in\mathbb{N}_{>0} \\ n\in\mathbb{N} \\ s\in\mathbb{N}_{>0} \end{array}$$

$3*2^t$ with $t \in \mathbb{N}_{>0}$

it will never be $\equiv 4(\text{mod}6)$

$4+6m \neq 2^t*(3+6n)$

Given Definition 2.5 we can state that ODD numbers have roots {1,3,5}(mod6). ODD numbers ≡ {1.5}(mod6) multiplied 2^{tmax} are ≡ 4 (mod6), while multiples of 3 are not, therefore at the exit from block 2 we will only have ODD numbers ≡ {1.5}(mod6).

By multiplying the ODD numbers ≡{1,3,5}(mod6), <u>which are all the ODD numbers</u>, by 3+1, having seen Lemma 2.9. we obtain:

$$3*(6p+1)+1 \Rightarrow 18p+4 \equiv 4(\text{mod}9) \Rightarrow 18p+4 \equiv 4\ (\text{mod}6)$$
$$3*(6p+3)+1 \Rightarrow 18p+10 \equiv 1(\text{mod}9) \Rightarrow 18p+10 \equiv 4\ (\text{mod}6)$$
$$3*(6p+5)+1 \Rightarrow 18p+16 \equiv 7(\text{mod}9) \Rightarrow 18p+16 \equiv 4\ (\text{mod}6)$$

Unfolding the equation 3.12.6.:

$$\left(\frac{j*2^{t}-1}{3}+2^{t+1}*n\right)*3+1=2^{t}*(j+6n)$$

$$j*2^{t}+3*2^{t+1}*n=2^{t}*(j+6n)$$

$$j*2^{t}+6n*2^{t}=j*2^{t}+6n*2^{t}$$

we obtain $\boxed{\mathbf{4+6m=j*2^{tmax}+6n*2^{tmax}}}$

$\mathbf{j*2^{t}=1*4*2^{t-2}}$ con $tmax=2d$, $tmax,d\in\mathbb{N}_{>0}$, $j=1$ $\Rightarrow$ $\mathbf{4*2^{t-2}\equiv 4(mod6)}$

$\mathbf{j*2^{t}=5*2*2^{t-1}}$ con $tmax=1+2p$, $tmax\in\mathbb{N}_{>0}, p\in\mathbb{N}$, $j=5\Rightarrow$ $\mathbf{10*2^{t-1}\equiv 4(mod6)}$

$$\mathbf{6n*2^{t}\equiv 0(mod6)}$$ □

Table 3.16.1. We highlight with the same colour the 32 cycles of Dout ≡ {1,5}(mod6) which are repeated ad infinitum.

4+18p	(mod6)	tmax	$\frac{4+18p}{2^{tmax}}$	(mod6)
4	4	2	1	1
22	4	1	11	5
40	4	3	5	5
58	4	1	29	5
76	4	2	19	1
94	4	1	47	5
112	4	4	7	1
130	4	1	65	5
148	4	2	37	1
166	4	1	83	5
184	4	3	23	5
202	4	1	101	5
220	4	2	55	1
238	4	1	119	5
256	4	8	1	1
274	4	1	137	5
292	4	2	73	1
310	4	1	155	5
328	4	3	41	5
346	4	1	173	5
364	4	2	91	1
382	4	1	191	5
400	4	4	25	1
418	4	1	209	5
436	4	2	109	1
454	4	1	227	5
472	4	3	59	5
490	4	1	245	5
508	4	2	127	1
526	4	1	263	5
544	4	5	17	5
562	4	1	281	5
580	4	2	145	1
598	4	1	299	5
616	4	3	77	5
634	4	1	317	5
652	4	2	163	1
670	4	1	335	5
688	4	4	43	1
706	4	1	353	5
724	4	2	181	1
742	4	1	371	5
760	4	3	95	5
778	4	1	389	5
796	4	2	199	1
814	4	1	407	5
832	4	6	13	1
850	4	1	425	5
868	4	2	217	1
886	4	1	443	5
904	4	3	113	5
922	4	1	461	5
940	4	2	235	1
958	4	1	479	5
976	4	4	61	1
994	4	1	497	5
1012	4	2	253	1
1030	4	1	515	5
1048	4	3	131	5
1066	4	1	533	5
1084	4	2	271	1
1102	4	1	551	5
1120	4	5	35	5
1138	4	1	569	5

10+18p	(mod6)	tmax	$\frac{10+18p}{2^{tmax}}$	(mod6)
10	4	1	5	5
28	4	2	7	1
46	4	1	23	5
64	4	6	1	1
82	4	1	41	5
100	4	2	25	1
118	4	1	59	5
136	4	3	17	5
154	4	1	77	5
172	4	2	43	1
190	4	1	95	5
208	4	4	13	1
226	4	1	113	5
244	4	2	61	1
262	4	1	131	5
280	4	3	35	5
298	4	1	149	5
316	4	2	79	1
334	4	1	167	5
352	4	5	11	5
370	4	1	185	5
388	4	2	97	1
406	4	1	203	5
424	4	3	53	5
442	4	1	221	5
460	4	2	115	1
478	4	1	239	5
496	4	4	31	1
514	4	1	257	5
532	4	2	133	1
550	4	1	275	5
568	4	3	71	5
586	4	1	293	5
604	4	2	151	1
622	4	1	311	5
640	4	7	5	5
658	4	1	329	5
676	4	2	169	1
694	4	1	347	5
712	4	3	89	5
730	4	1	365	5
748	4	2	187	1
766	4	1	383	5
784	4	4	49	1
802	4	1	401	5
820	4	2	205	1
838	4	1	419	5
856	4	3	107	5
874	4	1	437	5
892	4	2	223	1
910	4	1	455	5
928	4	5	29	5
946	4	1	473	5
964	4	2	241	1
982	4	1	491	5
1000	4	3	125	5
1018	4	1	509	5
1036	4	2	259	1
1054	4	1	527	5
1072	4	4	67	1
1090	4	1	545	5
1108	4	2	277	1
1126	4	1	563	5
1144	4	3	143	5

16+18p	(mod6)	tmax	$\frac{16+18p}{2^{tmax}}$	(mod6)
16	4	4	1	1
34	4	1	17	5
52	4	2	13	1
70	4	1	35	5
88	4	3	11	5
106	4	1	53	5
124	4	2	31	1
142	4	1	71	5
160	4	5	5	5
178	4	1	89	5
196	4	2	49	1
214	4	1	107	5
232	4	3	29	5
250	4	1	125	5
268	4	2	67	1
286	4	1	143	5
304	4	4	19	1
322	4	1	161	5
340	4	2	85	1
358	4	1	179	5
376	4	3	47	5
394	4	1	197	5
412	4	2	103	1
430	4	1	215	5
448	4	6	7	1
466	4	1	233	5
484	4	2	121	1
502	4	1	251	5
520	4	3	65	5
538	4	1	269	5
556	4	2	139	1
574	4	1	287	5
592	4	4	37	1
610	4	1	305	5
628	4	2	157	1
646	4	1	323	5
664	4	3	83	5
682	4	1	341	5
700	4	2	175	1
718	4	1	359	5
736	4	5	23	5
754	4	1	377	5
772	4	2	193	1
790	4	1	395	5
808	4	3	101	5
826	4	1	413	5
844	4	2	211	1
862	4	1	431	5
880	4	4	55	1
898	4	1	449	5
916	4	2	229	1
934	4	1	467	5
952	4	3	119	5
970	4	1	485	5
988	4	2	247	1
1006	4	1	503	5
1024	4	10	1	1
1042	4	1	521	5
1060	4	2	265	1
1078	4	1	539	5
1096	4	3	137	5
1114	4	1	557	5
1132	4	2	283	1
1150	4	1	575	5

Lemma 3.16.2. Given a "mother" set consisting of at least 2 "children" subsets, inductive basis, e.g. {2+4n,4+4n}={2+2p} :∀n,p∈ℕ, the equality is true if the roots of the subsets are equal in number and value to the numbers expressed in the "mother" sequence starting from the first: 2+2*0=2, 2+2*1=4 ⇨ {2, 4}, if the differences of the roots, i.e. the interval between them which we will call phase shift, coincides with the module of the "mother" sequence: 4-2=2 ⇨ {2}, and if the modules of the "children" sequences are equal and coincide with the product of the "mother" module by the number of "child" sequences: 4=2*2 ⇨ {4}.

Proof

Zero step: Directed graph 2.3.2.

Inductive step: Table 3.16.1

{4+18p,10+18p,16+18p}={4+6p}

4+6*0=**4**, 4+6*1=**10**, 4+6*2=**16**; 16-10=**6**, 10-4=**6**; **18**=6*3 □

Lemma 3.16.3. $x = 1+2m$

$$3*(1+2m)+1 = 4+6m$$

$\frac{4+6m}{2^{tmax}} = 1+6n$ after dividing a power of 2 EVEN

$\frac{4+6m}{2^{tmax}} = 5+6n$ after dividing a power of 2 ODD.

Proof

Leading to the second member of equation 2^t, with t=tmax, we can write for t=ODD, $t=\{1+2p\}$, $p\in\mathbb{N}$, m =ODD, $m=1+2^t n+c$, $m\in\mathbb{N}_{>0}$, $n\in\mathbb{N}$:

t=1, c=0, $4+6*(1+2^t n+c) = 2^t*(\mathbf{5+6n})$

$$4+6+12n = 10+12n$$

$$10+12n = 10+12n$$

$$10*2^0+(2^1+10*2^0)*n = 10*2^0+(2+10*2^0)*n$$

We can write for t= ODD>1: m = EVEN, $m= 1+2^t n+c$, $m\in\mathbb{N}_{>0}$, $n\in\mathbb{N}$ $c=ODD\in\mathbb{N}_{>0}$, $c= \sum_{t=3}^{t} 5*2^{t-3}$:

t=3, c=5, $4+6*(1+2^3 n+5) = 2^3*(\mathbf{5+6n})$

$$40+48n=40+48n$$

$$10*2^2+(2^3+10*2^2)*n = 10*2^2+(2^3+10*2^2)*n$$

t=5, c=25, $4+6*(1+2^5 n+25) = 2^5*(\mathbf{5+6n})$

$$160+192n = 160+192n$$

$$10*2^4+(2^5+10*2^4)*n = 10*2^4+(2^5+10*2^4)*n$$

t=7, c=105, $4+6*(1+2^7 n+105) = 2^7*(\mathbf{5+6n})$

$$640+768n = 640+768n$$

$$10*2^6+(2^7+10*2^6)*n = 10*2^6+(2^7+10*2^6)*n$$

t=9, c=425, $4+6*(1+2^9 n+425) = 2^9*(\mathbf{5+6n})$

$$2560+3072n = 2560+3072n$$

$$10*2^8+(2^9+10*2^8)*n = 10*2^8+(2^9+10*2^8)*n$$

t=11, c=1705, $4+6*(1+2^{11} n+1705) = 2^{11}*(\mathbf{5+6n})$

$$10240+12288n = 10240+12288n$$

$$10*2^{10}+(2^{11}+10*2^{10})*n = 10*2^{10}+(2^{11}+10*2^{10})*n \Rightarrow$$

$$10*2^{t-1}+(2^t+10*2^{t-1})*n = \mathbf{5*2^t+(2^t+5*2^t)*n} \Rightarrow$$

$$5*2^t+(2^t*(1+5))*n = 5*2^t+2^t*6n \Rightarrow 5*2^t+2^t*6n = \mathbf{2^t*(5+6n)}$$

$$\mathbf{4+6m} = 4+6*(1+2^t n+c) \Rightarrow 5*2^t+(2^t+5*2^t)*n = \mathbf{2^t*(5+6n)}$$

t = ODD, t=tmax of 4+6m ; $n,c\in\mathbb{N}$; $t,m\in\mathbb{N}_{>0}$;

c =0 if t=1; $c= \sum_{t=3}^{t} 5*2^{t-3}$, if t>1, $t=\{1+2p\}$, $p\in\mathbb{N}$

We can write for the m,d,t=EVEN, $m=2^t n+d$, $t\in\mathbb{N}_{>0}$, $t\in\{2p\}$, $p\in\mathbb{N}_{>0}$, m,n,d∈ℕ:

$$4+6*m = 2^t*(\mathbf{1+6n})$$
$$4+6*(2^t n + d) = 2^t*(\mathbf{1+6n})$$

t=2, d=0,

$$4+6*2^2 n = 2^2*(\mathbf{1+6n})$$
$$4+24n=4+24n$$
$$2^2+(2^3+2^4)*n = 2^2+(2^3+2^4)*n$$

$d= \sum_{t=4}^{t} 2^{t-3}$

t=4, d=2,

$$4+6*(2^4 n+2) = 2^4*(\mathbf{1+6n})$$
$$16+96n=16+96n$$
$$2^4+(2^5+2^6)*n = 2^4+(2^5+2^6)*n$$

t=6, d=10,

$$4+6*(2^6 n+10) = 2^6*(\mathbf{1+6n})$$
$$64+384n=64+384n$$
$$2^6+(2^7+2^8)*n = 2^6+(2^7+2^8)*n$$

t=8, d=42,

$$4+6*(2^8 n+42) = 2^8*(\mathbf{1+6n})$$
$$256+1536n=256+1536n$$
$$2^8+(2^9+2^{10})*n = 2^8+(2^9+2^{10})*n$$

t=10, d=170

$$4+6*(2^{10} n+170) = 2^{10}*(\mathbf{1+6n})$$
$$1024+6144n=1024+6144n$$

$$2^{10}+(2^{11}+2^{12})*n = \mathbf{2^t+(2^{t+1}+2^{t+2})*n} \Rightarrow$$
$$2^t+(2^t*2^1+2^t*2^2)*n = 2^t+(2^t*2+2^t*4)*n \Rightarrow$$
$$2^t+(2^t*(2+4))*n = 2^t+(2^t*6)*n \Rightarrow 2^t+(2^t*6)*n = \mathbf{2^t*(1+6n)}$$

$$\mathbf{4+6m} = 4+6*(2^t n +d) \Rightarrow 2^t+(2^{t+1}+2^{t+2})*n = \mathbf{2^t*(1+6n)}$$

t = EVEN, t=tmax di 4+6m, n,m,d∈ℕ, $t\in\mathbb{N}_{>0}$,

d =0 if t=2; $d= \sum_{t=4}^{t} 2^{t-3}$, if t>2; $t\in\{2p\}$, $p\in\mathbb{N}_{>0}$

We deduce from the above that tmax=1 if m=ODD

Din=1+2m ⇒ Din=1+2*(1+2p) ⇒ Din=3+4p, p∈ℕ,
which is the sequence of Din that *3+1 will give EVEN≡ 4 (mod6) divisible by tmax=1.

$$(3+4p)*3+1=10+12p \Rightarrow 4+6*(1+2p)=10+12p,\ p\in\mathbb{N},$$

The 2 equations with t= EVEN and ODD can be expressed using a single equation that shares the same independent variable in both sides:

Equation 3.16.4.

$$\boxed{4+6*(f+g+2^t n) = 2^t*(j+6n)}$$

$$m=f+g+2^t n$$
$$j = t \bmod 2 * 4 + 1,\ j \in \{1,5\}$$
$$f= t \bmod 2,\ f \in \{0,1\}$$
$$g=0 \text{ if } t\in\{1,2\},\ \ g= \sum_{tz}^{t} j * 2^{tz-3},\ \text{if } t>2$$
$$t_z=4 \text{ if } t=\text{EVEN} \Rightarrow t\in\{2p\},\ \ p\in\mathbb{N}_{>0}$$
$$t_z=3 \text{ if } t=\text{ODD} \Rightarrow t\in\{1+2p\},\ p\in\mathbb{N}$$
$$t\in\mathbb{N}_{>0},\ t=\text{tmax of } 4+6*(f+g+2^t n)$$
$$(f+g+2^t n)=m,\quad g,n,m\in\mathbb{N}$$

Equation that show how for any value of the independent variable n we will have a value of t that confirms equality, in agreement with what has been proof in Lemma 3.2. and the Directed Graph 3.7.1.

We can express Dout ≡ {1,5} (mod6) as:

$$1+6n = \mathbf{1+3*2n}$$
$$5+6n = 2+3+3*2n = \mathbf{2+3*(1+2n)}$$
$$\{Dout \equiv 1 \pmod 6\} = \{Dout \equiv 1 \pmod 3\}$$
$$\{Dout \equiv 5 \pmod 6\} = \{Dout \equiv 2 \pmod 3\}$$

Dout = r+3λ, $\lambda\in\mathbb{N}$, $\lambda= 2n$ se $j=1$, $\lambda=1+2n$ se $j=5$,

$r=1+\text{tmax} \bmod 2$, $r=\{1,2\}$, tmax di $4+6m$, $\text{tmax} \in\mathbb{N}_{>0}$

$$\mathbf{4+6m = 2^{tmax}*(r+3\lambda)} \Rightarrow \mathbf{2+3m= 2^{tmax-1}*(r+3\lambda)},\qquad m\in\mathbb{N}$$

$$\lambda = \frac{\frac{4+6m}{2^{tmax}}-r}{3} \Rightarrow \lambda = \frac{\frac{2+3m}{2^{tmax-1}}-r}{3},\ \lambda = \text{integer}$$

we can then write:

Equation 3.16.5.

$2^{tmax}*(r+3\lambda)=2^{tmax}*(j+6n) \Rightarrow \mathbf{r+3\lambda=j+6n} \Rightarrow \lambda=2n+\{0,1\}\Rightarrow$

$\lambda = 2n+ \text{tmax} \bmod 2 \Rightarrow \lambda = 2n+r-1 \Rightarrow r+3*(2n+r-1)=j+6n \Rightarrow$

$$\mathbf{4r-3=j},\ \Rightarrow \begin{cases} j=5 \Rightarrow r=2 \Rightarrow 4*2-3=5 \\ j=1 \Rightarrow r=1 \Rightarrow 4*1-3=1 \end{cases}$$

□

Distribution of equations 3.17.

Cycle 32 of equations:

Table 3.17.1.

In the sequence of equations, with EVEN and ODD exponent following each other following a cycle 32, there is a variation starting with the number 213 as the EVEN>5 exponent becomes ODD>5 and this occurs at the eleventh ordinal number of cycle 32 which is ≡ 21 (mod32) with Din-Din$_{-1}$ cadence varying between 128 and 256:

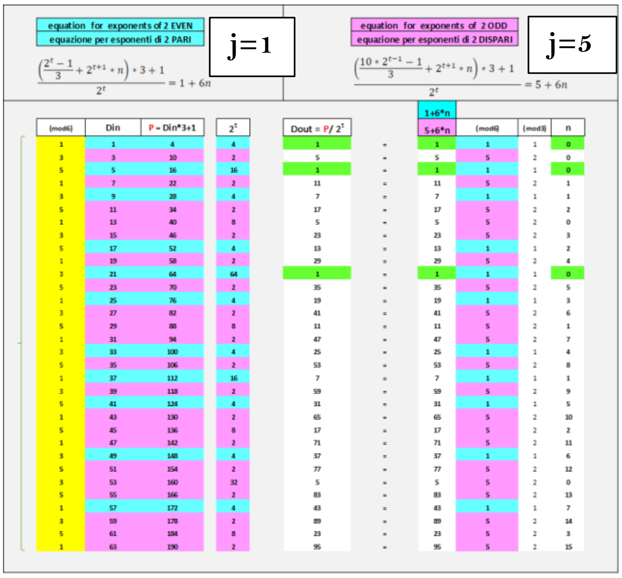


Table 3.17.2.

Din-Din$_{-1}$	Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	$P/2^{tmax}$
	6	213	21	640	1	7	5
256	1	469	21	1408	4	7	11
256	5	725	21	2176	7	7	17
128	7	853	21	2560	4	9	5
128	0	981	21	2944	1	7	23
256	4	1237	21	3712	4	7	29
256	8	1493	21	4480	7	7	35
256	3	1749	21	5248	1	7	41
128	5	1877	21	5632	7	9	11
128	7	2005	21	6016	4	7	47
256	2	2261	21	6784	7	7	53
256	6	2517	21	7552	1	7	59
256	1	2773	21	8320	4	7	65
128	3	2901	21	8704	1	9	17
128	5	3029	21	9088	7	7	71
256	0	3285	21	9856	1	7	77
128	2	3413	21	10240	7	11	5
128	4	3541	21	10624	4	7	83
256	8	3797	21	11392	7	7	89
128	1	3925	21	11776	4	9	23
128	3	4053	21	12160	1	7	95
256	7	4309	21	12928	4	7	101
256	2	4565	21	13696	7	7	107
256	6	4821	21	14464	1	7	113
128	8	4949	21	14848	7	9	29

Lemma 3.18. Starting from $j*2^t$, every 2^t ordinal numbers of the sequence of numbers ≡ 4 (mod6), we will find a number divisible by 2^t with $t \in \mathbb{N}_{>0}$, which is equivalent to stating: we can generate, by varying j,t,n all numbers ≡ 4 (mod6) taking $j*2^t$ as the root and $6*2^t$ as the modulus.

$$t=1 \Rightarrow 6*2^t=12,\quad t=2 \Rightarrow 6*2^t=24,\quad t=3 \Rightarrow 6*2^t=48,\ldots\ t=n \Rightarrow 6*2^n,\quad t,n \in \mathbb{N}_{>0}$$

Proof All this is in agreement with what we saw in Definition 3.5 where we deduced that $6*2^t$ was the increase of numbers ≡ 4 (mod6), and in agreement with the previous Lemmas which showed that $4+6m=j*2^t+6n*2^t$.

If we extend the following table to infinity and attribute values from 1 to 4 to t we will have gaps in the t=tmax column corresponding to Din ≡ 21 (mod32) which correspond to the values of the numbers ≡ 4 (mod6) divisible by tmax>4. In other words, if we sift the values of t<5 only Din ≡ 21 (mod32) will remain.

Table 3.18.1. we assume Equation 3.12.6.:

$$4+6m=j*2^t+6n*2^t \;; \qquad \frac{j*2^t-1}{3}+2^{t+1}*n=\text{Din}$$

Din	Din mod32	P1-P4-P7	P mod9	t=tmax	$\frac{P}{2^{tmax}}$	j	$j2^t$	$\frac{j*2^t-1}{3}$	$2^{t+1}*n$	n	Din
1	1	4	4	2	1	1	4	1	0	0	1
3	3	10	1	1	5	5	10	3	0	0	3
5	5	16	7	4	1	1	16	5	0	0	5
7	7	22	4	1	11	5	10	3	4	1	7
9	9	28	1	2	7	1	4	1	8	1	9
11	11	34	7	1	17	5	10	3	8	2	11
13	13	40	4	3	5	5	40	13	0	0	13
15	15	46	1	1	23	5	10	3	12	3	15
17	17	52	7	2	13	1	4	1	16	2	17
19	19	58	4	1	29	5	10	3	16	4	19
21	21	64	1	6	1	1	64	21	0	0	21
23	23	70	7	1	35	5	10	3	20	5	23
25	25	76	4	2	19	1	4	1	24	3	25
27	27	82	1	1	41	5	10	3	24	6	27
29	29	88	7	3	11	5	40	13	16	1	29
31	31	94	4	1	47	5	10	3	28	7	31
33	1	100	1	2	25	1	4	1	32	4	33
35	3	106	7	1	53	5	10	3	32	8	35
37	5	112	4	4	7	1	16	5	32	1	37
39	7	118	1	1	59	5	10	3	36	9	39
41	9	124	7	2	31	1	4	1	40	5	41
43	11	130	4	1	65	5	10	3	40	10	43
45	13	136	1	3	17	5	40	13	32	2	45
47	15	142	7	1	71	5	10	3	44	11	47
49	17	148	4	2	37	1	4	1	48	6	49
51	19	154	1	1	77	5	10	3	48	12	51
53	21	160	7	5	5	5	160	53	0	0	53
55	23	166	4	1	83	5	10	3	52	13	55
57	25	172	1	2	43	1	4	1	56	7	57
59	27	178	7	1	89	5	10	3	56	14	59
61	29	184	4	3	23	5	40	13	48	3	61
63	31	190	1	1	95	5	10	3	60	15	63

Directed graph 3.18.4.

We observe how by multiplying the point-cells $j*2^t$ by 4 we obtain the rows of the subsequent patterns and how pattern 1 repeated infinite times allows the formula $j*2^t+6n*2^t$ to reach all the numbers of the set {4+6m}. By eliminating the sequence that occurs most frequently, i.e. which has the smallest increase, we obtain the following patterns:

Table 3.18.4.1.

Din	Din mod32	P1-P4-P7	P mod9	t=tmax	$\frac{P}{2^{tmax}}$	j	$j2^t$	$\frac{j*2^t-1}{3}$	$2^{t+1}*n$	n	Din
21	21	64	1	6	1	1	64	21	0	0	21
53	21	160	7	5	5	5	160	53	0	0	53
85	21	256	4	8	1	1	256	85	0	0	85
213	21	640	1	7	5	5	640	213	0	0	213
341	21	1024	7	10	1	1	1024	341	0	0	341
853	21	2560	4	9	5	5	2560	853	0	0	853
1365	21	4096	1	12	1	1	4096	1365	0	0	1365
3413	21	10240	7	11	5	5	10240	3413	0	0	3413
5461	21	16384	4	14	1	1	16384	5461	0	0	5461

We highlight how the roots $j*2^t$ from 64 onwards, i.e. the starting points of the sequences that fill the gaps left by the previous ones starting from the third repetition of scheme 1, take on values ≡ 21 (mod32).

All this is in agreement with what has been show with Direct Graph 3.7.1. □

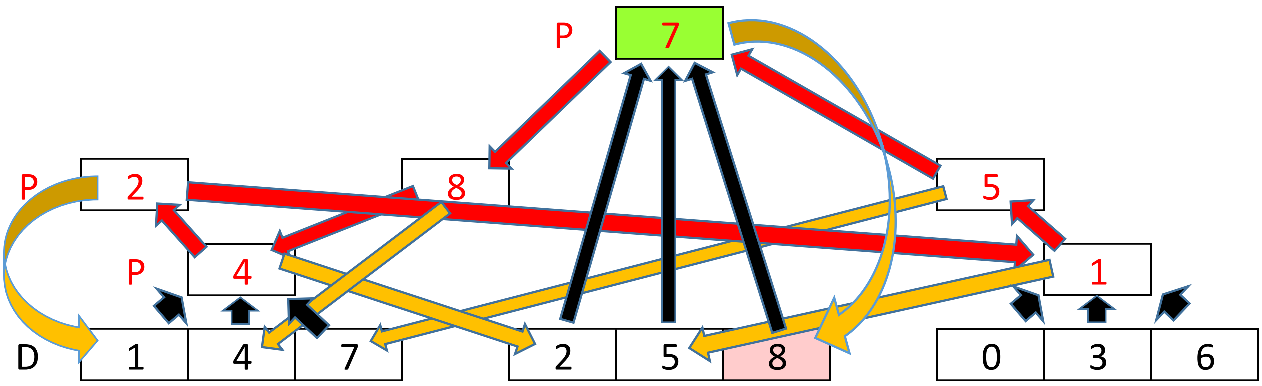
Lemma 3.19. Operation of the algorithm.

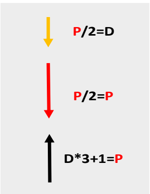
The only way for the algorithm to grow the number is to divide by 2 with exponent 1. However, this condition is not arbitrary but dictated by numerical possibilities. The deterministic alternation of increasing and decreasing routines determines the progress of the function.

Given Lemmas 2.7., 2.9., 2.10. and Flowchart 3, and the invariance of the modulo 9 residue class with respect to addition and multiplication, we observe how these behave by subjecting them to the 2 conditions.

Proof

Directed graph 3.19.1.





Flowchart of block 2

In Direct Graph 3.19.1. we highlight how the numerical antigravity generated by the algorithm converges on P7.

We highlight how the PARI nodes [1,4,7] have 4 incoming and 2 outgoing connections and how after crossing the "passage" nodes, with one incoming and one outgoing connection, the flow heads to node P7 which "resonates" with D8 or reaches P4 via P8, and from P4 to P7 through D2 or P1. After receiving multiples of 3, P1 remains inactive until it receives the flow from P2 which reaches P7 through P5 and D5.

Definition 3.19.2. **“Pump upwards mechanism”.**

To achieve a trend that causes the number to fall towards 1, you need to find the exponents of 2 "virtuous" ones, i.e. >1.
This happens when, in an increasing phase, the «Peak» number is reached (the highest value reached in all the cycles relating to the assigned number) which is always an EVEN ≡ 7(mod9), as demonstrated by Direct Graph 3.19.1. This number is obtained thanks to the "pump upwards" mechanism implemented by D8∗3+1=P7, P7/2=D8 which occurs several times until a P8 is generated. <u>It can be said that the algorithm "seeks" the way back to 1 in the highest numbers.</u>

Table 3.19.2.1.

D8∗3+1=P7/2=D8 is repeated until P8 is generated.

Table 3.19.2.2.

We highlight how the ODD numbers follow the same cycle 16 as D8: Extrapolating the D8 from Table 3.19.2.2. we obtain Table 3.19.2.1.

number of repetitions (3x+1)/2	D8	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2
1	17	26					
2	35	53	80				
1	53	80					
3	71	107	161	242			
1	89	134					
2	107	161	242				
1	125	188					
4	143	215	323	485	728		
1	161	242					
2	179	269	404				
1	197	296					
3	215	323	485	728			
1	233	350					
2	251	377	566				
1	269	404					
5	287	431	647	971	1457	2186	
1	305	458					
2	323	485	728				
1	341	512					
3	359	539	809	1214			
1	377	566					
2	395	593	890				
1	413	620					
4	431	647	971	1457	2186		
1	449	674					
2	467	701	1052				
1	485	728					
3	503	755	1133	1700			
1	521	782					
2	539	809	1214				
1	557	836					
6	575	863	1295	1943	2915	4373	6560
1	593	890					
2	611	917	1376				
1	629	944					
3	647	971	1457	2186			
1	665	998					
2	683	1025	1538				
1	701	1052					
4	719	1079	1619	2429	3644		
1	737	1106					
2	755	1133	1700				
1	773	1160					
3	791	1187	1781	2672			
1	809	1214					
2	827	1241	1862				
1	845	1268					
5	863	1295	1943	2915	4373	6560	

number of repetitions (3x+1)/2	Din	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2
1	1	2							
2	3	5	8						
1	5	8							
3	7	11	17	26					
1	9	14							
2	11	17	26						
1	13	20							
4	15	23	35	53	80				
1	17	26							
2	19	29	44						
1	21	32							
3	23	35	53	80					
1	25	38							
2	27	41	62						
1	29	44							
5	31	47	71	107	161	242			
1	33	50							
2	35	53	80						
1	37	56							
3	39	59	89	134					
1	41	62							
2	43	65	98						
1	45	68							
4	47	71	107	161	242				
1	49	74							
2	51	77	116						
1	53	80							
3	55	83	125	188					
1	57	86							
2	59	89	134						
1	61	92							
6	63	95	143	215	323	485	728		
1	65	98							
2	67	101	152						
1	69	104							
3	71	107	161	242					
1	73	110							
2	75	113	170						
1	77	116							
4	79	119	179	269	404				
1	81	122							
2	83	125	188						
1	85	128							
3	87	131	197	296					
1	89	134							
2	91	137	206						
1	93	140							
5	95	143	215	323	485	728			
1	97	146							
2	99	149	224						
1	101	152							
3	103	155	233	350					
1	105	158							
2	107	161	242						
1	109	164							
4	111	167	251	377	566				
1	113	170							
2	115	173	260						
1	117	176							
3	119	179	269	404					
1	121	182							
2	123	185	278						
1	125	188							
7	127	191	287	431	647	971	1457	2186	
1	129	194							
2	131	197	296						
1	133	200							
3	135	203	305	458					
1	137	206							
2	139	209	314						
1	141	212							
4	143	215	323	485	728				
1	145	218							
2	147	221	332						
1	149	224							
3	151	227	341	512					
1	153	230							
2	155	233	350						
1	157	236							
5	159	239	359	539	809	1214			
1	161	242							
2	163	245	368						
1	165	248							
3	167	251	377	566					
1	169	254							
2	171	257	386						
1	173	260							
4	175	263	395	593	890				
1	177	266							
2	179	269	404						
1	181	272							
3	183	275	413	620					
1	185	278							
2	187	281	422						
1	189	284							
6	191	287	431	647	971	1457	2186		
1	193	290							
2	195	293	440						
1	197	296							
3	199	299	449	674					
1	201	302							
2	203	305	458						
1	205	308							
4	207	311	467	701	1052				
1	209	314							
2	211	317	476						
1	213	320							
3	215	323	485	728					
1	217	326							
2	219	329	494						
1	221	332							
5	223	335	503	755	1133	1700			
1	225	338							
2	227	341	512						
1	229	344							
3	231	347	521	782					
1	233	350							
2	235	353	530						
1	237	356							
4	239	359	539	809	1214				
1	241	362							
2	243	365	548						
1	245	368							
3	247	371	557	836					
1	249	374							
2	251	377	566						
1	253	380							
8	255	383	575	863	1295	1943	2915	4373	6560
1	257	386							
2	259	389	584						
1	261	392							
3	263	395	593	890					
1	265	398							
2	267	401	602						
1	269	404							
4	271	407	611	917	1376				
1	273	410							
2	275	413	620						
1	277	416							
3	279	419	629	944					
1	281	422							
2	283	425	638						
1	285	428							
5	287	431	647	971	1457	2186			

Table 3.19.2.2. highlights how $\frac{Din*3+1}{2} \equiv \{2,5,8\}(mod9)$ as can be seen in Direct Graph 3.19.1. The resulting ODD once again subjected to the two conditions will generate D8 until reaching a P7 which is divisible by 2^{tmax} with tmax>1. In this last case the routine stops and we display in the tables $\frac{P7}{2} = P8$.

Table 3.19.3.

n°cycles (3x+1)/2	D8	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2
7	883583	1325375	1988063	2982095	4473143	6709715	10064573	15096860								
1	883601	1325402														
2	883619	1325429	1988144													
1	883637	1325456														
3	883655	1325483	1988225	2982338												
1	883673	1325510														
2	883691	1325537	1988306													
1	883709	1325564														
4	883727	1325591	1988387	2982581	4473872											
1	883745	1325618														
2	883763	1325645	1988468													
1	883781	1325672														
3	883799	1325699	1988549	2982824												
1	883817	1325726														
2	883835	1325753	1988630													
1	883853	1325780														
5	883871	1325807	1988711	2983067	4474601	6711902										
1	883889	1325834														
2	883907	1325861	1988792													
1	883925	1325888														
3	883943	1325915	1988873	2983310												
1	883961	1325942														
2	883979	1325969	1988954													
1	883997	1325996														
4	884015	1326023	1989035	2983553	4475330											
1	884033	1326050														
2	884051	1326077	1989116													
1	884069	1326104														
3	884087	1326131	1989197	2983796												
1	884105	1326158														
2	884123	1326185	1989278													
1	884141	1326212														
6	884159	1326239	1989359	2984039	4476059	6714089	10071134									
1	884177	1326266														
2	884195	1326293	1989440													
1	884213	1326320														
3	884231	1326347	1989521	2984282												
1	884249	1326374														
2	884267	1326401	1989602													
1	884285	1326428														
4	884303	1326455	1989683	2984525	4476788											
1	884321	1326482														
2	884339	1326509	1989764													
1	884357	1326536														
3	884375	1326563	1989845	2984768												
1	884393	1326590														
2	884411	1326617	1989926													
1	884429	1326644														
5	884447	1326671	1990007	2985011	4477517	6716276										
1	884465	1326698														
2	884483	1326725	1990088													
1	884501	1326752														
3	884519	1326779	1990169	2985254												
1	884537	1326806														
2	884555	1326833	1990250													
1	884573	1326860														
4	884591	1326887	1990331	2985497	4478246											
1	884609	1326914														
2	884627	1326941	1990412													
1	884645	1326968														
3	884663	1326995	1990493	2985740												
1	884681	1327022														
2	884699	1327049	1990574													
1	884717	1327076														
15	884735	1327103	1990655	2985983	4478975	6718463	10077695	15116543	22674815	34012223	51018335	76527503	114791255	172186883	258280325	387420488

Table 3.19.4.

Din(mod9)	Din=D8	Din(mod32)	P1-P4-P7	$P/2^{tmax}$	tmax	n°cycles (3x+1)/2
8	17	17	52	13	2	1
8	35	3	106	53	1	2
8	53	21	160	5	5	1
8	71	7	214	107	1	3
8	89	25	268	67	2	1
8	107	11	322	161	1	2
8	125	29	376	47	3	1
8	143	15	430	215	1	4
8	161	1	484	121	2	1
8	179	19	538	269	1	2
8	197	5	592	37	4	1
8	215	23	646	323	1	3
8	233	9	700	175	2	1
8	251	27	754	377	1	2
8	269	13	808	101	3	1
8	287	31	862	431	1	5
8	305	17	916	229	2	1
8	323	3	970	485	1	2
8	341	21	1024	1	10	1
8	359	7	1078	539	1	3
8	377	25	1132	283	2	1
8	395	11	1186	593	1	2
8	413	29	1240	155	3	1
8	431	15	1294	647	1	4
8	449	1	1348	337	2	1
8	467	19	1402	701	1	2
8	485	5	1456	91	4	1
8	503	23	1510	755	1	3
8	521	9	1564	391	2	1
8	539	27	1618	809	1	2
8	557	13	1672	209	3	1
8	575	31	1726	863	1	6
8	593	17	1780	445	2	1
8	611	3	1834	917	1	2
8	629	21	1888	59	5	1
8	647	7	1942	971	1	3
8	665	25	1996	499	2	1
8	683	11	2050	1025	1	2
8	701	29	2104	263	3	1
8	719	15	2158	1079	1	4
8	737	1	2212	553	2	1
8	755	19	2266	1133	1	2
8	773	5	2320	145	4	1
8	791	23	2374	1187	1	3
8	809	9	2428	607	2	1
8	827	27	2482	1241	1	2
8	845	13	2536	317	3	1
8	863	31	2590	1295	1	5

The first column of Table 3.19.3., which explains the number of repetitions of D8∗3+1=P7, $\frac{P7}{2}$, highlights how this routine also follows **cycle 16**. There are no routines that make the number grow faster than the "pump up mechanism". When pump-up routines become frequent, the number grows faster. 50% of the numbers ODD∗3+1 have 2^1 as their divisor, while the remaining 50% will become EVEN divisible by a power of 2>1. ODD ≡ 8 (mod9) are $\frac{1}{9}$ of the ODD numbers i.e. $11,\bar{1}\%$ of the same, therefore there are $5,\bar{5}\%$ of ODD numbers which are D8 which activate the pump upwards mechanism.

Definition 3.19.5. k = number of repetitions of $\frac{3x+1}{2}$ until an EVEN number ≡ 8(mod9) is found.

Table 3.19.4. highlights how the 2 cycles 16 of tmax and k are repeated. In cycle 16 of k the variable occurs again at the sixteenth ordinal which is D8 ≡ 31 (mod32) as can be seen in table 3.19.3 and 3.19.4.: {883583, 883871, 884159, 884447, 884735 , 287, 575, 863 } ≡ 31 (mod32)

$$tmax=1 \Rightarrow k > 1, \quad tmax>1 \Rightarrow k = 1,$$

all this show the complementarity of the two cycles which depend on the divisibility by 2.

Equation 3.19.6.

$D8+9*2^{k+1}*w=D8_{+w}$, $w \in \mathbb{N}$, $w+1=$ ordinal of the D8 that share the same k.

Tables 3.19.7.

e.g. $17+9*2^2*14=521$

ordinal number	number of repetitions (3x+1)/2	D8	(3x+1)/2
1	1	17	26
2	1	53	80
3	1	89	134
4	1	125	188
5	1	161	242
6	1	197	296
7	1	233	350
8	1	269	404
9	1	305	458
10	1	341	512
11	1	377	566
12	1	413	620
13	1	449	674
14	1	485	728
15	1	521	782

e.g. $359+9*2^4*9=1655$

ordinal number	number of repetitions (3x+1)/2	D8	(3x+1)/2	(3x+1)/2	(3x+1)/2
	3	71	107	161	242
	3	215	323	485	728
1	3	359	539	809	607
2	3	503	755	1133	425
3	3	647	971	1457	1093
4	3	791	1187	1781	167
5	3	935	1403	2105	1579
6	3	1079	1619	2429	911
7	3	1223	1835	2753	2065
8	3	1367	2051	3077	577
9	3	1511	2267	3401	2551
10	3	1655	2483	3725	1397

n°cycles (3x+1)/2	D8	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2	(3x+1)/2
1	17	26								
1	53	80								
1	89	134								
2	35	53	80							
2	107	161	242							
2	179	269	404							
3	71	107	161	242						
3	215	323	485	728						
3	359	539	809	1214						
4	143	215	323	485	728					
4	431	647	971	1457	2186					
4	719	1079	1619	2429	3644					
5	287	431	647	971	1457	2186				
5	863	1295	1943	2915	4373	6560				
5	1439	2159	3239	4859	7289	10934				
6	575	863	1295	1943	2915	4373	6560			
6	1727	2591	3887	5831	8747	13121	19682			
6	4031	6047	9071	13607	20411	30617	45926			
7	1151	1727	2591	3887	5831	8747	13121	19682		
7	3455	5183	7775	11663	17495	26243	39365	59048		
7	5759	8639	12959	19439	29159	43739	65609	98414		
8	2303	3455	5183	7775	11663	17495	26243	39365	59048	
8	6911	10367	15551	23327	34991	52487	78731	118097	177146	
8	11519	17279	25919	38879	58319	87479	131219	196829	295244	
9	4607	6911	10367	15551	23327	34991	52487	78731	118097	177146
9	13823	20735	31103	46655	69983	104975	157463	236195	354293	531440
9	23039	34559	51839	77759	116639	174959	262439	393659	590489	885734

...

Equation 3.19.8. We define $D8_{start}=$ 1st D8 which has a given k. Let's write the equation that allows us to obtain all the $D8_{start}$ following the 17:

$$D8_{start}=17+\sum_2^k 9*2^{k-1}, \quad k \in \mathbb{N}_{>1}$$

which is equivalent to the analytical expression:

$$\begin{cases} a_0=17 \\ a_b=a_{b-1}+a_{b-1}+1, \quad b \in \mathbb{N}_{>0} \end{cases} \Rightarrow \begin{cases} a_0=17 \\ a_b=a_{b-1}*2+1, \quad b \in \mathbb{N}_{>0} \end{cases}$$

Combining equations 3.19.6 and 3.19.8 we obtain:

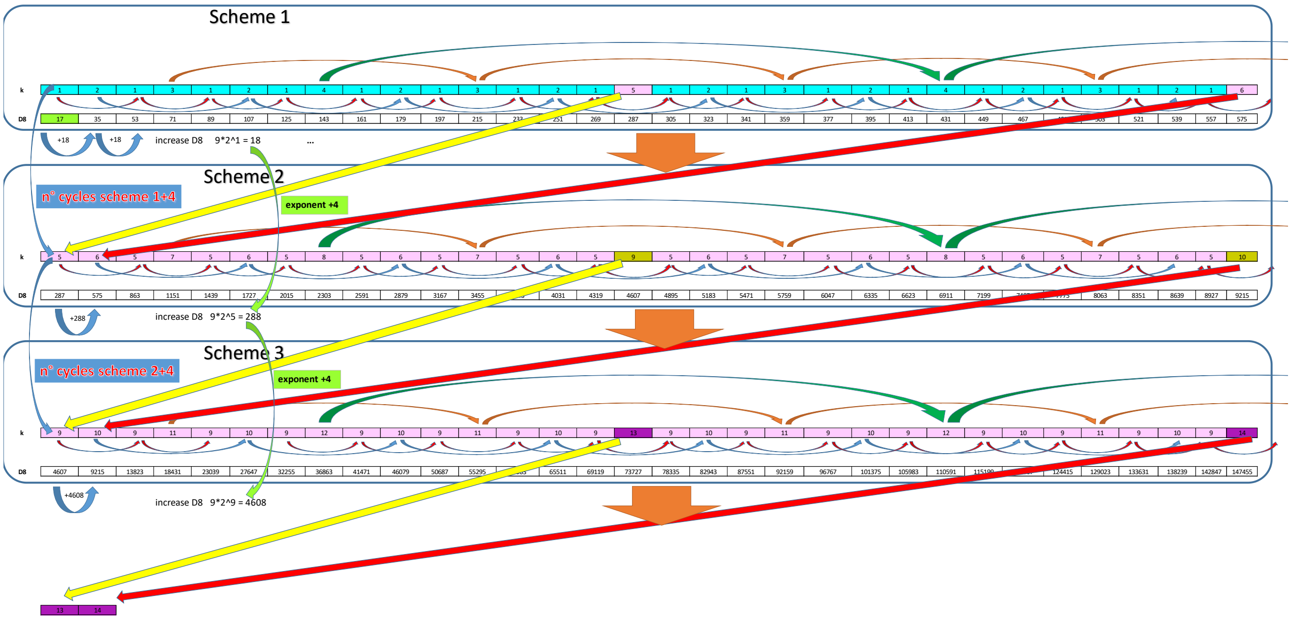
Equation 3.19.9.

$$D8=9*2^{k+1}*w+17+\sum_2^k 9*2^{k-1}, \quad w \in \mathbb{N}, \quad k \in \mathbb{N}_{>0},$$

with $k=1 \Rightarrow \sum_2^k 9*2^{k-1}=0$

which allows us to obtain all $ODD \equiv 8 \pmod 9$.

Directed graph 3.19.10.



Direct Graph 3.19.10., obtained thanks to Equation 3.19.9., shows how scheme 1 repeats infinitely and highlights the determinism of the "pump upwards mechanism", and therefore how the same before or then it is interrupted by an exponent of 2 >1. As we observed in Directed Graph 3.7.1. we obtain the values of k by adding 4 to the value of the previous scheme. We obtain the equations of cycle 16 of Din which, as we have highlighted in Tables 3.19.2.1 and 3.19.2.2., is identical to cycle 16 of k expressed by D8.

Equation 3.19.11.

$Din_{+w}=Din_{start}+2^{k+1}*w$, $w\in\mathbb{N}$, $w+1=$ ordinal of the Din that share the same k.

Equation 3.19.12.

We define Din_{start} = 1st Din that has a certain k.
Let's write the equation that allows us to obtain all the Din_{start}:

$$Din_{start}=\sum_{1}^{k} 2^{k-1}, \qquad k\in\mathbb{N}_{>0}$$

which is equivalent to the analytical expression:

$$\begin{cases} a_0 = 1 \\ a_b = a_{b-1} + a_{b-1} + 1, & b\in\mathbb{N}_{>0} \end{cases} \Rightarrow \begin{cases} a_0 = 1 \\ a_b = a_{b-1}*2+1, & b\in\mathbb{N}_{>0} \end{cases}$$

$\{\sum_{1}^{k} 2^{k-1}\}=\{1,3,7,15,31\ldots\}$ we can write $3=1+2^1$, $7=3+2^2$, $15=7+2^3\ldots\Rightarrow$
$1=2*2^0-1$, $3=2*2^1-1$, $7=2*2^2-1\ldots$ $\Rightarrow$ $Din_{start}=2*2^{k-1}-1$ $\Rightarrow$

$$Din_{start}=2^k-1, \quad k\in\mathbb{N}_{>0}$$

Combining equations 3.19.11. and 3.19.12. we obtain:

Equation 3.19.13. $Din= 2^{k+1}*w+\sum_1^k 2^{k-1}, \quad w\in\mathbb{N},\ k\in\mathbb{N}_{>0} \ \Rightarrow$

Equation 3.19.14.

$$Din= 2^k-1+ 2^{k+1}*w \ \Rightarrow$$

$$Din= 2^k*(1+2w)-1, \quad w\in\mathbb{N},\ k\in\mathbb{N}_{>0} \ \Rightarrow$$

$$k= \log_2\left(\frac{Din+1}{1+2w}\right)$$

$$1+2w=\frac{Din+1}{2^k} \ \Rightarrow \quad w=\frac{\frac{Din+1}{2^k}-1}{2} \ \Rightarrow \quad w=\frac{Din+1}{2^{k+1}}-\frac{1}{2} \ \Rightarrow$$

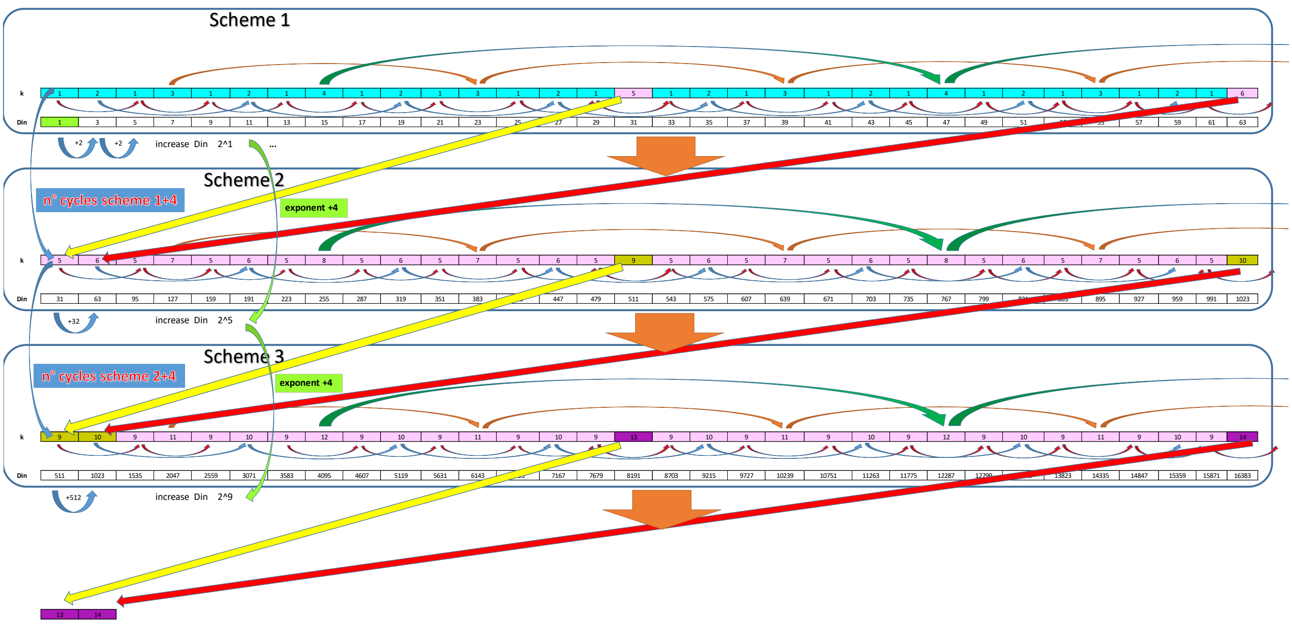
$$w=\frac{1+2m+1}{2^{k+1}}-\frac{1}{2} \ \Rightarrow \quad \left(w+\frac{1}{2}\right)*2^{k+1}=2+2m \ \Rightarrow$$

$$m=\left(w+\frac{1}{2}\right)*2^k-1 \ \Rightarrow \quad w=\frac{m+1}{2^k}-\frac{1}{2} \ \Rightarrow$$

$$m=2^{k-1}-1+\ 2^k*w \ \Rightarrow \quad w=\frac{m+1-2^{k-1}}{2^k}$$

Equation that allows us to obtain all the Din as a function of k and the powers of 2 as demonstrated by the following Directed graph:

Directed graph 3.19.15.



Equation 3.19.15.1. We can express the Din as:

$$Din= 1+2m+\sum_{\theta=1}^{\theta} 30*(m+1)*2^{4*(\theta-1)}, \quad m\in\mathbb{N},\ \theta\in\mathbb{N}$$

with $\theta=0$ we will have the ODD numbers of scheme 1 $\Rightarrow$ Din=1+2m
with $\theta=1$ we will have the ODD numbers of scheme 2
...
We highlight the equivalence of the 2 directed graphs: 3.19.10. and 3.19.15.
The reason for the progression of D8 is: $9*2^{1+4\theta}$ which is the reason for the progression of ODD numbers multiplied by 9 with $\theta\in\mathbb{N}$.

Table 3.19.16.

k	Din	k	D8
1	1	1	17
2	3	2	35
1	5	1	53
3	7	3	71
1	9	1	89
2	11	2	107
1	13	1	125
4	15	4	143
1	17	1	161
2	19	2	179
1	21	1	197
3	23	3	215
1	25	1	233
2	27	2	251
1	29	1	269
5	31	5	287

k	Din	k	D8
1	1889	1	17009
2	1891	2	17027
1	1893	1	17045
3	1895	3	17063
1	1897	1	17081
2	1899	2	17099
1	1901	1	17117
4	1903	4	17135
1	1905	1	17153
2	1907	2	17171
1	1909	1	17189
3	1911	3	17207
1	1913	1	17225
2	1915	2	17243
1	1917	1	17261
7	1919	7	17279

k	Din	k	D8
1	4065	1	36593
2	4067	2	36611
1	4069	1	36629
3	4071	3	36647
1	4073	1	36665
2	4075	2	36683
1	4077	1	36701
4	4079	4	36719
1	4081	1	36737
2	4083	2	36755
1	4085	1	36773
3	4087	3	36791
1	4089	1	36809
2	4091	2	36827
1	4093	1	36845
12	4095	12	36863

Equation 3.19.17. we deduce from Table 3.19.16.:

$$D8= Din*9+8$$

We can then express Equation 3.19.17. as:

Equazione 3.19.18. $D8=(2^k*(1+2w)-1)*9+8 \Rightarrow \quad D8=9*2^k*(1+2w)-1 \quad \Rightarrow$

$$D8=2^k*(9+18w)-1, \quad w\in\mathbb{N}, \; k\in\mathbb{N}_{>0} \qquad \square$$

Definition 3.20. Cycle 16 determinism of tmax and k. Directed Graphs 3.7.1. and 3.19.15. highlight the trend of tmax and k as Din varies.

Definition 3.20.1.

Since in the "pump up" cycle 2^1 is a possible value of 2^{tmax} we can generalize by redefining: **k = number of repetitions of $\frac{3x+1}{2^{tmax}}$ until finding 3x+1= P7** with tmax becoming >1 $\Rightarrow$ $\frac{\mathbf{P7}}{\mathbf{2^{tmax}}}$ **=Dout**:

3 .20.1.1. We thus obtain Equation 3.1.1. of block 2:

$$\frac{\left(\frac{\left(\frac{\left(\frac{D8*3+1}{2}\right)*3+1}{2}\right)*3+1}{2}\ldots\right)*3+1}{2}=P8 \qquad \frac{\left(\frac{\left(\frac{\left(\frac{Din*3+1}{2^{tmax}}\right)*3+1}{2^{tmax}}\right)*3+1}{2^{tmax}}\ldots\right)*3+1}{2^{tmax}}=Dout$$

We indicate with K (upper case) the sum of the k (lower case) with which we obtain Dout=1. $\Rightarrow$ $$\frac{\left(\frac{\left(\frac{Din*3+1}{2^{tmax}}\right)*3+1}{2^{tmax}}\ldots\right)*3+1}{2^{tmax}}=1,\quad Din,tmax\in\mathbb{N}_{>0}$$

Table 3.20.2 We obtain Din through Equation 3.19.14. and we use Equation 3.1.2., we assume some values of k for example:

K=2	Din	$\frac{3x+1}{2}$	$\frac{3x+1}{2^{tmax}}$	tmax
2	3	5	1	4
2	11	17	13	2
2	19	29	11	3
2	27	41	31	2
2	35	53	5	5
2	43	65	49	2
2	51	77	29	3
2	59	89	67	2
2	67	101	19	4
2	75	113	85	2
2	83	125	47	3
2	91	137	103	2
2	99	149	7	6
2	107	161	121	2
2	115	173	65	3
2	123	185	139	2
2	131	197	37	4
2	139	209	157	2
2	147	221	83	3
2	155	233	175	2
2	163	245	23	5
2	171	257	193	2
2	179	269	101	3
2	187	281	211	2
2	195	293	55	4
2	203	305	229	2
2	211	317	119	3
2	219	329	247	2
2	227	341	1	10
2	235	353	265	2
2	243	365	137	3
2	251	377	283	2
2	259	389	73	4
2	267	401	301	2
2	275	413	155	3
2	283	425	319	2
2	291	437	41	5
2	299	449	337	2
2	307	461	173	3
2	315	473	355	2
2	323	485	91	4
2	331	497	373	2
2	339	509	191	3
2	347	521	391	2
2	355	533	25	6
2	363	545	409	2
2	371	557	209	3
2	379	569	427	2

K=3	Din	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2^{tmax}}$	tmax
3	7	11	17	13	2
3	23	35	53	5	5
3	39	59	89	67	2
3	55	83	125	47	3
3	71	107	161	121	2
3	87	131	197	37	4
3	103	155	233	175	2
3	119	179	269	101	3
3	135	203	305	229	2
3	151	227	341	1	10
3	167	251	377	283	2
3	183	275	413	155	3
3	199	299	449	337	2
3	215	323	485	91	4
3	231	347	521	391	2
3	247	371	557	209	3
3	263	395	593	445	2
3	279	419	629	59	5
3	295	443	665	499	2
3	311	467	701	263	3
3	327	491	737	553	2
3	343	515	773	145	4
3	359	539	809	607	2
3	375	563	845	317	3
3	391	587	881	661	2
3	407	611	917	43	6
3	423	635	953	715	2
3	439	659	989	371	3
3	455	683	1025	769	2
3	471	707	1061	199	4
3	487	731	1097	823	2
3	503	755	1133	425	3
3	519	779	1169	877	2
3	535	803	1205	113	5
3	551	827	1241	931	2
3	567	851	1277	479	3
3	583	875	1313	985	2
3	599	899	1349	253	4
3	615	923	1385	1039	2
3	631	947	1421	533	3
3	647	971	1457	1093	2
3	663	995	1493	35	7
3	679	1019	1529	1147	2
3	695	1043	1565	587	3
3	711	1067	1601	1201	2
3	727	1091	1637	307	4
3	743	1115	1673	1255	2
3	759	1139	1709	641	3

K=4	Din	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2^{tmax}}$	tmax
4	15	23	35	53	5	5
4	47	71	107	161	121	2
4	79	119	179	269	101	3
4	111	167	251	377	283	2
4	143	215	323	485	91	4
4	175	263	395	593	445	2
4	207	311	467	701	263	3
4	239	359	539	809	607	2
4	271	407	611	917	43	6
4	303	455	683	1025	769	2
4	335	503	755	1133	425	3
4	367	551	827	1241	931	2
4	399	599	899	1349	253	4
4	431	647	971	1457	1093	2
4	463	695	1043	1565	587	3
4	495	743	1115	1673	1255	2
4	527	791	1187	1781	167	5
4	559	839	1259	1889	1417	2
4	591	887	1331	1997	749	3
4	623	935	1403	2105	1579	2
4	655	983	1475	2213	415	4
4	687	1031	1547	2321	1741	2
4	719	1079	1619	2429	911	3
4	751	1127	1691	2537	1903	2
4	783	1175	1763	2645	31	8
4	815	1223	1835	2753	2065	2
4	847	1271	1907	2861	1073	3
4	879	1319	1979	2969	2227	2
4	911	1367	2051	3077	577	4
4	943	1415	2123	3185	2389	2
4	975	1463	2195	3293	1235	3
4	1007	1511	2267	3401	2551	2
4	1039	1559	2339	3509	329	5
4	1071	1607	2411	3617	2713	2
4	1103	1655	2483	3725	1397	3
4	1135	1703	2555	3833	2875	2
4	1167	1751	2627	3941	739	4
4	1199	1799	2699	4049	3037	2
4	1231	1847	2771	4157	1559	3
4	1263	1895	2843	4265	3199	2
4	1295	1943	2915	4373	205	6
4	1327	1991	2987	4481	3361	2
4	1359	2039	3059	4589	1721	3
4	1391	2087	3131	4697	3523	2
4	1423	2135	3203	4805	901	4
4	1455	2183	3275	4913	3685	2
4	1487	2231	3347	5021	1883	3
4	1519	2279	3419	5129	3847	2

K=8	Din	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2}$	$\frac{3x+1}{2^{tmax}}$	tmax
8	255	383	575	863	1295	1943	2915	4373	205	6
8	767	1151	1727	2591	3887	5831	8747	13121	9841	2
8	1279	1919	2879	4319	6479	9719	14579	21869	8201	3
8	1791	2687	4031	6047	9071	13607	20411	30617	22963	2
8	2303	3455	5183	7775	11663	17495	26243	39365	7381	4
8	2815	4223	6335	9503	14255	21383	32075	48113	36085	2
8	3327	4991	7487	11231	16847	25271	37907	56861	21323	3
8	3839	5759	8639	12959	19439	29159	43739	65609	49207	2
8	4351	6527	9791	14687	22031	33047	49571	74357	6971	5
8	4863	7295	10943	16415	24623	36935	55403	83105	62329	2
8	5375	8063	12095	18143	27215	40823	61235	91853	34445	3
8	5887	8831	13247	19871	29807	44711	67067	100601	75451	2
8	6399	9599	14399	21599	32399	48599	72899	109349	20503	4
8	6911	10367	15551	23327	34991	52487	78731	118097	88573	2
8	7423	11135	16703	25055	37583	56375	84563	126845	47567	3
8	7935	11903	17855	26783	40175	60263	90395	135593	101695	2
8	8447	12671	19007	28511	42767	64151	96227	144341	3383	7
8	8959	13439	20159	30239	45359	68039	102059	153089	114817	2
8	9471	14207	21311	31967	47951	71927	107891	161837	60689	3
8	9983	14975	22463	33695	50543	75815	113723	170585	127939	2
8	10495	15743	23615	35423	53135	79703	119555	179333	33625	4
8	11007	16511	24767	37151	55727	83591	125387	188081	141061	2
8	11519	17279	25919	38879	58319	87479	131219	196829	73811	3
8	12031	18047	27071	40607	60911	91367	137051	205577	154183	2
8	12543	18815	28223	42335	63503	95255	142883	214325	20093	5
8	13055	19583	29375	44063	66095	99143	148715	223073	167305	2
8	13567	20351	30527	45791	68687	103031	154547	231821	86933	3
8	14079	21119	31679	47519	71279	106919	160379	240569	180427	2
8	14591	21887	32831	49247	73871	110807	166211	249317	46747	4
8	15103	22655	33983	50975	76463	114695	172043	258065	193549	2
8	15615	23423	35135	52703	79055	118583	177875	266813	100055	3
8	16127	24191	36287	54431	81647	122471	183707	275561	206671	2
8	16639	24959	37439	56159	84239	126359	189539	284309	13327	6
8	17151	25727	38591	57887	86831	130247	195371	293057	219793	2
8	17663	26495	39743	59615	89423	134135	201203	301805	113177	3
8	18175	27263	40895	61343	92015	138023	207035	310553	232915	2
8	18687	28031	42047	63071	94607	141911	212867	319301	59869	4
8	19199	28799	43199	64799	97199	145799	218699	328049	246037	2
8	19711	29567	44351	66527	99791	149687	224531	336797	126299	3
8	20223	30335	45503	68255	102383	153575	230363	345545	259159	2
8	20735	31103	46655	69983	104975	157463	236195	354293	33215	5
8	21247	31871	47807	71711	107567	161351	242027	363041	272281	2
8	21759	32639	48959	73439	110159	165239	247859	371789	139421	3
8	22271	33407	50111	75167	112751	169127	253691	380537	285403	2
8	22783	34175	51263	76895	115343	173015	259523	389285	72991	4
8	23295	34943	52415	78623	117935	176903	265355	398033	298525	2
8	23807	35711	53567	80351	120527	180791	271187	406781	152543	3
8	24319	36479	54719	82079	123119	184679	277019	415529	311647	2

Table 3.20.3.

If we extend the same procedure up to, for example, k=17 we obtain the following table which highlights the tmax values of the first 3 cycles 16 as a function of k. The values of tmax <6 are repeated horizontally following a cycle 8: $k_r=k+8*r$ with $r\in\mathbb{N}$, $k\in\mathbb{N}_{>0}$ while the values >5, respecting the horizontal and vertical position, become alternatively 6 or >6 .

K=1	K=2	K=3	K=4	K=5	k=6	K=7	K=8	K=9	K=10	K=11	K=12	K=13	K=14	K=15	K=16	K=17
2	4	2	5	2	4	2	6	2	4	2	5	2	4	2	7	2
4	2	5	2	4	2	6	2	4	2	5	2	4	2	7	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	5	2	4	2	6	2	4	2	5	2	4	2	7	2	4	2
6	2	4	2	5	2	4	2	7	2	4	2	5	2	4	2	6
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	4	2	6	2	4	2	5	2	4	2	9	2	4	2	5	2
4	2	10	2	4	2	5	2	4	2	6	2	4	2	5	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	6	2	4	2	5	2	4	2	8	2	4	2	5	2	4	2
5	2	4	2	6	2	4	2	5	2	4	2	7	2	4	2	5
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	4	2	5	2	4	2	7	2	4	2	5	2	4	2	6	2
4	2	5	2	4	2	9	2	4	2	5	2	4	2	6	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	5	2	4	2	7	2	4	2	5	2	4	2	6	2	4	2
8	2	4	2	5	2	4	2	6	2	4	2	5	2	4	2	7
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	4	2	8	2	4	2	5	2	4	2	6	2	4	2	5	2
4	2	6	2	4	2	5	2	4	2	9	2	4	2	5	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	10	2	4	2	5	2	4	2	6	2	4	2	5	2	4	2
5	2	4	2	12	2	4	2	5	2	4	2	6	2	4	2	5
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	4	2	5	2	4	2	6	2	4	2	5	2	4	2	8	2
4	2	5	2	4	2	6	2	4	2	5	2	4	2	10	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	5	2	4	2	6	2	4	2	5	2	4	2	7	2	4	2
6	2	4	2	5	2	4	2	8	2	4	2	5	2	4	2	6
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	4	2	6	2	4	2	5	2	4	2	7	2	4	2	5	2
4	2	7	2	4	2	5	2	4	2	6	2	4	2	5	2	4
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3
2	6	2	4	2	5	2	4	2	7	2	4	2	5	2	4	2
5	2	4	2	6	2	4	2	5	2	4	2	11	2	4	2	5
2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2
3	2	3	2	3	2	3	2	3	2	3	2	3	2	3	2	3

It is clear that the minimum tmax is 2. P8 resulting from the division of 3x+1 by 2 will in turn be divided at least once by 2 therefore tmax>1 to generate an ODD. The average of the tmax values, which generate Dout, extracted from the adjacent table, is 3.003676471. The same will increase due to the tmax >6 which will vary upwards.

Lemma 3.20.4.

We define k := sum of the least significant 1 of the ODD number, expressed with the binary positional system, up to the first least significant 0:

$$2^r <= N_{10} \text{ e } 2^{(r+1)} > N_{10}$$

$$N_{10} = \sum_{a=0}^{r} 2^a * x_a \, , \quad x_a \in \{0,1\}$$

$$k(N_{\{10\}}) := \min\{a \in \{0,1,\ldots,r\} : x_a = 0\}$$

Having acquired Definition 3.20.4. and Equation 3.19.14. :

$$Din = 2^{k+1} * w + 2^k - 1$$

e.g. $863_{10} = 2^{5+1} * 13 + 2^5 - 1 \implies 863_{10} = 832_{10} + 31_{10}$

$832_{10} = 1101000000_2$

$31_{10} = 11111_2$

$863_{10} = 1101011111_2$

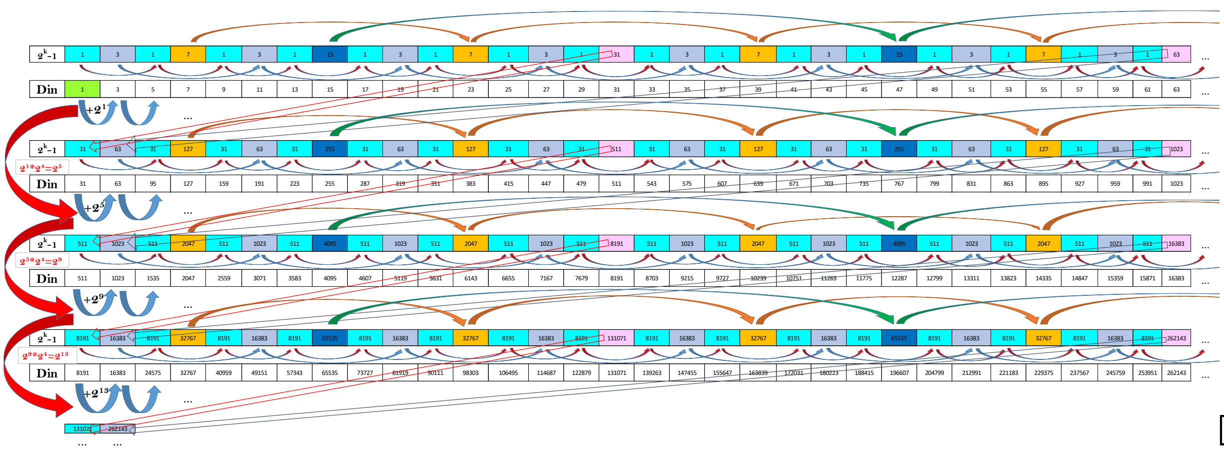
Proof

Table 3.20.5.

We highlight cycle 16 of 2^k-1 which proposes the same values in each cycle, with the exclusion of the sixteenth ordinal, and how the sum of the 1, in each row of the column $(2^k-1)_2$ expresses exactly the k next to it. Equation 3.19.14. can be expressed as: $Din=2^k*(1+2w)-1$. Theorem 2.3. proves that the equation is true and that it returns ODD numbers. In the following directed graph, we demonstrate how the equation reaches all ODD numbers. The reason for Din progression is 2^1 in the first scheme, $2^1*2^4=2^5$ in the second, $2^5*2^4=2^9$ in the third... Net of the values, the first scheme repeats infinitely.

Din_{10}	tmax	$Dout_{10}$	2^{k+1}	m	$2^{k+1}*m$	2^k-1	Din_2	$(2^{k+1}*m)_2$	$(2^k-1)_2$	k
1	2	1	4	0	0	1	1	0	1	1
3	1	5	8	0	0	3	11	0	11	2
5	4	1	4	1	4	1	101	100	1	1
7	1	11	16	0	0	7	111	0	111	3
9	2	7	4	2	8	1	1001	1000	1	1
11	1	17	8	1	8	3	1011	1000	11	2
13	3	5	4	3	12	1	1101	1100	1	1
15	1	23	32	0	0	15	1111	0	1111	4
17	2	13	4	4	16	1	10001	10000	1	1
19	1	29	8	2	16	3	10011	10000	11	2
21	6	1	4	5	20	1	10101	10100	1	1
23	1	35	16	1	16	7	10111	10000	111	3
25	2	19	4	6	24	1	11001	11000	1	1
27	1	41	8	3	24	3	11011	11000	11	2
29	3	11	4	7	28	1	11101	11100	1	1
31	1	47	64	0	0	31	11111	0	11111	5
33	2	25	4	8	32	1	100001	100000	1	1
35	1	53	8	4	32	3	100011	100000	11	2
37	4	7	4	9	36	1	100101	100100	1	1
39	1	59	16	2	32	7	100111	100000	111	3
41	2	31	4	10	40	1	101001	101000	1	1
43	1	65	8	5	40	3	101011	101000	11	2
45	3	17	4	11	44	1	101101	101100	1	1
47	1	71	32	1	32	15	101111	100000	1111	4
49	2	37	4	12	48	1	110001	110000	1	1
51	1	77	8	6	48	3	110011	110000	11	2
53	5	5	4	13	52	1	110101	110100	1	1
55	1	83	16	3	48	7	110111	110000	111	3
57	2	43	4	14	56	1	111001	111000	1	1
59	1	89	8	7	56	3	111011	111000	11	2
61	3	23	4	15	60	1	111101	111100	1	1
63	1	95	128	0	0	63	111111	0	111111	6
65	2	49	4	16	64	1	1000001	1000000	1	1
67	1	101	8	8	64	3	1000011	1000000	11	2
69	4	13	4	17	68	1	1000101	1000100	1	1
71	1	107	16	4	64	7	1000111	1000000	111	3
73	2	55	4	18	72	1	1001001	1001000	1	1
75	1	113	8	9	72	3	1001011	1001000	11	2
77	3	29	4	19	76	1	1001101	1001100	1	1
79	1	119	32	2	64	15	1001111	1000000	1111	4
81	2	61	4	20	80	1	1010001	1010000	1	1
83	1	125	8	10	80	3	1010011	1010000	11	2
85	8	1	4	21	84	1	1010101	1010100	1	1
87	1	131	16	5	80	7	1010111	1010000	111	3
89	2	67	4	22	88	1	1011001	1011000	1	1
91	1	137	8	11	88	3	1011011	1011000	11	2
93	3	35	4	23	92	1	1011101	1011100	1	1
95	1	143	64	1	64	31	1011111	1000000	11111	5
...	...	...	...	...	...	...	...	...	...	...
353	2	265	4	88	352	1	101100001	101100000	1	1
355	1	533	8	44	352	3	101100011	101100000	11	2
357	4	67	4	89	356	1	101100101	101100100	1	1
359	1	539	16	22	352	7	101100111	101100000	111	3
361	2	271	4	90	360	1	101101001	101101000	1	1
363	1	545	8	45	360	3	101101011	101101000	11	2
365	3	137	4	91	364	1	101101101	101101100	1	1
367	1	551	32	11	352	15	101101111	101100000	1111	4
369	2	277	4	92	368	1	101110001	101110000	1	1
371	1	557	8	46	368	3	101110011	101110000	11	2
373	5	35	4	93	372	1	101110101	101110100	1	1
375	1	563	16	23	368	7	101110111	101110000	111	3
377	2	283	4	94	376	1	101111001	101111000	1	1
379	1	569	8	47	376	3	101111011	101111000	11	2
381	3	143	4	95	380	1	101111101	101111100	1	1
383	1	575	256	1	256	127	101111111	100000000	1111111	7

Directed graph 3.20.6.



q.e.d. with Teorem 3.1.

4 Analytical expressions and equations

Theorem 4.1.

4x+1 generates the succession of infinite ODD inputs which follow one another with the interval 3x+1 and share the same ODD output.

The 3x+1 interval is the "measure", the distance between an incoming ODD and the next one, it is a power of 2 or a multiple of a power of 2:

$$\text{Din}_{+1} - \text{Din} = \text{Din}*3+1 \;\Rightarrow$$

$$\text{Din}_{+1} = \text{Din} + \text{Din}*3+1 \;\Rightarrow\; \text{Din}+\text{Din}*3+1 = \mathbf{Din*4+1}$$

equation for exponents of 2 EVEN

equazione per esponenti di 2 PARI

t=>2

n= 295

$$\frac{\left(\frac{2^t-1}{3}+2^{t+1}*n\right)*3+1}{2^t} = 1+6n$$

(mod6)	Din	P = Din*3+1	2^t	Dout = $P/2^t$		Dout = 1+6*n	(mod6)	t	Din_{+1}-Din
3	2361	7084	4	1771	=	1771	1	2	7084
1	9445	28336	16	1771	=	1771	1	4	28336
5	37781	113344	64	1771	=	1771	1	6	113344
3	151125	453376	256	1771	=	1771	1	8	453376
1	604501	1813504	1024	1771	=	1771	1	10	1813504
5	2418005	7254016	4096	1771	=	1771	1	12	7254016
3	9672021	29016064	16384	1771	=	1771	1	14	29016064
1	38688085	116064256	65536	1771	=	1771	1	16	116064256
5	154752341	464257024	262144	1771	=	1771	1	18	464257024
3	619009365	1857028096	1048576	1771	=	1771	1	20	1857028096
1	2476037461	7428112384	4194304	1771	=	1771	1	22	7428112384
5	9904149845	29712449536	16777216	1771	=	1771	1	24	29712449536

$$\text{Din}_{+1} = 4\left(\frac{2^t-1}{3}+2^{t+1}*n\right)+1 = \frac{2^{t+2}-4+3\cdot 2^{t+3}\cdot n+3}{3} = \frac{1*2^{t+2}-1}{3}+2^{t+3}*n$$

t EVEN

$$\text{Din}_{+1} = 4\left(\frac{10*2^{t-1}-1}{3}+2^{t+1}*n\right)+1 = \frac{5\cdot 2^{t+2}-4+3\cdot 2^{t+3}\cdot n+3}{3} = \frac{5*2^{t+2}-1}{3}+2^{t+3}*n$$

t ODD

$$\text{Din}_{+1} = \frac{j*2^{t+2}-1}{3}+2^{t+3}*n\,,\quad t\in\mathbb{N}_{>0}\,,\;\; n\in\mathbb{N}$$

Proof. We insert $\text{Din}_{+1} = \text{Din}*4+1$ into the equation $\frac{\text{Din}*3+1}{2^{tmax}} = \text{Dout}$, where in the denominator we'll have $2^{2b+tmax}$ which is $=2^{tmax}$ with b=0, that becomes $=4*2^{tmax}$ with b=1:

$$\frac{(\text{Din}*4+1)*3+1}{2^{2b+tmax}} = \text{Dout} \Rightarrow \frac{\text{Din}*12+4}{4*2^{tmax}} = \text{Dout} \Rightarrow \frac{(\text{Din}*3+1)*4}{4*2^{tmax}} = \text{Dout} \Rightarrow \frac{\text{Din}*3+1}{2^{tmax}} = \text{Dout}$$

If we insert Din_{+2} we will have b=2:

$$\frac{((Din*4+1)*4+1)*3+1}{2^{2b+tmax}} = Dout \Rightarrow \frac{(Din*3+1)*16}{16*2^{tmax}} = Dout \Rightarrow \frac{Din*3+1}{2^{tmax}} = Dout$$

If we insert Din_{+3} we will have b=3:

$$\frac{(((Din*4+1)*4+1)*4+1)*3+1}{2^{2b+tmax}} = Dout \Rightarrow \frac{(Din*3+1)*64}{64*2^{tmax}} = Dout \Rightarrow \frac{Din*3+1}{2^{tmax}} = Dout$$

Analytical equation 4.1.1. $$\frac{2^{2b}*(Din*3+1)}{2^{2b}*2^{tmax}} = Dout$$

It's clear that both the numerator and the denominator are multiplied by 2^{2b}, which represents the values of the powers of 2 EVEN. By the induction principle the equation is valid for every Din_{+b} with Din,b∈ℕ.

Powers of 2 with EVEN exponent >0 can be expressed recursively as:

$$\begin{cases} a_0 = 1 \\ a_b = a_{b-1} * 4, \quad b \in \mathbb{N}_{>0} \end{cases}$$

and the powers of 2 EVEN as: 2^{2b}, b∈ℕ

The sequence of Din sharing the same Dout can be expressed recursively:

$$\begin{cases} a_0 = 1+2m, \quad m \in \mathbb{N} \\ a_b = a_{b-1} * 4 + 1, \quad b \in \mathbb{N}_{>0} \end{cases}$$

Analytic expression 4.1.2.

$$2^{2b}*((1+2m)*3+1), \quad b,m \in \mathbb{N}$$

produces the sequence of numbers ≡ 4(mod6) which, divided by $2^{2b+tmax}$, generate the same Dout.

$$2^{2b}*((1+2m)*3+1)=((((((1+2m)*4+1)*4+1)*4+1)*4+1)\ldots*4+1)*3+1 \Rightarrow$$

b= number of repetitions of the expression *4+1
we obtain:

Analytical equation 4.1.3. $Din_{+b}=\frac{2^{2b}*((1+2m)*3+1)-1}{3}$

Analytic expression 4.1.4. $2^{2b}*(1+2m)+\sum_{i=0}^{b-1}2^{2i}$

with $m\in\mathbb{N}$; $b\in\mathbb{N}$, if b=0 $\Rightarrow\sum_{i=0}^{b-1}2^{2i}=0$; $i=0\div b-1$

since $\sum_{i=0}^{b-1}2^{2i}=\frac{2^{2b}-1}{3}$ $\Rightarrow$

Analytical equation 4.1.5. $Din_{+b}=2^{2b}*(1+2m)+\frac{2^{2b}-1}{3}$ $\Rightarrow$

by unwinding the equation, we obtain 4.1.3.
Let's insert 4.1.3. in 4.1.1.:

$\frac{\left(\frac{2^{2b}*((1+2m)*3+1)-1}{3}\right)*3+1}{2^{2b+tmax}}=j+6n$ $\Rightarrow$ we obtain the Analytical Equation 4.1.1.:

$$\frac{2^{2b}*((1+2m)*3+1)}{2^{2b+tmax}}=j+6n$$

with
- $m,n\in\mathbb{N}$
- $b\in\mathbb{N}$
- tmax calculated with b =0, tmax of $(1+2m)*3+1$, $tmax\in\mathbb{N}_{>0}$
- $j=t\bmod 2*4+1$, $j\in\{1,5\}$

Assuming **m**, by varying **b** we generate the sequence of ODD input sharing the same ODD output and by varying **n** all possible sequences.

simplifying 2^{2b} we will have: $\frac{4+6m}{2^{tmax}}=j+6n$, which coincides with b=0.

Analytical equation 4.1.6. wich is the first member of the equation 3.12.1.2.

$$Din_{+b}=\frac{j*2^{t}-1}{3}+2^{t+1}*n,\quad t\in\mathbb{N}_{>0},\quad n\in\mathbb{N}$$

We obtain tmax as a function of b by inserting the same in 4.1.3.:

$$Din_{+b}=\frac{2^{2b}*\left(\left(\frac{j*2^{t}-1}{3}+2^{t+1}*n\right)*3+1\right)-1}{3} \Rightarrow Din_{+b}=\frac{\left(j*2^{t}+6n*2^{t}\right)*2^{2b}-1}{3}$$

we set t=tmin to work with b, tmin=2 if j=1 and tmin=1 if j=5,
so we find:

Analytic equation 4.2.

$$Din_{+b}=\frac{(j+6n)*2^{2b+tmin}-1}{3}, \quad :\forall n,b\in\mathbb{N}$$

By varying b we generate the succession of possible Din.

Since Dout=j+6n we obtain:

$$Dout=\frac{Din_{+b}*3+1}{2^{2b+tmin}} \Rightarrow \frac{Din_{+b}*3+1}{2^{tmax}}$$

so we can write: tmax=2b+tmin, $tmax\in\mathbb{N}_{>0}$ □

Lemma 4.3. m as a function of n and tmax

We derive m from equation 4.1.1. which, removed 2^{2b}, is equivalent to 3.12.5.:

$$2^{2b}*(4+6m)=2^{2b+tmax}*(j+6n) \Rightarrow m=\frac{2^{tmax}*(j+6n)-4}{6} \Rightarrow$$

$$m=2^{tmax}*\frac{(j+6n)}{6}-\frac{4}{6}, \quad m\in\mathbb{N} \Rightarrow m=2^{tmax}*\left(n+\frac{j}{6}\right)-\frac{2}{3} \Rightarrow$$

Analytic equation 4.3.1

$$Din=1+2*\frac{2^{tmax}*(j+6n)-4}{6} \Rightarrow Din_{+b}=1+2*\left(2^{2b+tmin}*\left(n+\frac{j}{6}\right)-\frac{2}{3}\right)$$

Proof

What is stated is true because we derived m by unfolding Equation 4.1.1. and because of what we proved in Lemmas 3.12.-3.16. and in Theorem 4.1.
□

Theorem 4.3.2. Equation 3.12.5. it is capable of generating all natural numbers, therefore it **proves the Collatz conjecture**.

Proof

m can be expressed with con j=1, t=PARI, $t \in \mathbb{N}_{>0}$, if t=2 $\Rightarrow \sum_{t=4}^{t} j * 2^{t-3} = 0$

$$m = \frac{2^t*(1+6n)-4}{6} \Rightarrow m = 2^t n + \sum_{t=4}^{t} j * 2^{t-3}$$

m can be expressed with j=5, t=ODD, $t \in \mathbb{N}_{>0}$, if t=1 $\Rightarrow \sum_{t=3}^{t} j * 2^{t-3} = 0$

$$m = \frac{2^t*(5+6n)-4}{6} \Rightarrow m = 1 + 2^t n + \sum_{t=3}^{t} j * 2^{t-3}$$

$$m = \frac{2^t*(j+6n)-4}{6} \Rightarrow$$

with t=2p, $p \in \mathbb{N}_{>0}$,			with t=1+2d, $d \in \mathbb{N}$		
⇩			⇩		
t=2	⇨	m=0+4n,	t=1	⇨	m=1+2n,
t=4	⇨	m=2+16n,	t=3	⇨	m=6+8n,
t=6	⇨	m=10+64n,	t=5	⇨	m=26+32n,
t=8	⇨	m=42+256n,	t=7	⇨	m=106+128n,
t=10	⇨	m=170+1024n,	t=9	⇨	m=426+512n,
t=12	⇨	m=682+4096n,…	t=11	⇨	m=1706+2048n,…

t=EVEN
- root: $\frac{2^{t-1}-2}{3} = \{0,2,10,42,170,682, \dots\}$
- module: 2^t

t=ODD
- root: $\frac{2^{t+1}-1}{3} + \frac{2^{t-1}-1}{3} \Rightarrow \frac{2^{t+1}+2^{t-1}-2}{3} \Rightarrow \frac{j2^{t-1}-2}{3} = \{1,6,26,106,\dots\}$
- module: 2^t

so we obtain by unfolding Equation 3.12.5.:

Equation 4.3.2.1.

$$\frac{(1+2m)*3+1}{2^t} = j + 6n \Rightarrow m = \frac{2^t*(j+6n)-4}{6} \Rightarrow m = \frac{j*2^t+6n*2^t-4}{6} \Rightarrow$$

$$\mathbf{m = \frac{j*2^{t-1}-2}{3} + 2^t * n,} \quad t\in\mathbb{N}_{>0},\ m,n\in\mathbb{N}$$

In fact we equate m with that obtained in Lemma 4.3. we obtain:

$$2^t * \left(n + \frac{j}{6}\right) - \frac{2}{3} = \frac{j * 2^{t-1} - 2}{3} + 2^t * n$$

By inserting the successions obtained from Equation 4.3.2.1. into the nodes of the following directed graph, we obtain m. The reasons for the progressions will be 2^t, $t\in\mathbb{N}_{>0}$. The successions with higher modulus generate the natural numbers that the previous ones cannot reach. We will have two patterns that follow each other to infinity, as can be appreciated in Directed graph 4.3.2.2. which show, how Equation 4.3.2.1 is capable of generating all natural numbers:

Directed graph 4.3.2.2.

Equation 4.3.3.3.

$$n = \frac{ja2^{a-1}-2}{3} + 2^a * \left(\frac{jb*2^{b-1}-2}{3} + 2^b * \ldots \left(\frac{j\mu*2^{\mu-1}-2}{3} + 2^\mu * 0\right)\right) \Rightarrow$$

$$m = \frac{jt*2^{t-1}-2}{3} + 2^t * \left(\frac{ja*2^{a-1}-2}{3} + 2^a * \ldots \left(\frac{j\mu*2^{\mu-1}-2}{3}\right)\right) \Rightarrow$$

$$\{\mathbb{N}\} = \left\{ \frac{jt*2^{t-1}-2}{3} + 2^t * \left(\frac{ja*2^{a-1}-2}{3} + 2^a * \dots \left(\frac{j\mu*2^{\mu-1}-2}{3} \right) \right) \right\}, \quad t,a,\dots,\mu \in \mathbb{N}_{>0},$$

the dependent variable j will be multiplied $*2^{\{t,a,b\dots,\mu\}-1}$ and will be a function of the exponent of 2 of this factor:

$\{jt,ja,jb\dots,j\mu\} = \{t,a,b\dots,\mu\} \bmod 2 * 4+1, \quad \{jt,ja,jb\dots,j\mu\} \in \{1,5\}$

The nesting stops when $2^{\mu} * 0 = 0$. The number of iterations varies deterministically as m varies.

Assume for example Din=$\{341_{10}, 213_{10}, 409_{10}, 18870353_{10}\}$

Table 4.4.

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
10	170	1024	0	170	341	1024	1

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
1010	10101010	10000000000	0	10101010	101010101	10000000000	1

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
7	106	640	0	106	213	640	5
2	0	4	0	0	1	4	1

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
111	1101010	1010000000	0	1101010	101010101	1010000000	101
10	0	100	0	0	1	100	1

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
2	0	4	51	204	409	1228	307
1	1	2	25	51	103	310	155
1	1	2	12	25	51	154	77
2	0	4	3	12	25	76	19
1	1	2	1	3	7	22	11
1	1	2	0	1	3	10	5
2	0	4	0	0	1	4	1

t	root	(mod)	n	m	Din_2	4+6m	Dout
10	0	100	110011	11001100	110011001	10011001100	100110011
1	1	10	11001	110011	1100111	100110110	10011011
1	1	10	1100	11001	110011	10011010	1001101
10	0	100	11	1100	11001	1001100	10011
1	1	10	1	11	111	10110	1011
1	1	10	0	1	11	1010	101
10	0	100	0	0	1	100	1

t	root	(mod)	n	m	Din_{10}	4+6m	Dout
2	0	4	2358794	9435176	18870353	56611060	14152765
6	10	64	36856	2358794	4717589	14152768	221137
2	0	4	9214	36856	73713	221140	55285
3	6	8	1151	9214	18429	55288	6911
1	1	2	575	1151	2303	6910	3455
1	1	2	287	575	1151	3454	1727
1	1	2	143	287	575	1726	863
1	1	2	71	143	287	862	431
1	1	2	35	71	143	430	215
1	1	2	17	35	71	214	107
1	1	2	8	17	35	106	53
2	0	4	2	8	17	52	13
4	2	16	0	2	5	16	1

t	root	(mod)	n	m	Din_2	4+6m	Dout
10	0	100	1000111111111000001010	1000111111111000000101000	1000111111111000001010001	1101011111111010000011110100	110101111111010000111101
110	1010	1000000	1000111111111000	1000111111111000001010	10001111111110000010101	1101011111111010001000000	110101111111010001
10	0	100	10001111111110	1000111111111000	10001111111110001	110101111111010100	1101011111110101
11	110	1000	10001111111	10001111111110	100011111111101	1101011111111000	1101011111111
1	1	10	1000111111	10001111111	100011111111	1101011111110	110101111111
1	1	10	100011111	1000111111	10001111111	110101111110	11010111111
1	1	10	10001111	100011111	1000111111	11010111110	1101011111
1	1	10	1000111	10001111	100011111	1101011110	110101111
1	1	10	100011	1000111	10001111	110101110	11010111
1	1	10	10001	100011	1000111	11010110	1101011
1	1	10	1000	10001	100011	1101010	110101
10	0	100	10	1000	10001	110100	1101
100	10	10000	0	10	101	10000	1

We observe the binary column n: n_{+1} loses the decimal number t_{+1}, n_{+2} loses the decimal number t_{+2} of less representative binary digits, and so it goes until $n_{+(p-1)}$, with the exclusion of the last n_{+p}, which still becomes 0. We observe the binary column m: $m_{+1}=n, \dots m_{+p}=n_{+(p-1)}$, m_{+1} loses the decimal number t, m_{+2} loses the decimal number $t_{+1}\dots$ of less representative binary digits, and so it goes until m_{+p}. □

Theorem 4.5. Proof of the decrement of Dout.

We have proved by Direct Graph 4.3.2.2. that Equation 4.3.2.1. is true. We get n from Equation 4.3.2.1:

Equation 4.5.1.

$$n = \frac{m - \frac{j2^{tmax-1}-2}{3}}{2^{tmax}}, \quad tmax \in \mathbb{N}_{>0},\ n,m \in \mathbb{N}$$

is evident that n will always be < m, n becomes m_{+1}, n_{+1} becomes m_{+2}, ...n_{+p} becomes $m_{+(p+1)}$:

$$\frac{m - \frac{j2^{tmax-1}-2}{3}}{2^{tmax}} = n \Rightarrow \frac{\frac{m - \frac{j2^{t-1}-2}{3}}{2^{t}} - \frac{j2^{a-1}-2}{3}}{2^{a}} \ldots = 0, \quad t,a,\ldots = tmax$$

$Dout_{+1}$ at worst, with tmax=1, while becoming larger than Din, will always be smaller than Dout:

$$n = \frac{m - \frac{j2^{tmax-1}-2}{3}}{2^{tmax}} \Rightarrow n = \frac{m - \frac{5*2^{0}-2}{3}}{2^{1}} \Rightarrow n = \frac{m}{2} - \frac{5-2}{6} \Rightarrow n = \frac{m-1}{2} \Rightarrow \mathbf{m=1+2n} \Rightarrow$$

$$n_{+1} = \frac{\frac{m-1}{2}-1}{2} \Rightarrow n_{+1} = \frac{m-3}{4} \Rightarrow n_{+2} = \frac{\frac{m-3}{4}-1}{2} \Rightarrow n_{+2} = \frac{m-7}{8}$$

as can be appreciated in the following table:

Table 4.5.2.

j	tmax	root	2^t	n	m	Din	4+6m	Dout
5	1	1	2	35	71	143	430	215
5	1	1	2	17	35	71	214	107
5	1	1	2	2	17	35	106	53
1	2	0	4	8	2	17	52	13
1	4	2	16	0	8	5	16	1
1	2	0	4	0	0	1	4	1

With {n,m}=0 we are in loop 1,4,2,1...

if tmax remains 1 we will have n=m_{+1} and m_{+1}=Din_{+2}

$Dout = j + 6*n \Rightarrow Dout = j + 6*\frac{m-1}{2} \Rightarrow$ $\mathbf{Dout = 2 + 3*m} \Rightarrow$

$Dout = 2 + 3*(1+2n) \Rightarrow Dout = 5 + 6n$

$Dout_{+1} = j + 6*\frac{m-3}{4} \Rightarrow Dout_{+1} = j + \frac{3m-9}{2} \Rightarrow$ $\mathbf{Dout_{+1} = \frac{1+3m}{2}}$ □

Equation 4.6. We assume ODD numbers, multiply them *4+1 and iterate the operation:

$1+2m \Rightarrow \frac{2^2-1}{3} + 2^1 * m, \quad b = 1$

$(1+2m)*4+1 = 5+8m \Rightarrow \frac{2^4-1}{3} + 2^3 * m, \quad b = 2$

$(5+8m)*4+1 = 21+32m \Rightarrow \frac{2^6-1}{3} + 2^5 * m, \quad b = 3$

$(21+32m)*4+1 = 85+128m \Rightarrow \frac{2^8-1}{3} + 2^7 * m, \quad b = 4$

$(85+128m)*4+1 = 341+512m \Rightarrow \frac{2^{10}-1}{3} + 2^9 * m, \quad b = 5$

…

$$Din_{+(b-1)} = \frac{2^{2b}-1}{3} + 2^{2b-1} * m, \quad b \in \mathbb{N}_{>0},\ m \in \mathbb{N}$$

which is equivalent to writing:

Equation 4.7. $Din_{+b} = \frac{2^{2b}*(4+6m)-1}{3}, \quad b,m \in \mathbb{N}$

$1+2m = \frac{2^0*(4+6m)-1}{3}, \quad b = 0$

$5+8m = \frac{2^2*(4+6m)-1}{3}, \quad b = 1$

$21+32m = \frac{2^4*(4+6m)-1}{3}, \quad b = 2$

$85+128m = \frac{2^6*(4+6m)-1}{3}, \quad b = 3$

$341+512m = \frac{2^8*(4+6m)-1}{3}, \quad b = 4$

…

Equation 4.8. Analytic equation as a function of powers of 2.

Inserting m derived from 4.3.2.1. into 4.6. we get:

$$Din_{+(b-1)} = \frac{2^{2b}-1}{3} + 2^{2b-1} * \left(\frac{jt*2^{t-1}-2}{3} + 2^t * \left(\frac{ja*2^{a-1}-2}{3} + 2^a * \ldots \left(\frac{j\mu*2^{\mu-1}-2}{3}\right)\right)\right),$$

$b,t,a,\ldots,\mu \in \mathbb{N}_{>0}$,

with b= ordinal number of Din, which is included in the succession of Dins that share the same Dout.

Equation 4.9. Given Lemma 3.19., Theorem 3.20.4. and Equation 3.12.5. we can write:

$$(2^k*(1+2w)-1)*3+1=2^{tmax}*(j+6n),\ \ k,tmax\in\mathbb{N}_{>0},\ \ w,n\in\mathbb{N},\ \ w=\frac{m+1}{2^k}-\frac{1}{2}\quad\Rightarrow$$

$$tmax=\log_2\left(\frac{Din*3+1}{Dout}\right)\quad\Rightarrow$$

$$tmax=\log_2\left(\frac{(2^k*(1+2w)-1)*3+1}{j+6n}\right)$$

Equation 4.10.

$$k=\log_2\left(\frac{1+\frac{2^{tmax}*(j+6n)-1}{3}}{1+2w}\right)$$

5 Binary code operations

Theorem 5.1.

With the operations of addition and subtraction and by adding or eliminating binary digits it is possible to carry out the operations of division and multiplication and the most significant formulas that arise from the algorithm. Having examined the number expressed in binary code, we will acquire a predictive vision of the number of applicable conditions:

Division.

Assegnato un numero $PARI_{10}$ lo convertiamo in binario ed eliminando gli zeri meno significativi otteniamo un numero DISPARI, che equivale a dividere il numero per 2^{tmax}:

tmax = 2   w   k

e.g. $1500_{10}=10111011100_2\Rightarrow\frac{1500}{2^2}=375\Rightarrow 375_{10}=101110111_2$

we will then have $k=3$, $w=23_{10}\Rightarrow 23_{10}=10111_2\Rightarrow 23_{10}*2=46_{10}$

$46_{10}=101110_2\Rightarrow 46_{10}+1=47_{10}\Rightarrow 47_{10}=1+2w\Rightarrow$

$47_{10}=101111_2\Rightarrow 2^3*47-1=375_{10}\Rightarrow 375_{10}=2^k*(1+2w)-1$

$\frac{Din+1}{2^k}=1+2w\Rightarrow\frac{375+1}{2^3}=47_{10}$

tmax=3 w k

es. $13976_{10}=11011010011000_2 \Rightarrow \frac{13976}{2^3}=1747 \Rightarrow 1747_{10}=11011010011_2$

we will then have $k=2$, $w=218_{10} \Rightarrow 218_{10}=11011010_2 \Rightarrow 218_{10}*2=436_{10}$

$436_{10}=110110100_2 \Rightarrow 436_{10}+1=437_{10} \Rightarrow 437_{10}=1+2w \Rightarrow$
$437_{10}=110110101_2 \Rightarrow 2^2*437-1=1747_{10} \Rightarrow 1747_{10}=2^k*(1+2w)-1$
$\frac{Din+1}{2^k}=1+2w \Rightarrow \frac{1747+1}{2^2}=437_{10}$

Applying Equation 3.19.14. we get x. The most significant digit of a number x=ODD, expressed in binary code, after k will always be 0. The most significant digits after 0 are w, extrapolated maintaining the order, they will be considered as a new binary number. When x is composed of just 1 $x=2^k-1 \Rightarrow w=0$.

Proof. The directed graph 3.20.6. demonstrates how Equation 3.19.14. reach all ODD numbers. Theorem 3.20.4, Table 3.20.5. and Lemma 3.2. prove that what is stated is true.

$$Din=2^{k+1}*w+2^k-1 \Rightarrow Din-2^k-1=2^{k+1}*w \Rightarrow w=\frac{Din+1-2^k}{2^{k+1}},$$

Replace the 1s that make up k with of 0s, equivalent to subtracting 2^{k-1} from Din. Dividing the EVEN number obtained by 2^{k+1} means eliminating the least significant 0s from it, thus obtaining w. e.g.:
$375_{10}=2^{k+1}*w+2^k-1$, $2^k-1=2^3-1 \Rightarrow 2^{k+1}*w=375-7 \Rightarrow 368_{10}=101110000_2$,
$368_{10}=2^8+2^6+2^5+2^4$, $2^{k+1}=2^4 \Rightarrow w=\frac{2^{k+1}*w}{2^4} \Rightarrow \frac{368}{2^4}=23_{10} \Rightarrow$
$23_{10}=2^{8-4}+2^{6-4}+2^{5-4}+2^{4-4} \Rightarrow$

$2^r < Din_{10}$ e $2^{(r+1)} > Din_{10}$, $\quad Din_{10}=\sum_{a=0}^{r} 2^a * x_a$, $x_a \in \{0,1\}$

$$w=\frac{1-2^k+\sum_{a=0}^{r} 2^a*x_a}{2^{k+1}},\ x_a\in\{0,1\} \Rightarrow \sum_{a=k+1}^{r} 2^{a-(k+1)}*x_a=\frac{1-2^k+\sum_{a=0}^{r} 2^a*x_a}{2^{k+1}}$$

Dividing $Din+1-2^k$ by 2^{k+1} is equivalent to subtracting k+1 from the exponents of 2 with $x_a=1$, of the digits that make up w, due to the well-known property of powers that have the same base.

Multiplication. The condition $3*x+1$ can be written as $x*2+x+1$. Add a 0 to the right of the least significant digit of the number equivalent to multiplying by 2. **If we add the number itself to the product we obtain 3x:** e.g. $11_{10}*3_{10}=33_{10}$

$11_{10} = 1011_2 \Rightarrow 22_{10} = 10110_2$

10110 +

1011 =

$100001_2 \Rightarrow 33_{10} = 100001_2$

<u>If we insert 1 instead of 0 **we obtain the condition 3x+1**</u>:

10111 +

1011 =

$100010_2 \Rightarrow 34_{10} = 100010_2 \Rightarrow 34_{10} = 3*11+1,$

$13_{10} = 1101_2 \Rightarrow 26_{10} = 11010_2$

11010 +

1101 =

$100111_2 \Rightarrow 39_{10} = 100111_2 \Rightarrow 39_{10}+1=40_{10} \Rightarrow$

11011 +

1101 =

$101000_2 \Rightarrow 40_{10} = 101000_2 \Rightarrow 40_{10} = 3*13+1$

Adding 01 to the right of the least significant digit of the number is equivalent to multiplying it by 4+1:

$13_{10} = 1101_2 \Rightarrow 13*4+1=53_{10} \Rightarrow 110101_2 = 53_{10}$

consequently if the least significant digits of the number are 01, eliminating these means subtracting 1 and dividing the difference by 4:

$13_{10} = 1101_2 \Rightarrow 3_{10}=11_2 \Rightarrow \frac{13-1}{4} = 3_{10} \Rightarrow$

$110101_2 = 53_{10} \Rightarrow \frac{\frac{53-1}{4}-1}{4} = 3_{10}$

If we insert $\{3_{10}, 13_{10}, 53_{10}\ldots\} \Rightarrow 3_{10}*4_{10}+1_{10}=13_{10},\ 13_{10}*4_{10}+1_{10}=53_{10}$ in the equation of block 2 we obtain the output $5_{10}=101_2$:

$\frac{3*3+1}{2^1} = 5_{10} \Rightarrow 3*3+1=10_{10} \Rightarrow 10_{10}=1010_2 \Rightarrow$ tmax=1

$\frac{13*3+1}{2^3} = 5_{10} \Rightarrow 13*3+1=40_{10} \Rightarrow 40_{10}=101000_2 \Rightarrow$ tmax=3

$\frac{53*3+1}{2^5} = 5_{10} \Rightarrow 53*3+1=160_{10} \Rightarrow 160_{10}=10100000_2 \Rightarrow$ tmax=5

delete 01 from $101_2 \Rightarrow 1_2=1_{10}$ is equivalent to $\frac{5*3+1}{2^4} = 1_{10}$

delete 2 times 01 from $10101_2 \Rightarrow 1_2=1_{10}$ is equivalent to $\frac{21*3+1}{2^6} = 1_{10}$

delete 3 times 01 from $1010101_2 \Rightarrow 1_2=1_{10}$ is equivalent to $\frac{85*3+1}{2^8} = 1_{10}$

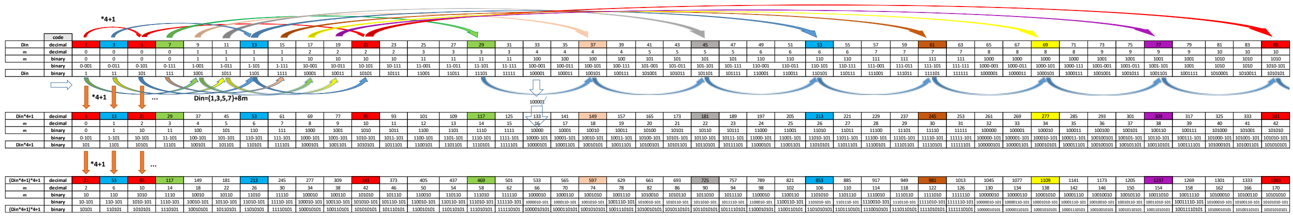
Multiplying an ODD number *4+1 is equivalent to increasing tmax by 2 which is equivalent to increasing b by 1 since: $2^{2b+tmin} = 2^{tmax}$

e.g. $((11*4+1)*4+1)*4+1=725$, $725_{10}=1011010101_2$, $11_{10}=1011_2$

$$\frac{725*3+1}{2^7}=17,\quad \frac{11*3+1}{2^1}=17$$

Theorem 4.1. below will prove the validity of 4x+1.

Directed graph 5.1.



We deduce from the directed graph and from what has been stated so far that after multiplying the numbers ODD*4+1 at least once we obtain:

- $101_2=5_{10}$ as the three least significant digits of the binary code number, 1 in front of 01 since we add 01 to the right of an ODD number which always has 1 as the least significant digit,
- that by expressing in binary code m of the sequences {1,3,5,7}+8*m, m∈ℕ, as the most significant digits, and 101_2 as the least significant, we obtain Din_{+m},
- that m of Din*4+1 is equal to Din stripped of the least significant digit, that m of (Din*4+1)*4+1 is equal to Din*4+1 stripped of the least significant digit …
- that the sequence of ODD numbers that have 101_2 as the three least significant digits is 5+8m and represents 25% of the ODD numbers, the same ones have tmax>2,
- that the sequence of ODD numbers that have $01_2=1_{10}$ as least significant digits is 1+8m and represents 25% of the ODD numbers, they have tmax=2,
- that the sequence of ODD numbers that have $11_2=3_{10}$ as least significant digits is 3+4m and represents 50% of the ODD numbers, they have tmax=1,
- that seen the Lemma 3.16.2. {1+8m,5+8m}={1+4m} and {1+4m,3+4m}={1+2m}.

Therefore by observing an ODD number expressed in binary code we are able to know tmax without multiplying *3+1:

Table 5.2.

$Din-Din_{-1}$	Din_{10}	Din_2	tmax	$Din-Din_{-1}$	Din_{10}	Din_2	tmax	$Din-Din_{-1}$	Din_{10}	Din_2	tmax	$Din-Din_{-1}$	Din_{10}	Din_2	tmax	$Din-Din_{-1}$	Din_{10}	Din_2	tmax	$Din-Din_{-1}$	Din_{10}	Din_2	tmax
	3	11	1		1	1	2		13	1101	3		5	101	4		53	110101	5		21	10101	6
4	7	111	1	8	9	1001	2	16	29	11101	3	32	37	100101	4	64	117	1110101	5	128	149	10010101	6
4	11	1011	1	8	17	10001	2	16	45	101101	3	32	69	1000101	4	64	181	10110101	5	128	277	100010101	6
4	15	1111	1	8	25	11001	2	16	61	111101	3	32	101	1100101	4	64	245	11110101	5	128	405	110010101	6
4	19	10011	1	8	33	100001	2	16	77	1001101	3	32	133	10000101	4	64	309	100110101	5	128	533	1000010101	6
4	23	10111	1	8	41	101001	2	16	93	1011101	3	32	165	10100101	4	64	373	101110101	5	128	661	1010010101	6
4	27	11011	1	8	49	110001	2	16	109	1101101	3	32	197	11000101	4	64	437	110110101	5	128	789	1100010101	6
4	31	11111	1	8	57	111001	2	16	125	1111101	3	32	229	11100101	4	64	501	111110101	5	128	917	1110010101	6
4	35	100011	1	8	65	1000001	2	16	141	10001101	3	32	261	100000101	4	64	565	1000110101	5	128	1045	10000010101	6
4	39	100111	1	8	73	1001001	2	16	157	10011101	3	32	293	100100101	4	64	629	1001110101	5	128	1173	10010010101	6
4	43	101011	1	8	81	1010001	2	16	173	10101101	3	32	325	101000101	4	64	693	1010110101	5	128	1301	10100010101	6
4	47	101111	1	8	89	1011001	2	16	189	10111101	3	32	357	101100101	4	64	757	1011110101	5	128	1429	10110010101	6
4	51	110011	1	8	97	1100001	2	16	205	11001101	3	32	389	110000101	4	64	821	1100110101	5	128	1557	11000010101	6
4	55	110111	1	8	105	1101001	2	16	221	11011101	3	32	421	110100101	4	64	885	1101110101	5	128	1685	11010010101	6
4	59	111011	1	8	113	1110001	2	16	237	11101101	3	32	453	111000101	4	64	949	1110110101	5	128	1813	11100010101	6
4	63	111111	1	8	121	1111001	2	16	253	11111101	3	32	485	111100101	4	64	1013	1111110101	5	128	1941	11110010101	6
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

Less significant digits:

11 ⇨ tmax=1, ⇨ b=0, c=1
01 ⇨ tmax=2, ⇨ b=1, c=0
1101 ⇨ tmax=3, ⇨ b=1, c=1
0101 ⇨ tmax=4, ⇨ b=2, c=0
110101 ⇨ tmax=5, ⇨ b=2, c=1
010101 ⇨ tmax=6, ⇨ b=3, c=0
... ...

tmax=2b+c

Generalising: we had defined b= number of repetitions of the expression $*4+1$, therefore we confirm b as the number of repetitions of 01_2 starting from the least significant digit. If after the repetitions of 01_2 we find 0_2 as the most significant digit, the counting stops and we will have $c=0_{10}$. If we find 11_2 as the most significant digit, the counting stops and we will have $c=1_{10}$. c=tmin-2 if tmax=EVEN, c=tmin if tmax=ODD ⇨tmin=2 if j=1 and tmin=1 if j=5 ⇨ **2b+tmin=2b+c**, b∈ℕ, tmax∈$\mathbb{N}_{>0}$, c={0,1}

Statement 5.3.

We can find the corresponding decimal number of any natural number>0 starting from the binary one:

starting from the most significant 1_2:

- if the digit on the right is a 0_2 we multiply $x*2$, with $x=1_{10}$
- if the digit on the right is a 1_2 we multiply $x*2+1$, with $x=1_{10}$
- if the two digits on the right are 01_2 we multiply $x*4+1$, with $x=1_{10}$

we repeat the conditions with x=product obtained, up to the least significant digit.

E.g. 1 0 0 1 0 01 $01_2=293_{10}$

$1*2=2$, $2*2=4$, $4*2+1=9$, $9*2=18$, $18*4+1=73$, $73*4+1=293_{10}$

It is trivial to observe that $x*2*2+1=4x+1$

The two conditions of the algorithm generate the possible connections that link the nodes of the Directed graphs of Lemma 2.11. The same graphs seen in 3 dimensions will be connected thanks to the deterministic possibilities intrinsic to the number itself. The privileged observation lens, offered by the binary code, shows the numerical "quantum" represented by k and tmax which are amalgamated by the algorithm by interacting with the independent variable. Thus the algorithm connects: all positive numbers to 1, also considering the inverse function any positive number to any positive number as shown in Equation 4.3.2.1.

Equation 5.4. Given Equations 3.19.14. and 4.1.6.:

$$2k*(1+2w)-1=\frac{j*2^{t}-1}{3}+2^{t+1}*n,\quad k,t\in\mathbb{N}_{>0},\quad n,w\in\mathbb{N}$$

w+1= nth ordinal of Din sharing the same k.
n+1= nth ordinal of Din sharing the same tmax.
we derive n:

$$n=\frac{2^{k}*(1+2w)-1-\frac{j*2^{tmax}-1}{3}}{2^{tmax+1}}\quad\Rightarrow\quad Dout=j+6*\left(\frac{2^{k}*(1+2w)-1-\frac{j*2^{tmax}-1}{3}}{2^{tmax+1}}\right)$$

this equation allows us to calculate n and therefore Dout by observing Din expressed with the binary system:

e.g. $Din=71_{10}\Rightarrow Din=1000111\Rightarrow k=3\Rightarrow tmax=1\Rightarrow j=5$

$w=100_{2}\Rightarrow w=4_{10}\Rightarrow n=\frac{2^{3}*(1+2*4)-1-\frac{5*2^{1}-1}{3}}{2^{2}}\Rightarrow n=17\Rightarrow$

$Dout=5+6*17=107$ which is equal to: $Dout=\frac{Din*3+1}{2^{tmax}}\Rightarrow\frac{71*3+1}{2^{1}}=107$

e.g. $Din=73_{10}\Rightarrow Din=1001001\Rightarrow k=1,\quad tmax=2\Rightarrow j=1$

$w=10010_{2}\Rightarrow w=18_{10}\Rightarrow n=\frac{2^{1}*(1+2*18)-1-\frac{1*2^{2}-1}{3}}{2^{3}}\Rightarrow n=9\Rightarrow$

$Dout=1+6*9=55$ which is equal to: $Dout=\frac{Din*3+1}{2^{tmax}}\Rightarrow\frac{73*3+1}{2^{2}}=55$ □

6 Theorem: there are no routines leading to infinity.

Proof

The Lemma 3.12. and Theorems 3.2 - 4.3.2. prove the validity of the equation $(1+2m)*3+1=(j+6n)*2^{tmax}$ we will then have $4+6m=j*2^{tmax}+6n*2^{tmax}$.

Equation 3.12.3. shows that eliminating multiples of 3 ODD: $\left\{\frac{10+18p-1}{3}\right\}=\{3+6p\}$, $p\in\mathbb{N}$, is equivalent to assuming $m=h+3u$, $h\in\{0,2\}$, so the equation becomes: $(1+2*(h+3u))*3+1=(j+6n)*2^{tmax}\Rightarrow(1+2h+6u)*3+1=(j+6n)*2^{tmax}$, $:\forall u,n\in\mathbb{N}$, $tmax\in\mathbb{N}_{>0}$ with $j_h=1+2h$, $j_h,j\in\{1,5\}$

Equation 6.1. $(j_h+6u)*3+1=(j+6n)*2^{tmax}$ ⇨

Din=1+2m becomes j_h+6u after applying the condition *3+1 and dividing by tmax, Lemma 3.1. Having seen Lemma 2.7. we can say that the algorithm is able to reach the number 1.

Table 6.2. on the right it highlights cycle 32 of tmax with $(j+6n)*2^{tmax}$ ≡{4,7}(mod9). tmax will vary at the twelfth and seventeenth ordinal of each cycle, taking as a minimum value{5}. The average of the t values in this case is 1.96875 and will increase due to the Din ≡ 21(mod32). E.g. the average of tmax of the first 512 Din is 2, and 1,998046875 considering 3136 Din.

The Directed graph 3.18.4. and equation 3.12.5 show how $(j+6n)*2^{tmax}$ represents all EVEN≡ 4(mod6).We will therefore have the first term of the equation which will be multiplied *3+1 and the second $*2^{tmax}$. Dout= j_h+6u will be equal to Dout=j+6n only in the case in which u,n=0 and j_h,j=1 and tmax=2, i.e. the algorithm enters the 1-4-2-1… loop. Since the average of 2^{tmax} is equal to 3,914288248 in the minimum cycle and as it grows it will exceed 4, j+6n will on average be lower than j_h+6u to respect equality. j_h and j can take on the same values {1,5} and the same u,n∈ℕ.

Din	Din(mod32)	P1-P4-P7	P(mod9)	tmax	$P/2^{tmax}$
1	1	4	4	2	1
5	5	16	7	4	1
7	7	22	4	1	11
11	11	34	7	1	17
13	13	40	4	3	5
17	17	52	7	2	13
19	19	58	4	1	29
23	23	70	7	1	35
25	25	76	4	2	19
29	29	88	7	3	11
31	31	94	4	1	47
35	3	106	7	1	53
37	5	112	4	4	7
41	9	124	7	2	31
43	11	130	4	1	65
47	15	142	7	1	71
49	17	148	4	2	37
53	21	160	7	5	5
55	23	166	4	1	83
59	27	178	7	1	89
61	29	184	4	3	23
65	1	196	7	2	49
67	3	202	4	1	101
71	7	214	7	1	107
73	9	220	4	2	55
77	13	232	7	3	29
79	15	238	4	1	119
83	19	250	7	1	125
85	21	256	4	8	1
89	25	268	7	2	67
91	27	274	4	1	137
95	31	286	7	1	143
97	1	292	4	2	73
101	5	304	7	4	19
103	7	310	4	1	155
107	11	322	7	1	161
109	13	328	4	3	41
113	17	340	7	2	85
115	19	346	4	1	173
119	23	358	7	1	179
121	25	364	4	2	91
125	29	376	7	3	47
127	31	382	4	1	191
131	3	394	7	1	197
133	5	400	4	4	25
137	9	412	7	2	103
139	11	418	4	1	209
143	15	430	7	1	215
145	17	436	4	2	109
149	21	448	7	6	7
151	23	454	4	1	227
155	27	466	7	1	233
157	29	472	4	3	59
161	1	484	7	2	121
163	3	490	4	1	245
167	7	502	7	1	251
169	9	508	4	2	127
173	13	520	7	3	65
175	15	526	4	1	263
179	19	538	7	1	269
181	21	544	4	5	17
185	25	556	7	2	139
187	27	562	4	1	281
191	31	574	7	1	287
193	1	580	4	2	145
197	5	592	7	4	37
199	7	598	4	1	299
203	11	610	7	1	305
205	13	616	4	3	77
209	17	628	7	2	157
211	19	634	4	1	317
215	23	646	7	1	323
217	25	652	4	2	163
221	29	664	7	3	83
223	31	670	4	1	335
227	3	682	7	1	341
229	5	688	4	4	43
233	9	700	7	2	175
235	11	706	4	1	353
239	15	718	7	1	359
241	17	724	4	2	181
245	21	736	7	5	23
247	23	742	4	1	371
251	27	754	7	1	377
253	29	760	4	3	95
257	1	772	7	2	193
259	3	778	4	1	389
263	7	790	7	1	395
265	9	796	4	2	199
269	13	808	7	3	101
271	15	814	4	1	407
275	19	826	7	1	413
277	21	832	4	6	13
281	25	844	7	2	211
283	27	850	4	1	425
287	31	862	7	1	431

The average of the first 144 $\frac{Din}{Dout}$ ratios, including multiples of 3, is 2.945891942 with average tmax equal to 1.993055556, the average of the first 96 $\frac{Din}{Dout}$ ratios, without multiples of 3 (Table 5.2.), becomes 2.982146051 with average tmax of 1.989583333.

The average ratio of the first 4716 $\frac{Din}{Dout}$, including multiples of 3, is 5.102624895 with average tmax equal to 2. They become 3144 $\frac{Din}{Dout}$ without multiples of 3, with an average ratio of 5.390693764 with average tmax of 1.999363868 . The Equations 5.1. and 4.5.1. and Theorem 4.5. they show how routines that lead to infinity cannot exist. □

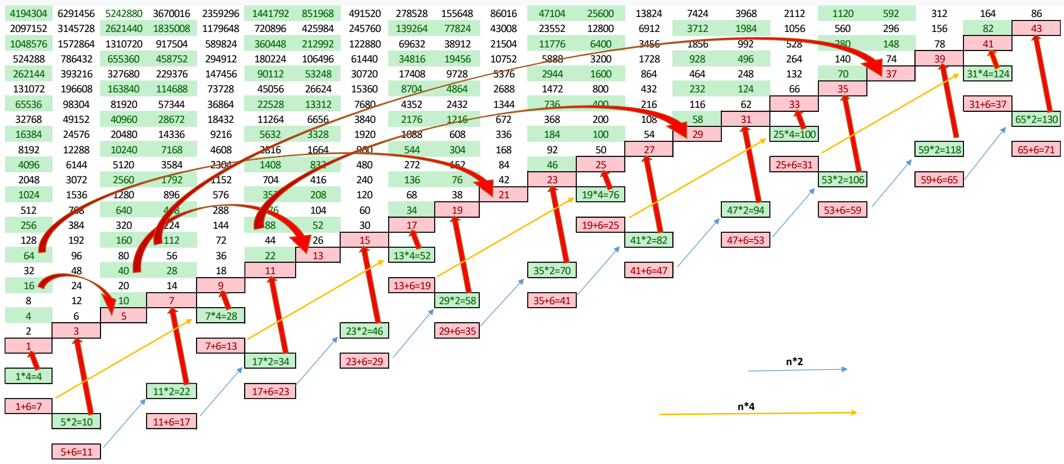
7 We go up the graph tree using the inverse function.

Starting from 1 the algorithm reaches all positive integers using the two conditions: $\frac{n-1}{3}$ and $2n$.

Scheme 7.1.

We detect the scheme used by the algorithm thanks to the table 7.1.1. The same continues infinitely and contains all the ODD numbers, pink cells at the base of each "branch" column, and all the EVEN numbers obtained by iterating the $2n$ condition. We highlight the $\frac{n-1}{3}$ condition represented by the red arrows. We multiply the numbers ODD $\equiv 1 \pmod 6$ by 2^2 and then iterate the operation with the products. We multiply the ODD numbers $\equiv 5 \pmod 6$ by 2^1 and then the products by 2^2, iterating the operation. Thus we obtain numbers $\equiv 4 \pmod 6$ which will return ODD which will in turn be multiplied following the same method. EVEN numbers, green cells, are $\equiv 4 \pmod 6$, EVEN numbers $\equiv \{0,2\} \pmod 6$ are not highlighted.

Table 7.1.1.



Let's start from the EVEN roots {4,10} thus obtained:

$1*2^2=4 \Rightarrow 4*2^2=16 \Rightarrow \frac{16-1}{3}=5 \Rightarrow 5*2^1=10 \Rightarrow \frac{10-1}{3}=3 \ldots$

$\frac{4-1}{3}=1,\ \frac{4+12-1}{3}=5,\ \frac{16+12-1}{3}=9,\ \frac{28+12-1}{3}=13,\ \frac{40+12-1}{3}=17,\ \frac{52+12-1}{3}=21\ldots$

$\frac{10-1}{3}=3,\ \frac{10+12-1}{3}=7,\ \frac{22+12-1}{3}=11,\ \frac{34+12-1}{3}=15,\ \frac{46+12-1}{3}=19,\ \frac{58+12-1}{3}=23\ldots$

We can then write the following equations with $n,m \in \mathbb{N}$:

$$\frac{4+12n-1}{3} = 1 + 4m, \quad \frac{10+12n-1}{3} = 3 + 4m$$

{4+12n , 10+12n} represents all EVEN≡ 4 (mod6)

12 ≡ 0 (mod6), {4,10}≡ 4 (mod6), [0]+[6]=[6]

$4+6m_0 = 4+12n_0 \Rightarrow 4 \equiv 4 \pmod 6$

$4+6m_{+1} = 10+12n_0 \Rightarrow 10 \equiv 4 \pmod 6$

$4+6m_{+2} = 4+12n_{+1} \Rightarrow 16 \equiv 4 \pmod 6$

$4+6m_{+3} = 10+12n_{+1} \Rightarrow 22 \equiv 4 \pmod 6$

$4+6m_{+4} = 4+12n_{+2} \Rightarrow 28 \equiv 4 \pmod 6$

$4+6m_{+5} = 10+12n_{+2} \Rightarrow 34 \equiv 4 \pmod 6$

...

m increases by 1 for every equation, n increases by 1 for every 2 equations because: [6]*[2]=[12]

{1+ 4n , 3+ 4n} represents all ODD ≡ 1 (mod2)

4 ≡ 0 (mod2), {1,3}≡ 1 (mod2), [0]+[2]=[2]

$1+2m_0 = 1+4n_0 \Rightarrow 1 \equiv 1 \pmod 2$

$1+2m_{+1} = 3+4n_0 \Rightarrow 3 \equiv 1 \pmod 2$

$1+2m_{+2} = 1+4n_{+1} \Rightarrow 5 \equiv 1 \pmod 2$

$1+2m_{+3} = 3+4n_{+1} \Rightarrow 7 \equiv 1 \pmod 2$

$1+2m_{+4} = 1+4n_{+2} \Rightarrow 9 \equiv 1 \pmod 2$

$1+2m_{+5} = 3+4n_{+2} \Rightarrow 11 \equiv 1 \pmod 2$

...

m increases by 1 for each equation, n increases by 1 for every 2 equations since: [2]*[2]=[4]

Having seen Lemma 3.16.2. we state:

since the phase shift of the roots, 10-4=6 coincides with the module of the "mother" sequence: 4+6m, and the two modules of the expressions are equal and coincide with double the phase shift, we can state that {4 +12n,10+12n}={4+6m} :∀n,m∈ℕ. Since the phase shift of the roots, 3-1=2 coincides with the module of the "mother" sequence 1+2m, and the two modules of the expressions are equal and coincide with double the phase shift, we can state that {1+ 4n , 3+ 4n} = {1+2m} :∀n,m∈ℕ.

Tables 7.1.2.1.

Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	Dout
3	3	3	10	1	1	5
7	7	7	22	4	1	11
2	11	11	34	7	1	17
6	15	15	46	1	1	23
1	19	19	58	4	1	29
5	23	23	70	7	1	35
0	27	27	82	1	1	41
4	31	31	94	4	1	47
8	35	3	106	7	1	53
3	39	7	118	1	1	59
7	43	11	130	4	1	65
2	47	15	142	7	1	71
6	51	19	154	1	1	77
1	55	23	166	4	1	83
5	59	27	178	7	1	89
0	63	31	190	1	1	95
4	67	3	202	4	1	101
8	71	7	214	7	1	107
3	75	11	226	1	1	113
7	79	15	238	4	1	119
2	83	19	250	7	1	125
6	87	23	262	1	1	131
1	91	27	274	4	1	137
5	95	31	286	7	1	143
...	...	...	...	...	...	...
0	1827	3	5482	1	1	2741
4	1831	7	5494	4	1	2747
8	1835	11	5506	7	1	2753
3	1839	15	5518	1	1	2759
7	1843	19	5530	4	1	2765
2	1847	23	5542	7	1	2771
6	1851	27	5554	1	1	2777
1	1855	31	5566	4	1	2783
5	1859	3	5578	7	1	2789
0	1863	7	5590	1	1	2795
4	1867	11	5602	4	1	2801
8	1871	15	5614	7	1	2807
3	1875	19	5626	1	1	2813
7	1879	23	5638	4	1	2819
2	1883	27	5650	7	1	2825
6	1887	31	5662	1	1	2831
1	1891	3	5674	4	1	2837
5	1895	7	5686	7	1	2843
0	1899	11	5698	1	1	2849
4	1903	15	5710	4	1	2855
8	1907	19	5722	7	1	2861
3	1911	23	5734	1	1	2867
7	1915	27	5746	4	1	2873
2	1919	31	5758	7	1	2879
...	...	...	...	...	...	...

Din mod9	Din	Din mod32	P1-P4-P7	P mod9	tmax	Dout
1	1	1	4	4	2	1
5	5	5	16	7	4	1
0	9	9	28	1	2	7
4	13	13	40	4	3	5
8	17	17	52	7	2	13
3	21	21	64	1	6	1
7	25	25	76	4	2	19
2	29	29	88	7	3	11
6	33	1	100	1	2	25
1	37	5	112	4	4	7
5	41	9	124	7	2	31
0	45	13	136	1	3	17
4	49	17	148	4	2	37
8	53	21	160	7	5	5
3	57	25	172	1	2	43
7	61	29	184	4	3	23
2	65	1	196	7	2	49
6	69	5	208	1	4	13
1	73	9	220	4	2	55
5	77	13	232	7	3	29
0	81	17	244	1	2	61
4	85	21	256	4	8	1
8	89	25	268	7	2	67
3	93	29	280	1	3	35
...	...	...	...	...	...	...
8	1601	1	4804	7	2	1201
3	1605	5	4816	1	4	301
7	1609	9	4828	4	2	1207
2	1613	13	4840	7	3	605
6	1617	17	4852	1	2	1213
1	1621	21	4864	4	8	19
5	1625	25	4876	7	2	1219
0	1629	29	4888	1	3	611
4	1633	1	4900	4	2	1225
8	1637	5	4912	7	4	307
3	1641	9	4924	1	2	1231
7	1645	13	4936	4	3	617
2	1649	17	4948	7	2	1237
6	1653	21	4960	1	5	155
1	1657	25	4972	4	2	1243
5	1661	29	4984	7	3	623
0	1665	1	4996	1	2	1249
4	1669	5	5008	4	4	313
8	1673	9	5020	7	2	1255
3	1677	13	5032	1	3	629
7	1681	17	5044	4	2	1261
2	1685	21	5056	7	6	79
6	1689	25	5068	1	2	1267
1	1693	29	5080	4	3	635
...	...	...	...	...	...	...

Table 7.1.2.1. highlights that:

Din= 3+4m with tmax=1,
P1,P4,P7=10+12n with tmax=1,

Din=1+4m with k=1
P1,P4,P7=4+12n with k=1

Table 7.2. Starting from Dout and following scheme 7.1.1. we obtain:

horizontal increment →

1+6*m	6	D	1	7	13	19	25	31	37	43	49	55	61	67	73
1+8*m	8	$\frac{P-1}{3}$	1	9	17	25	33	41	49	57	65	73	81	89	97
4+24*m	24	P=D*4	4	28	52	76	100	124	148	172	196	220	244	268	292
5+32m	32		5	37	69	101	133	165	197	229	261	293	325	357	389
16+96m	96	*4=	16	112	208	304	400	496	592	688	784	880	976	1072	1168
21+128m	128		21	149	277	405	533	661	789	917	1045	1173	1301	1429	1557
64+384m	384	*4=	64	448	832	1216	1600	1984	2368	2752	3136	3520	3904	4288	4672
85+512m	512		85	597	1109	1621	2133	2645	3157	3669	4181	4693	5205	5717	6229
256+1536m	1536	*4=	256	1792	3328	4864	6400	7936	9472	11008	12544	14080	15616	17152	18688
341+2048m	2048		341	2389	4437	6485	8533	10581	12629	14677	16725	18773	20821	22869	24917
1024+6144m	6144	*4=	1024	7168	13312	19456	25600	31744	37888	44032	50176	56320	62464	68608	74752
1365+8192m	8192		1365	9557	17749	25941	34133	42325	50517	58709	66901	75093	83285	91477	99669
4096+24576m	24576	*4=	4096	28672	53248	77824	102400	126976	151552	176128	200704	225280	249856	274432	299008

5+6*m	6		5	11	17	23	29	35	41	47	53	59	65	71	77
3+4*m	4		3	7	11	15	19	23	27	31	35	39	43	47	51
10+12*m	12	*2	10	22	34	46	58	70	82	94	106	118	130	142	154
13+16m	16		13	29	45	61	77	93	109	125	141	157	173	189	205
40+48m	48	*4=	40	88	136	184	232	280	328	376	424	472	520	568	616
53+64m	64		53	117	181	245	309	373	437	501	565	629	693	757	821
160+192m	192	*4=	160	352	544	736	928	1120	1312	1504	1696	1888	2080	2272	2464
213+256m	256		213	469	725	981	1237	1493	1749	2005	2261	2517	2773	3029	3285
640+768m	768	*4=	640	1408	2176	2944	3712	4480	5248	6016	6784	7552	8320	9088	9856
853+1024m	1024		853	1877	2901	3925	4949	5973	6997	8021	9045	10069	11093	12117	13141
2560+3072m	3072	*4=	2560	5632	8704	11776	14848	17920	20992	24064	27136	30208	33280	36352	39424
3413+4096m	4096		3413	7509	11605	15701	19797	23893	27989	32085	36181	40277	44373	48469	52565
10240+12288m	12288	*4=	10240	22528	34816	47104	59392	71680	83968	96256	108544	120832	133120	145408	157696

We deduce the following equations by looking at the table:

$(1+6m)*4*4^t=(4+24m)*4^t$
$(5+6m)*2*4^t=(10+12m)*4^t$
} m,t∈ℕ

Having seen Table 7.2. we have to prove that with $(4+24m)*4^t$ and $(10+12m)*4^t$ we reach all EVEN ≡ 4 (mod6). With t=0 we certainly reach 10+12m and 4+24m. Multiplying (4+6m)*4 we obtain 16+24m and since having seen Lemma 3.16.2. we can say:

{10+12m, 4+24m, 16+24m}={10+12n, 4+12n} ⇨

{10+12n, 4+12n} = {4+6p} :∀m,n,p∈ℕ

Let's see how the algorithm reaches EVEN≡ 16 (mod24).

We insert in the following table 12 sequences that appear in table 7.2.:

Table 7.3.

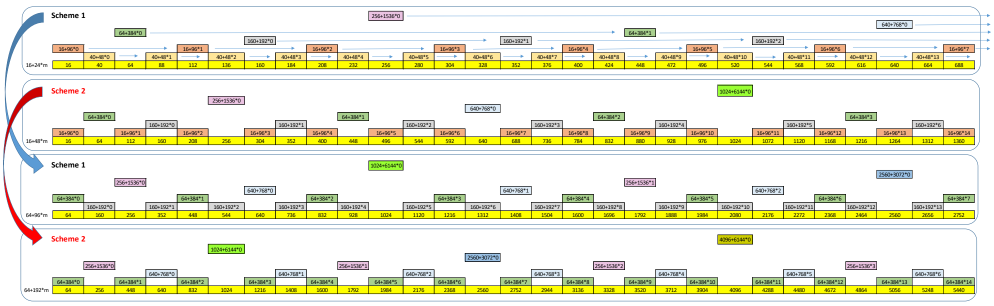
16+24m	16+96m	40+48m	64+384m	160+192m	256+384*m	256+1536m	640+768m	1024+6144	2560+3072m	4096+6144m	4096+24576m	10240+12288m	16384+98304m	40960+49152m	65536+98304m
16	16	40	64	160	256	256	640	1024	2560	4096	4096	10240	16384	40960	65536
40	112	88	448	352	640	1792	1408	7168	5632	10240	28672	22528	114688	90112	163840
64	208	136	832	544	1024	3328	2176	13312	8704	16384	53248	34816	212992	139264	262144
88	304	184	1216	736	1408	4864	2944	19456	11776	22528	77824	47104	311296	188416	360448
112	400	232	1600	928	1792	6400	3712	25600	14848	28672	102400	59392	409600	237568	458752
136	496	280	1984	1120	2176	7936	4480	31744	17920	34816	126976	71680	507904	286720	557056
160	592	328	2368	1312	2560	9472	5248	37888	20992	40960	151552	83968	606208	335872	655360
184	688	376	2752	1504	2944	11008	6016	44032	24064	47104	176128	96256	704512	385024	753664
208	784	424	3136	1696	3328	12544	6784	50176	27136	53248	200704	108544	802816	434176	851968
232	880	472	3520	1888	3712	14080	7552	56320	30208	59392	225280	120832	901120	483328	950272
256	976	520	3904	2080	4096	15616	8320	62464	33280	65536	249856	133120	999424	532480	1048576
280	1072	568	4288	2272	4480	17152	9088	68608	36352	71680	274432	145408	1097728	581632	1146880
304	1168	616	4672	2464	4864	18688	9856	74752	39424	77824	299008	157696	1196032	630784	1245184
328	1264	664	5056	2656	5248	20224	10624	80896	42496	83968	323584	169984	1294336	679936	1343488
352	1360	712	5440	2848	5632	21760	11392	87040	45568	90112	348160	182272	1392640	729088	1441792
376	1456	760	5824	3040	6016	23296	12160	93184	48640	96256	372736	194560	1490944	778240	1540096
400	1552	808	6208	3232	6400	24832	12928	99328	51712	102400	397312	206848	1589248	827392	1638400
424	1648	856	6592	3424	6784	26368	13696	105472	54784	108544	421888	219136	1687552	876544	1736704
448	1744	904	6976	3616	7168	27904	14464	111616	57856	114688	446464	231424	1785856	925696	1835008
472	1840	952	7360	3808	7552	29440	15232	117760	60928	120832	471040	243712	1884160	974848	1933312
496	1936	1000	7744	4000	7936	30976	16000	123904	64000	126976	495616	256000	1982464	1024000	2031616
520	2032	1048	8128	4192	8320	32512	16768	130048	67072	133120	520192	268288	2080768	1073152	2129920
544	2128	1096	8512	4384	8704	34048	17536	136192	70144	139264	544768	280576	2179072	1122304	2228224
568	2224	1144	8896	4576	9088	35584	18304	142336	73216	145408	569344	292864	2277376	1171456	2326528
592	2320	1192	9280	4768	9472	37120	19072	148480	76288	151552	593920	305152	2375680	1220608	2424832
616	2416	1240	9664	4960	9856	38656	19840	154624	79360	157696	618496	317440	2473984	1269760	2523136
640	2512	1288	10048	5152	10240	40192	20608	160768	82432	163840	643072	329728	2572288	1318912	2621440
664	2608	1336	10432	5344	10624	41728	21376	166912	85504	169984	667648	342016	2670592	1368064	2719744
688	2704	1384	10816	5536	11008	43264	22144	173056	88576	176128	692224	354304	2768896	1417216	2818048
712	2800	1432	11200	5728	11392	44800	22912	179200	91648	182272	716800	366592	2867200	1466368	2916352
736	2896	1480	11584	5920	11776	46336	23680	185344	94720	188416	741376	378880	2965504	1515520	3014656
760	2992	1528	11968	6112	12160	47872	24448	191488	97792	194560	765952	391168	3063808	1564672	3112960
784	3088	1576	12352	6304	12544	49408	25216	197632	100864	200704	790528	403456	3162112	1613824	3211264
808	3184	1624	12736	6496	12928	50944	25984	203776	103936	206848	815104	415744	3260416	1662976	3309568
832	3280	1672	13120	6688	13312	52480	26752	209920	107008	212992	839680	428032	3358720	1712128	3407872
856	3376	1720	13504	6880	13696	54016	27520	216064	110080	219136	864256	440320	3457024	1761280	3506176
880	3472	1768	13888	7072	14080	55552	28288	222208	113152	225280	888832	452608	3555328	1810432	3604480
904	3568	1816	14272	7264	14464	57088	29056	228352	116224	231424	913408	464896	3653632	1859584	3702784
928	3664	1864	14656	7456	14848	58624	29824	234496	119296	237568	937984	477184	3751936	1908736	3801088
952	3760	1912	15040	7648	15232	60160	30592	240640	122368	243712	962560	489472	3850240	1957888	3899392
976	3856	1960	15424	7840	15616	61696	31360	246784	125440	249856	987136	501760	3948544	2007040	3997696
1000	3952	2008	15808	8032	16000	63232	32128	252928	128512	256000	1011712	514048	4046848	2056192	4096000
1024	4048	2056	16192	8224	16384	64768	32896	259072	131584	262144	1036288	526336	4145152	2105344	4194304
1048	4144	2104	16576	8416	16768	66304	33664	265216	134656	268288	1060864	538624	4243456	2154496	4292608

4+6m
*4
16+24m
*4
64+96m
*4
256+384m
*4
1024+1536m
*4
4096+6144m
*4
16384+24576m
*4
65536+98304m

If we progressively order the numbers of columns 2,3,4,5 we obtain 16+24m with the exclusion of the highlighted numbers. Absence that is repeated every 16 ordinals starting from 256. If we progressively order the numbers of columns 7,8,9,10 we obtain $256+384m=16^2+24*16m$ with the exclusion of the highlighted numbers.

Absence that is repeated every 16 ordinals starting from 4096. If we progressively order the numbers of columns 12,13,14,15 we obtain $4096+6144m=16^3+24*16^2m$ with the exclusion of the highlighted numbers. Absence that is repeated every 16 ordinals starting from $65536=16^4$. Iterating the operations infinitely, we will always have 4 sequences that generate EVEN≡ 16 (mod24) which the previous ones do not reach. It is clear that the root of the sequences with the missing numbers, thus generated, is 16^t which is a number that can certainly be reached by multiplying $1*2^{4t}$ with $t \in \mathbb{N}_{>0}$. Thus the algorithm generates the sequence 16+24m, reaching all numbers EVEN≡ 4 (mod6) thanks to the 2 conditions.

Directed graph 7.3.1. Using the method already seen, we eliminate the most frequent sequences and obtain the following graph which highlights 2 patterns that alternate, demonstrating how all numbers of the sequence 16+24m are reached.



Using the three sequences: {10+12m, 4+24m, 16+24m} and the inverse function $\frac{(4+6m)-1}{3} = 1 + 2m$ we reach all the ODD numbers. The Theorem 2.3. shows how by multiplying *2 iteratively the ODD numbers we obtain EVEN numbers. Starting from 1 and iterating the 2 inverse conditions the algorithm reaches all positive integers.

Directed graph 7.3.2. Observing the sequences of ODD numbers of the Table 7.2 e of the Lemma 3.12.2. we obtain:

Iterating the $*4^t$ operation we generate successions that are added and begin at the points that the previous ones do not reach. The patterns {1,2,3,4,5,6} repeat endlessly.

We note that the possible sequences generate duplicates, with the sole exclusion of multiples of 3, alternating a number generated by the sequence 1+6m with the duplicate generated by the inverse function $\frac{2^t*(1+2m)-1}{3}$ followed by a number generated by the sequence 5+6m with the duplicate generated by the inverse function $\frac{2^t*(1+2m)-1}{3}$. This mechanism allows the algorithm to connect all the numbers of the 2 subsets: {1+6m,5+6m}, as shown Equation 6.1.

Table 7.4.

In the following table we take the ODD numbers from Table 7.2. highlighting the modular expressions that determine the sequences used by the algorithm: 1+6m, 5+6m, 3+4m, 1+8m, the fifth expression 21+24m is the result of the remaining ones. The first 4 generate all ODD with the exclusion of those generated by 21+24m. We can see how the sequences that determine 21+24m, column 6, follow a cycle 16 which is repeated with the single variable at the ninth ordinal.

Let's assume the equation 4.1.2. with b=1:

$$2^{2b}*((1+2m)*3+1)=((1+2m)*4+1)*3+1 \quad \Rightarrow \quad \frac{2^2*(4+6m)-1}{3} = (1+2m)*4+1 \Rightarrow$$

$$\text{we assume } m=2+3n \Rightarrow 1+2m=5+6n \quad \Rightarrow \quad \frac{64+72n-1}{3} = (5+6n)*4+1 \Rightarrow$$

$$21+24n = 21+24n, \quad n\in\mathbb{N}$$

which proof that the algorithm is able to reach all multiples of $3 \equiv 21 \pmod{24}$ and given the Lemma 3.16.2. all ODD numbers:

Data table 4.5.2.		multiples of 3 ODD = 3+6m						
ODD numbers		3+12m	9+24m	21+24m				
1+6m	5+6m	3+4m	1+8m	21+24m	sequence		1+2m	
1	5	3	1	21	21+128m		1	1+6m
7	11	7	9	45	13+16m		3	3+4m
13	17	11	17	69	5+32m		5	5+6m
19	23	15	25	93	13+16m		7	1+6m
25	29	19	33	117	53+64m		9	1+8m
31	35	23	41	141	13+16m		11	5+6m
37	41	27	49	165	5+32m		13	1+6m
43	47	31	57	189	13+16m		15	3+4m
49	53	35	65	213	213+256m		17	5+6m
55	59	39	73	237	13+16m		19	1+6m
61	65	43	81	261	5+32m		21	21+24m
67	71	47	89	285	13+16m		23	5+6m
73	77	51	97	309	53+64m		25	1+6m
79	83	55	105	333	13+16m		27	3+4m
85	89	59	113	357	5+32m		29	5+6m
91	95	63	121	381	13+16m		31	1+6m
97	101	67	129	405	21+128		33	1+8m
103	107	71	137	429	13+16m		35	5+6m
109	113	75	145	453	5+32m		37	1+6m
115	119	79	153	477	13+16m		39	3+4m
121	125	83	161	501	53+64m		41	5+6m
127	131	87	169	525	13+16m		43	1+6m
133	137	91	177	549	5+32m		45	21+24m
139	143	95	185	573	13+16m		47	5+6m
145	149	99	193	597	85+512m		49	1+6m
151	155	103	201	621	13+16m		51	3+4m
157	161	107	209	645	5+32m		53	5+6m
163	167	111	217	669	13+16m		55	1+6m
169	173	115	225	693	53+64m		57	1+8m
175	179	119	233	717	13+16m		59	5+6m
181	185	123	241	741	5+32m		61	1+6m
187	191	127	249	765	13+16m		63	3+4m
193	197	131	257	789	21+128		65	5+6m
199	203	135	265	813	13+16m		67	1+6m
205	209	139	273	837	5+32m		69	21+24m
211	215	143	281	861	13+16m		71	5+6m
217	221	147	289	885	53+64m		73	1+6m
223	227	151	297	909	13+16m		75	1+8m
229	233	155	305	933	5+32m		77	5+6m
235	239	159	313	957	13+16m		79	1+6m
241	245	163	321	981	213+256m		81	1+8m
247	251	167	329	1005	13+16m		83	5+6m
253	257	171	337	1029	5+32m		85	1+6m
259	263	175	345	1053	13+16m		87	3+4m
265	269	179	353	1077	53+64m		89	5+6m
271	275	183	361	1101	13+16m		91	1+6m
277	281	187	369	1125	5+32m		93	21+24m
283	287	191	377	1149	13+16m		95	5+6m
289	293	195	385	1173	21+128m		97	1+6m
295	299	199	393	1197	13+16m		99	1+8m
301	305	203	401	1221	5+32m		101	5+6m
307	311	207	409	1245	13+16m		103	1+6m
313	317	211	417	1269	53+64m		105	1+8m
319	323	215	425	1293	13+16m		107	5+6m
325	329	219	433	1317	5+32m		109	1+6m
331	335	223	441	1341	13+16m		111	3+4m
337	341	227	449	1365	1365+8192m		113	5+6m
343	347	231	457	1389	13+16m		115	1+6m
349	353	235	465	1413	5+32m		117	21+24m
355	359	239	473	1437	13+16m		119	5+6m
361	365	243	481	1461	53+64m		121	1+6m
367	371	247	489	1485	13+16m		123	1+8m
373	377	251	497	1509	5+32m		125	5+6m
379	383	255	505	1533	13+16m		127	1+6m

{9+24m, 21+24m}={9+12m}
roots 21-9=12
12*2=24 subset module

{3+12m, 9+12m}={3+6m}
roots 9-3=6
6*2=12 module

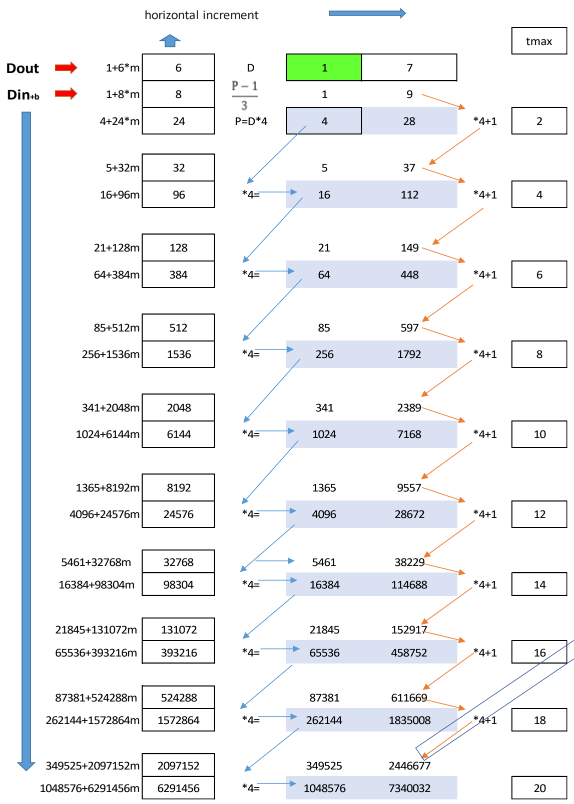
{1+6m, 3+6m,5+6m}={1+2n}
roots 5-3=2, roots 3-1=2
2*3 subset = 6 subset module

	Data table 4.5.2.	multiples of 3 ODD = 3+6m						
	ODD numbers		3+12m	9+24m	21+24m		sequence	
m=24+32p	1+6m	5+6m	3+4m	1+8m	597+768p	p	used	q
24	145	149	99	193	597	0	85+512q	1
56	337	341	227	449	1365	1	1365+8192q	0
88	529	533	355	705	2133	2	85+512q	4
120	721	725	483	961	2901	3	853+1024q	2
152	913	917	611	1217	3669	4	85+512q	7
184	1105	1109	739	1473	4437	5	341+2048q	2
216	1297	1301	867	1729	5205	6	85+512q	10
248	1489	1493	995	1985	5973	7	853+1024q	5
280	1681	1685	1123	2241	6741	8	85+512q	13
312	1873	1877	1251	2497	7509	9	3413+4096q	1
344	2065	2069	1379	2753	8277	10	85+512q	16
376	2257	2261	1507	3009	9045	11	853+1024q	8
408	2449	2453	1635	3265	9813	12	85+512q	19
440	2641	2645	1763	3521	10581	13	341+2048q	5
472	2833	2837	1891	3777	11349	14	85+512q	22
504	3025	3029	2019	4033	12117	15	853+1024q	11
536	3217	3221	2147	4289	12885	16	85+512q	25
568	3409	3413	2275	4545	13653	17	13653+16384q	0
600	3601	3605	2403	4801	14421	18	85+512q	28
632	3793	3797	2531	5057	15189	19	853+1024q	14
664	3985	3989	2659	5313	15957	20	85+512q	31
696	4177	4181	2787	5569	16725	21	341+2048q	8
728	4369	4373	2915	5825	17493	22	85+512q	34
760	4561	4565	3043	6081	18261	23	853+1024q	17
792	4753	4757	3171	6337	19029	24	85+512q	37
824	4945	4949	3299	6593	19797	25	3413+4096q	4
856	5137	5141	3427	6849	20565	26	85+512q	40
888	5329	5333	3555	7105	21333	27	853+1024q	20
920	5521	5525	3683	7361	22101	28	85+512q	43
952	5713	5717	3811	7617	22869	29	341+2048q	11
984	5905	5909	3939	7873	23637	30	85+512q	46
1016	6097	6101	4067	8129	24405	31	853+1024q	23
1048	6289	6293	4195	8385	25173	32	85+512q	49
1080	6481	6485	4323	8641	25941	33	1365+8192q	3

	Data table 4.5.2.	multiples of 3 ODD = 3+6m						
	ODD numbers		3+12m	9+24m	21+24m		sequence	
m=24+32*p	1+6m	5+6m	3+4m	1+8m	597+768p	p	used	q
56	337	341	227	449	1365	1	1365+8192q	0
568	3409	3413	2275	4545	13653	17	13653+16384q	0
1080	6481	6485	4323	8641	25941	33	1365+8192q	3
1592	9553	9557	6371	12737	38229	49	5461+32768q	1
2104	12625	12629	8419	16833	50517	65	1365+8192q	6
2616	15697	15701	10467	20929	62805	81	13653+16384q	3
3128	18769	18773	12515	25025	75093	97	1365+8192q	9
3640	21841	21845	14563	29121	87381	113	87381+524288q	0
4152	24913	24917	16611	33217	99669	129	1365+8192q	12
4664	27985	27989	18659	37313	111957	145	13653+16384q	6
5176	31057	31061	20707	41409	124245	161	1365+8192q	15
5688	34129	34133	22755	45505	136533	177	5461+32768q	4
6200	37201	37205	24803	49601	148821	193	1365+8192q	18
6712	40273	40277	26851	53697	161109	209	13653+16384q	9
7224	43345	43349	28899	57793	173397	225	1365+8192q	21
7736	46417	46421	30947	61889	185685	241	54613+65536m	2
8248	49489	49493	32995	65985	197973	257	1365+8192q	24
8760	52561	52565	35043	70081	210261	273	13653+16384q	12
9272	55633	55637	37091	74177	222549	289	1365+8192q	27
9784	58705	58709	39139	78273	234837	305	5461+32768q	7
10296	61777	61781	41187	82369	247125	321	1365+8192q	30
10808	64849	64853	43235	86465	259413	337	13653+16384q	15
11320	67921	67925	45283	90561	271701	353	1365+8192q	33
11832	70993	70997	47331	94657	283989	369	21845+131072q	2
12344	74065	74069	49379	98753	296277	385	1365+8192q	36
12856	77137	77141	51427	102849	308565	401	13653+16384q	18
13368	80209	80213	53475	106945	320853	417	1365+8192q	39
13880	83281	83285	55523	111041	333141	433	5461+32768q	10
14392	86353	86357	57571	115137	345429	449	1365+8192q	42
14904	89425	89429	59619	119233	357717	465	13653+16384q	21
15416	92497	92501	61667	123329	370005	481	1365+8192q	45
15928	95569	95573	63715	127425	382293	497	54613+65536m	5
16440	98641	98645	65763	131521	394581	513	1365+8192q	48
16952	101713	101717	67811	135617	406869	529	13653+16384q	24

$$21+24*(24+32*p)=597+768p$$

the numbers $597+768p$ are $\equiv 21(\mathrm{mod}32)$



	Data table 4.5.2.	multiples of 3 ODD = 3+6m						
	ODD numbers		3+12m	9+24m	21+24m		sequence	
m=24+32*p	1+6m	5+6m	3+4m	1+8m	597+768p	p	used	q
3640	21841	21845	14563	29121	87381	113	87381+524288q	0
11832	70993	70997	47331	94657	283989	369	21845+131072m	2
20024	120145	120149	80099	160193	480597	625	218453+262144q	1
28216	169297	169301	112867	225729	677205	881	21845+131072q	5
36408	218449	218453	145635	291265	873813	1137	873813+1048576q	0
44600	267601	267605	178403	356801	1070421	1393	21845+131072q	8
52792	316753	316757	211171	422337	1267029	1649	218453+262144q	4
60984	365905	365909	243939	487873	1463637	1905	21845+131072q	11
69176	415057	415061	276707	553409	1660245	2161	87381+524288q	3
77368	464209	464213	309475	618945	1856853	2417	21845+131072m	14
85560	513361	513365	342243	684481	2053461	2673	218453+262144q	7
93752	562513	562517	375011	750017	2250069	2929	21845+131072q	17
101944	611665	611669	407779	815553	2446677	3185	349525+2097152q	1
110136	660817	660821	440547	881089	2643285	3441	21845+131072m	20
118328	709969	709973	473315	946625	2839893	3697	218453+262144q	10
126520	759121	759125	506083	1012161	3036501	3953	21845+131072q	23
134712	808273	808277	538851	1077697	3233109	4209	87381+524288q	6
142904	857425	857429	571619	1143233	3429717	4465	21845+131072m	26
151096	906577	906581	604387	1208769	3626325	4721	218453+262144q	13
159288	955729	955733	637155	1274305	3822933	4977	21845+131072q	29
167480	1004881	1004885	669923	1339841	4019541	5233	873813+1048576q	3
175672	1054033	1054037	702691	1405377	4216149	5489	21845+131072m	32
183864	1103185	1103189	735459	1470913	4412757	5745	218453+262144q	16
192056	1152337	1152341	768227	1536449	4609365	6001	21845+131072q	35
200248	1201489	1201493	800995	1601985	4805973	6257	87381+524288q	9
208440	1250641	1250645	833763	1667521	5002581	6513	21845+131072m	38
216632	1299793	1299797	866531	1733057	5199189	6769	218453+262144q	19
224824	1348945	1348949	899299	1798593	5395797	7025	21845+131072q	41
233016	1398097	1398101	932067	1864129	5592405	7281	5592405+33554432q	0
241208	1447249	1447253	964835	1929665	5789013	7537	21845+131072m	44
249400	1496401	1496405	997603	1995201	5985621	7793	218453+262144q	22
257592	1545553	1545557	1030371	2060737	6182229	8049	21845+131072q	47
265784	1594705	1594709	1063139	2126273	6378837	8305	87381+524288q	12
273976	1643857	1643861	1095907	2191809	6575445	8561	21845+131072m	50

8 Proof of conjecture

EVEN numbers become ODD following the reiterated $\frac{n}{2}$ condition given Theorem 2.3.

Thanks to the impedance adaptation implemented by (1+2m)*3+1=4+6m which makes the Din divisible by the power of 2, the algorithm connects all the ODD integers to the possible Dout.

We therefore have an infinite number of Dout=j+6n and for each j+6n infinite $Din_{+b}=1+2*\left(2^{2b+tmin}*\left(n+\frac{j}{6}\right)-\frac{2}{3}\right)$ which using the 2 conditions bring us back to Dout himself.

Furthermore, Dout infinite number does not include multiples of 3 ODD. All this generates a sort of funnel, exactly the tree of the Collatz graph and the Flowchart 4. By increasing the number processed, the average of the exponents tmax increases. The deterministic sequences of conditions together with these mechanisms generate the inexorable descent of Dout to 1.

Proof All positive integers are present in the tree of the Collatz graph and are connected to 1 thanks to the 2 conditions, as demonstrated in the Lemmas and Theorems up to 4.1

Theorem 4.3.2. demonstrate how by carrying out equation 3.12.5. we reach all natural numbers thanks to Equation 4.3.2.1. and the powers of 2. The Proof 2.12. confirms that $\forall n \equiv 1 \pmod{2} \Leftrightarrow 3n+1 \equiv 4 \pmod{6}$.

The Theorems 4.5.-6. they demonstrate that there are no routines that lead to infinity. The directed graph 7.3.2. shows how all possible Dout are connected thanks to the inverse function. We have shown how Dout are connected to all ODD numbers and ODD numbers to EVEN>0 numbers, and EVEN numbers to 1.

9 Conclusion

All positive integers are present in the Collatz graph, reachable by iterating the 2 conditions and therefore connected to 1 thanks to the powers of 2, given what is shown.

The Collatz conjecture is true.

Thanks to the extraordinary Collatz Theorem we have found an equation that links all natural numbers. We are convinced that this proof will make a contribution to Chaos Theory, which we would cheekily call "Intrinsic Order Theory", and will help predict some physical phenomena. Jealously hidden, each number has a return code. We hope this may suggest the following thought:

"EVERYTHING BACK TO 1, he may never have left it"

Reference:

https://en.wikipedia.org/wiki/Collatz_conjecture

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