A New Identity For Prime Counting Function

Dedicated to my father, my first teacher

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“No Hardy, 1729 it is a very interesting number, it is the smallest number expressible as a sum of two cubes in two different ways.”

Srinivasa Ramanujan

Abstract

In this article, the author proves on a new identity (or equation) which asserts that for every natural number $n$ the sum of the prime-counting function $\pi(2n)$ and the con-counting function $\bar{\pi}(2n)$ equals $n$, explicitly and simply $\forall n \in \mathbb{N}^* \text{ we have } \pi(2n) + \bar{\pi}(2n) = n$. The new identity (or equation) may have many applications in Number Theory and its related to one of the famous problems in Mathematics.

Notation and reminder

\[ \mathbb{N}^* = \{1,2,3,4,5,6,7, \ldots \} \text{ The natural numbers.} \]
\[ \mathbb{N}_{en} = \{2,4,6,8,10,12,14, \ldots \} \text{ The even numbers.} \]
\[ \mathbb{N}_{con} = \{9,15,21,25,27,33,35, \ldots \} \text{ The composite odd numbers.} \]
\[ \mathbb{P} = \{2,3,5,7,11,13,17, \ldots \} \text{ The prime numbers.} \]
\[ \mathbb{P}^* = \{3,5,7,11,13,17,19, \ldots \} \text{ The odd prime numbers.} \]
\[ \forall : \text{ The universal quantifier.} \]
\[ \text{Card } A : \text{ The number of elements in } A. \]
\[ A \cap B : \text{All elements that are members of both } A \text{ and } B. \]
\[ A \cup B : \text{All elements that are members of both } A \text{ or } B. \]
\[ \emptyset : \text{The empty set is the unique set having no elements.} \]
Introduction

**Definition 1** (The prime-counting function \(\pi(x)\)).
\[ \forall x > 0 \text{ we have } \pi(x) = \text{Card}[0, x] \cap \mathbb{P} = \text{Card}\{p \leq x : p \in \mathbb{P}\}. \]
In other words, \(\pi(x)\) is the number of primes less than or equal to \(x\).

In 1838, Dirichlet observed that \(\pi(x)\) can be well approximated by the logarithmic integral function \(\text{li}(x) = \int_2^x \frac{dt}{\log t}\) or \(\pi(x) \sim \text{li}(x) (x \to +\infty)\).

The celebrated prime number theorem, proved independently by de la Vallée Poussin and Hadamard in 1896, states that \(\pi(x) \sim \frac{x}{\log x} (x \to +\infty)\).

**Definition 2** (The prime-counting function \(\pi(2n)\)).
\[ \forall n \in \mathbb{N}^* \text{ we have } \pi(2n) = \text{Card}[1, 2n] \cap \mathbb{P} = \text{Card}\{p \leq 2n : p \in \mathbb{P}\}. \]
In other words, \(\pi(2n)\) is the number of primes less than or equal to \(2n\).

**Definition 3** (The con-counting function \(\bar{\pi}(2n)\)).
\[ \forall n \in \mathbb{N}^* \text{ we have } \bar{\pi}(2n) = \text{Card}[1, 2n] \cap \mathbb{N}_{\text{con}} = \text{Card}\{p \leq 2n : p \in \mathbb{N}_{\text{con}}\}. \]
In other words, \(\bar{\pi}(2n)\) is the number of composite odd numbers less than \(2n\).

**Definition 4** (The en-counting function \(\bar{\bar{\pi}}(2n)\)).
\[ \forall n \in \mathbb{N}^* \text{ we have } \bar{\bar{\pi}}(2n) = \text{Card}[1, 2n] \cap \mathbb{N}_{\text{en}} = \text{Card}\{p \leq 2n : p \in \mathbb{N}_{\text{en}}\}. \]
In other words, \(\bar{\bar{\pi}}(2n)\) is the number of even numbers less than or equal to \(2n\).

For instance:

For \(n = 1\) we have \(\pi(2) = 1\) and \(\bar{\pi}(2) = 0\) and \(\bar{\bar{\pi}}(2) = 1\).

For \(n = 2\) we have \(\pi(4) = 2\) and \(\bar{\pi}(4) = 0\) and \(\bar{\bar{\pi}}(4) = 2\).

For \(n = 3\) we have \(\pi(6) = 3\) and \(\bar{\pi}(6) = 0\) and \(\bar{\bar{\pi}}(6) = 3\).

For \(n = 4\) we have \(\pi(8) = 4\) and \(\bar{\pi}(8) = 0\) and \(\bar{\bar{\pi}}(8) = 4\).

For \(n = 5\) we have \(\pi(10) = 4\) and \(\bar{\pi}(10) = 1\) and \(\bar{\bar{\pi}}(10) = 5\).

For \(n = 6\) we have \(\pi(12) = 5\) and \(\bar{\pi}(12) = 1\) and \(\bar{\bar{\pi}}(12) = 6\).

For \(n = 7\) we have \(\pi(14) = 6\) and \(\bar{\pi}(14) = 1\) and \(\bar{\bar{\pi}}(14) = 7\).

For \(n = 8\) we have \(\pi(16) = 6\) and \(\bar{\pi}(16) = 2\) and \(\bar{\bar{\pi}}(16) = 8\).

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**Lemma.** \( \forall n \in \mathbb{N}^* \) we have \( \bar{\pi}(2n) = n \).

**Proof.** Indeed, \( \forall n \in \mathbb{N}^* \) we have \( \operatorname{Card}[1, 2n] \cap \mathbb{N}^* = 2n \), this means that the number of integers odd or even in the interval \([1, 2n]\) is equal to \(2n\), and \( \operatorname{Card}[1, 3, \ldots, 2n - 1] = \operatorname{Card}[2, 4, \ldots, 2n] = \bar{\pi}(2n) \), this means that the number of odd numbers equal to the number of even numbers in \([1, 2n]\), and \( \operatorname{Card}[1, 2n] \cap \mathbb{N}^* = \operatorname{Card}[1, 3, \ldots, 2n - 1] + \bar{\pi}(2n) = 2n \), thus \( \bar{\pi}(2n) = n \).

**Theorem.** \( \forall n \in \mathbb{N}^* \) we have \( \pi(2n) + \bar{\pi}(2n) + \bar{\pi}(2n) = 2n \).

**Proof.** Indeed, \( \forall n \in \mathbb{N}^* \) we have

\[
[1, 2n] \cap \mathbb{N}^* = \{1\} \cup \{[1, 2n] \cap \mathbb{N}_{en}\} \cup \{[1, 2n] \cap \mathbb{P}^*\} \cup \{[1, 2n] \cap \mathbb{N}_{con}\}
\]

where \(\{1\} \cap \{[1, 2n] \cap \mathbb{N}_{en}\} \cap \{[1, 2n] \cap \mathbb{P}^*\} \cap \{[1, 2n] \cap \mathbb{N}_{con}\} = \emptyset\)

then, \( \operatorname{Card}[1, 2n] \cap \mathbb{N}^* = \operatorname{Card}[1] + \operatorname{Card}[1, 2n] \cap \mathbb{N}_{en} + \operatorname{Card}[1, 2n] \cap \mathbb{P}^* + \operatorname{Card}[1, 2n] \cap \mathbb{N}_{con} = 2n \),

then \(1 + \bar{\pi}(2n) + \pi(2n) - 1 + \bar{\pi}(2n) = 2n\)

, thus \( \pi(2n) + \bar{\pi}(2n) + \bar{\pi}(2n) = 2n \).

**For instance:**

For \( n = 1 \) we have \( \pi(2) + \bar{\pi}(2) + \bar{\pi}(2) = 1 + 0 + 1 = 2 = 2.1 \)

For \( n = 2 \) we have \( \pi(4) + \bar{\pi}(4) + \bar{\pi}(4) = 2 + 0 + 2 = 4 = 2.2 \)

For \( n = 3 \) we have \( \pi(6) + \bar{\pi}(6) + \bar{\pi}(6) = 3 + 0 + 3 = 6 = 2.3 \)

For \( n = 4 \) we have \( \pi(8) + \bar{\pi}(8) + \bar{\pi}(8) = 4 + 0 + 4 = 8 = 2.4 \)

For \( n = 5 \) we have \( \pi(10) + \bar{\pi}(10) + \bar{\pi}(10) = 4 + 1 + 5 = 10 = 2.5 \)

For \( n = 6 \) we have \( \pi(12) + \bar{\pi}(12) + \bar{\pi}(12) = 5 + 1 + 6 = 12 = 2.6 \)

For \( n = 7 \) we have \( \pi(14) + \bar{\pi}(14) + \bar{\pi}(14) = 6 + 1 + 7 = 14 = 2.7 \)

For \( n = 8 \) we have \( \pi(16) + \bar{\pi}(16) + \bar{\pi}(16) = 6 + 2 + 8 = 16 = 2.8 \)

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Corollary (New identity). ∀\( n \in \mathbb{N}^* \) we have \( \pi(2n) + \bar{\pi}(2n) = n \).

Proof. ∀\( n \in \mathbb{N}^* \) we have \( \pi(2n) + \bar{\pi}(2n) + \bar{\pi}(2n) = 2n \) and \( \bar{\pi}(2n) = n \), then \( \pi(2n) + \bar{\pi}(2n) + n = 2n \), thus \( \pi(2n) + \bar{\pi}(2n) = n \).

Remark. \( \bar{\pi}(2n) = 0 \) when \( n \leq 4 \)
\( \bar{\pi}(2n) \geq 1 \) when \( n > 4 \).

For instance:

For \( n = 1 \) we have \( \pi(2) + \bar{\pi}(2) = 1 + 0 = 1 \)
For \( n = 2 \) we have \( \pi(4) + \bar{\pi}(4) = 2 + 0 = 2 \)
For \( n = 3 \) we have \( \pi(6) + \bar{\pi}(6) = 3 + 0 = 3 \)
For \( n = 4 \) we have \( \pi(8) + \bar{\pi}(8) = 4 + 0 = 4 \)
For \( n = 5 \) we have \( \pi(10) + \bar{\pi}(10) = 4 + 1 = 5 \)
For \( n = 6 \) we have \( \pi(12) + \bar{\pi}(12) = 5 + 1 = 6 \)
For \( n = 7 \) we have \( \pi(14) + \bar{\pi}(14) = 6 + 1 = 7 \)
For \( n = 8 \) we have \( \pi(16) + \bar{\pi}(16) = 6 + 2 = 8 \)

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References


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