The Condition for the Real Part of Dirichlet Function to be 1/2

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Abstract
Find the trigonometric function $\sin(n/2\pi)$ that satisfies the Dirichlet feature, and then analyze the conditions for making the real part of the $L$-function $1/2$.

Keywords
Exceptional zero/Dirichlet feature

We know that the Dirichlet feature is a function $\chi(n)$ defined on an integer with the range of $C$, and now we have found a trigonometric function $\sin\left(\frac{n}{2}\pi\right) = \chi(n)$, then:

1) $\chi(n + N) = \sin\left(\frac{n + N}{2}\pi\right) = \chi(n) = \sin\left(\frac{n}{2}\pi\right)$ \quad ($N$ is an even period constant)

2) $\chi(1) = \sin\left(\frac{1}{2}\pi\right) = 1$

3) $\chi(nm) = \sin\left(\frac{nm}{2}\pi\right) = \chi(n)\chi(m) = \left(\sin\left(\frac{n}{2}\pi\right)\right)\left(\sin\left(\frac{m}{2}\pi\right)\right)$ \quad ($n$ and $m$ are integers)

4) $\chi(x) = 0 \quad (\gcd(x, N) \neq 1) \quad \Rightarrow \chi(x) = \chi(d + e \cdot i)$

Let $x = d + e \cdot i$ \quad $\Rightarrow \chi(x) = \chi(d + e \cdot i) = \sin\left(\frac{d + e}{2}i\right) \cdot \pi = 0$ \quad ($x, d, e \in C$)

So $d/2 = 1/2 + k \Rightarrow d = 1 + 2k \Rightarrow \chi(x) = \sin\left(\frac{(1/2 + k) + e}{2}i\right) \cdot \pi = 0$ \quad ($k$ is an integer)

It can be seen that $\sin\left(\frac{n}{2}\pi\right)$ satisfies all four conditions, and it is a Dirichlet feature.

The Dirichlet $L$-function is: $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$

When $n$ is an integer, then $\chi(n) = \sin\left(\frac{n}{2}\pi\right) = 1$ or $L(s, \chi) = 0$, these integer points are both the trivial zeros of the $L$-function and the Riemannian Zeta function; When $x = 1 + 2k + e \cdot i = s/2$ and $2k = 0$, then $x = 1 + e \cdot i = s/2$ and $L(s, \chi) = 0$, these points on the complex plane are both non-trivial zeros of the $L$-function and Riemannian Zeta function. But when $2k \neq 0$, $x = 1 + 2k + e \cdot i$, it is not a non-trivial zero of the Riemannian Zeta function but a non-trivial zero of the $L$-function.

Conclusion:
When $2k = 0$, then $\chi(x) = 0$, the real parts of the non-trivial zeros of the $L$-function and the Riemannian Zeta function are both all $1/2$. The Riemann hypothesis holds, but the Zero point conjecture is not completely true.

References
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