Skolems solution for integer-linear-recurrences, with commensurable arguments for characteristic-roots of the same modulus

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Abstract: - For a homogeneous linear-recurrence $f_n$ with integer coefficients and integer starting points, we derive a deterministic algorithm that finds the upper bound of the last non-periodic position $n$ where $f_n=0$, for a large family of special cases. First, when 0 is a given irrational constant, then we show that, an eventual lower bound of minimum(absolute(cos(mr0))), over positive integers $m$ less than $n$, for large positive integers $n$, is $(2^\theta / (sqrt(5) n))$. Our deterministic algorithm is based on the key concept that this lower bound decreases at a lower rate than the $n^\theta$ power of the ratio of root-moduli since the ratio is lesser than 1. Our deterministic algorithm is developed for the special cases where $G(x)$, the characteristic polynomial of $f_n$, has either equal absolute values of arguments or commensurable arguments of those complex roots whose moduli are equal. In an attempt to extend this algorithm as a general solution to Skolems problem, we obtain the lower bound of the distance between a zero and the next $(2^{m+1})b$ zero, in the weighted sum of $m$ continuous cosine functions, where the weights are given real-algebraic constants.

1. Introduction

Given integer constants \{b_1, b_2, ..., b_L, a_0, a_1, a_2, ..., a_{L-1}\}, we denote our homogenous integer linear recurrence sequence $f_n$ as follows:
   1. $f_0 = 0$, for integers $n < 0$
   2. $f_n = a_n$, for integers $0 \leq n \leq (L-1)$
   3. $f_n = b_1 f_{n-1} + b_2 f_{n-2} + ... + b_L f_{n-L}$, for integers $n \geq L$

It is known from the Skolem-Mahler-Lech (SML) theorem that the set of zeros of $f_n$ is the union of an infinite periodic set and a finite set. In other words, the set $Z$ of $n$ for which $f_n = 0$, is such that $Z = (P \cup Q)$, where $P$ is an infinite set of positive integers that is a finite set of infinite arithmetic progressions, and where $Q$ is a finite set of positive integers. It is also known how to describe $P$ and to determine whether or not $P$ is empty, but an upper bound on the largest integer in $Q$ is still an open problem. Consequently, it remains an open problem to determine whether or not $Q$ is empty within finite time, and this open problem may be simply reworded as determining whether or not $f_n = 0$ for some integer $n > 0$ [1][2][3][5][7].

A lot of research has gone into deterministicly finding the last non-periodic zero in $Q$, for many special cases. References [9] and [10] are some examples, and the reference [7] gives a good recent literature review of the special cases covered.

In this paper, we discuss what constitutes the zeros in sets $P$ and $Q$. We then derive a deterministic algorithm to describe the set $Z$ for a given $f_n$, for some more special cases. We finally present a result that attempts to generalize the algorithm to cover all cases.

2. Approach

2.1 Notations
Further to the notation of $f_n$ given in Section 1, we use the following additional notations:
   1. A number $x$ is said to be algebraic if it is the root of some polynomial with integer coefficients.
   2. If $x$ is a complex number $= y + iz$, where $i=square_root(-1)$, then modulus($x$) = square_root($y^2 + z^2$). Also, real($x$) = $y$ and imaginary($x$) = $z$. Also, $x = (\text{modulus}(x) \ e^{i \theta}) = \text{modulus}(x)(\cos(\theta) + i \sin(\theta))$.
   3. If $x$ and $y$ are two positive real numbers, then quotient($x/y$) = floor($x/y$) = the largest non-negative integer below $x/y$. Also, fraction($x/y$) = ($x/y$ - quotient($x/y$)).
   4. If $x$ is a real number, then absolute($x$) = $x$ if $x \geq 0$, and absolute($x$) = $-x$ if $x < 0$.
   5. BODMAS (Brackets, Order of Powers or Roots, Division, Multiplication, Addition, Subtraction) order is followed in all expressions where the order is not explicitly mentioned by brackets. For example, the expression $ax + b^0 y -$
Theorem is Hence \( f \) is a product function over \([0, L-1)\),
Theorem A 16.

1. \( \pi \) or \( \pi \) is the irrational ratio between the lengths of the circumference and diameter of a circle, which is approximately 3.14159.
2. sqrt(x) is the square root of x.
3. G(x) is the characteristic polynomial of \( f \). Basically, \( G(x) = (1 - b_1 x - b_2 x^2 - b_3 x^3 - \ldots - b_L x^L) \).
4. (Zero of \( f \)) = (n, such that \( p \cdot 0 \), where integer \( n > 0 \)). Similarly, zero of a function \( f(x) \), where \( x \) is a continuous real variable, is such that \( f(x) = 0 \).
5. Two non-zero real numbers \( x \) and \( y \) are said to be commensurable, if and only if, \( x/y \) is rational, i.e. iff \( (x/y) = (p/q) \), where \( p \) and \( q \) are both integers. If \( (x/y) \) is irrational, then \( x \) and \( y \) are said to be incommensurable.
6. If \( A \) and \( B \) are two boolean statements, then:
   - \( A\rightarrow B \) denotes "If \( A \) is true, then \( B \) is true"
   - \( A\rightarrow B \) denotes "\( A \) is true, if and only if, \( B \) is true".
7. A variable indexed by an integer, is represented in either or both of two ways:- with underscore or with subscript.
   For example, \( \gamma_i \) is used interchangeably with \( \gamma_i \).
8. Let \( f(x) \) be a function of \( x \). Then LowerBound(\( f(x) \)) is a function of \( x \), using which one can get the minimum value of \( f(x) \) for a given \( x \). Similarly, for UpperBound(\( f(x) \)). But Maximum(\( x \)) or Max(\( x \)) is a real value that is the global maximum of \( f(x) \) over the entire domain of \( x \). Similarly, for Minimum(\( x \)) or Min(\( x \)).

Theorem 1 below is a reworded form of a known result, on the stateless expression of \( f_n \) [6].

**Theorem 1:** For each integer \( n \geq 0 \), \( f_n = \text{SUM}( (n^r r_k s_{n,k} ), \text{over all integers k in [1,L]), over all integers j in [0,L-1]) }, \) where:

1. \( s_{n,k} = \text{SUM}( (d_{n,k} \cos(n \theta_k + \beta_{n,k} ) ) \), over all integers k in [1,L]).
2. Each of \( \{d_{n,k}, r_n, \cos(\theta_k), \sin(\theta_k), \cos(\beta_{n,k}), \sin(\beta_{n,k})\} \) is a constant real algebraic number, and \( r_n > 0 \)

**Proof:** It is well-known that \( f_n = C_1(n) r_n^3 e^{i(n \theta_1)} + C_2(n) r_n^3 e^{i(n \theta_2)} + \ldots + C_L(n) r_n^3 e^{i(n \theta_L)} \), where for each integer k in [1, L]:

1. \( C_k(n) \) is a constant univariate polynomial in \( n \) with degree \( = L-1 \), with constant algebraic coefficients. So \( C_k(n) = (d_{n,k} e^{i(\beta_{n,k})} + (d_{n,k} e^{i(\beta_{n,k})}) n + (d_{n,k} e^{i(\beta_{n,k})}) n^2 + \ldots + (d_{n,k} e^{i(\beta_{n,k})}) n^{L-1} \), where each of \( \{d_{n,k}, \cos(\beta_{n,k}), \sin(\beta_{n,k})\} \) is a constant real algebraic number for each integer j in [0,L-1].
2. \( (r_n e^{i(\theta_0)}) \) is a constant algebraic root of \( G(x) \) and \( (r_n e^{i(-\theta_0)}) \) is its conjugate, where each of \( \{r_n, \cos(\theta_0), \sin(\theta_0)\} \) is a constant real algebraic number.

Since the imaginary parts of \( f_n \) have to disappear, and since two or more roots may be of equal modulus, and of same or different arguments, we can further group the expression of \( f_n \) as a sum of product terms, where each product term is a product between the term \( (n^r r_n^3) \) and a weighted sum of trigonometric functions. Thus, \( f_n \) can be expressed as follows:

\[ f_n = \text{SUM}( (n^r r_n^3 s_{n,k} ), \text{over all integers k in [1,L]), over all integers j in [0,L-1]), \text{where s_{n,k} = SUM( (d_{n,k} \cos(n \theta_k + \beta_{n,k} ) ), over all integers k in [1,L]).} \]

**Hence Proved Theorem 1.**

Before proceeding further, let us remember that our aim is to describe the set \( Z \) of the zeros of \( f_n \), which we earlier mentioned is the union of sets P and Q. Theorem 2 describes what constitutes the set of zeros in P and Q.

**Theorem 2:** Assuming the SML Theorem is true, the finite set of non-periodic zeros \( Q \) of \( f_n \) is the set of zeros of the expression \( f_n = \text{SUM}( (n^r r_n^3 s_{n,k} ), \text{over all integers k in [1,L]), over all integers j in [0,L-1]), \) after removing cosines of rational multiples of \( \pi \) that are common factors from every \( s_{n,k} \)

**Proof:** If we take out the common cosine factors from all \( s_{n,k} \), then we may rewrite:

\[ f_n = \cos(\pi n \theta_{real,1} + \gamma_{real,1}) \cos(\pi n \theta_{real,2} + \gamma_{real,2}) \ldots \cos(\pi n \theta_{real,L} + \gamma_{real,L}) \]
Cosine of the product of real, V-1 real, and V real, (where V = L), over all integers k in [1,L]), over all integers j in [0,L-1]), where U and V are some positive integer constants, each of which is a rational constant, and each \( \phi \) is a real constant. It becomes clear that the set \( P \) of zeros of \( f_n \) is due to the first product \( \cos(\pi(nU+\phi)) \) of \( \cos(\pi(nU+\phi)) \) \( \cos(\pi(nU+\phi)) \) resulting in a finite union of infinite-sized arithmetic progressions. The second product \( \cos(\pi(nU+\phi)) \) \( \cos(\pi(nU+\phi)) \) \( \cos(\pi(nU+\phi)) \) might contribute to a maximum of V zeros to Q that are also easy to find. But it is the third product \( \cos(\pi(nU+\phi)) \) \( \cos(\pi(nU+\phi)) \) \( \cos(\pi(nU+\phi)) \) that makes the most interesting contribution to the finite set of zeros of Q assuming that the SML Theorem is true (i.e. assuming that the set of non-periodic zeros if finite for a given \( f_n \)).

**Hence Proved Theorem 2.**

Since it is already known how to define the set \( P \) of zeros of \( f_n \), we shall assume without loss of generality, that when we write \( f_n = \text{SUM}(n' r^n s_{n,k}) \), over all integers k in [1,L]), over all integers j in [0,L-1]), in the remaining part of this paper, all common cosine factors have already been factored out and removed, and that the zeros of this expression of \( f_n \) is what forms the set of non-periodic zeros Q that we are trying to find.

We now make the next important Theorem 3, which serves as the foundation for the solution to many special cases in Skolem's problem to the upper bound of the largest non-periodic zero of \( f_n \).

**Theorem 3:** Let \( \lambda_\alpha \) and \( \lambda_\beta \) be two positive incommensurable constants, where \( \lambda_\alpha > \lambda_\beta \). Consider the following algorithm:

**Algorithm description starts**

```plaintext
i = 0;
\n\alpha = 0;
\n\beta = 0;
\n\alpha = \lambda_\alpha;
\nREPEAT:
\nIf (\( n \alpha < n \beta \beta \))
\n\{ \n\alpha = n + 1;
\}
\nElse
\n\{ \n\beta = n + 1;
\}
\nIf (\( \text{absolute}(\alpha \lambda - n \beta) < p_i \))
\n\{ \n\iota = i + 1;
\}
\n\p_i = \text{absolute}(\alpha \lambda - n \beta);
\nGoto REPEAT;

Algorithm description ends
```

Then the following statements are true:

1. The following recurrence sequence can be used to find \( p_i \):
   \[
p_0 = \lambda_\alpha \\
p_1 = \lambda_\beta \\
p_{i+2} = \text{remainder}(p_i, p_{i+1}), \text{for integers } i \geq 0.
   \]
2. When \( \lambda = (\alpha / g) \), then \( p_i = (\alpha / g) \) for all integers \( i \geq 0 \).
3. \( p_i \geq (\alpha / g) \), for all values of \( \lambda_\alpha \) and \( \lambda_\beta \).
4. The \( i \)-th term in the Fibonacci sequence, is the lower bound for both \( n_\alpha \) and \( n_\beta \). That is, everytime \( \text{absolute}(\alpha \lambda - n \beta) < p_i \) in the algorithm, each of \( \{n_\alpha, n_\beta\} \geq t_1 \), where \( t_{i+2} = t_{i+1} + t_i \), for integers \( i \geq 0 \), and \( t_0 = 0 \) and \( t_1 = 1 \).

**Proof:** The algorithm described in this Theorem can be viewed as finding how close a zero of \( \sin(\theta_\alpha x) \) comes to a zero of \( \sin(\theta_\beta x) \), where \( \theta_\alpha = 2\pi(2\lambda_\alpha) = \pi / \lambda_\alpha \) and \( \theta_\beta = 2\pi(2\lambda_\beta) = \pi / \lambda_\beta \), as the real variable \( x \) is increased continuously from 0, and where the counter \( i \) is incremented by 1 each time the present absolute distance becomes lesser than the previous absolute distance, between these zeros. Consider the recurrence sequence:

\[
p_0 = \lambda_\alpha \\
p_1 = \lambda_\beta \\
p_{i+2} = \text{remainder}(p_i, p_{i+1}), \text{for integers } i \geq 0.
   \]

Writing out the first few terms, we get:

\[
p_0 = \lambda_\alpha \\
p_1 = \lambda_\beta
   \]
\[ p_2 = \text{remainder}(p_0, p_1) \]
\[ p_3 = \text{remainder}(p_1, p_2) \]
\[ p_4 = \text{remainder}(p_2, p_3) \]
\[ p_5 = \text{remainder}(p_3, p_4) \]

... and so on.

Denoting \( \text{floor}(p_i / p_{i+1}) = M_i \), we can write:

\[ p_0 = \lambda_i \cdot 0 \cdot \lambda_i^0 \]
\[ p_1 = \lambda_i \cdot 0 \cdot \lambda_i^0 \]
\[ p_2 = p_0 - M_0 \cdot p_1 = \lambda_i \cdot \lambda_i^0 - M_0 \cdot \lambda_i^0 \]
\[ p_3 = p_1 - M_1 \cdot p_2 = \lambda_i^0(1 + M_0 M_1) - \lambda_i(M_i) \]
\[ p_4 = p_2 - M_2 \cdot p_3 = \lambda_i^0(1 + M_0 M_2 + M_2 M_3) - \lambda_i(M_i + M_2 M_3) \]
\[ p_5 = p_3 - M_3 \cdot p_4 = \lambda_i^0(1 + M_0 M_3 + M_2 M_3 + M_2 M_3) - \lambda_i(M_i + M_3 + M_2 M_3) \]

...

and so on.

It follows that if \( p_i \) has \( x_{A_i} \) product terms in the "sum of products" coefficient of \( \lambda_i \) and \( x_{B_i} \) product terms in the "sum of products" coefficient of \( \lambda_i^0 \), then \( x_{A_{i+2}} = x_{A_{i+1}} + x_{A_i} \), and \( x_{B_{i+2}} = x_{B_{i+1}} + x_{B_i} \). The following patterns are straightforward by induction:

1. The number of product terms in the coefficient of \( \lambda_i^0 \) follows the well-known Fibonacci sequence \( t_i: 0, 1, 1, 2, 3, 5, 8, 13, \) etc, where \( t_{i+2} = t_{i+1} + t_i \) starting from \( p_0 \).
2. The number of product terms in the coefficient of \( \lambda_i \) follows the Fibonacci sequence \( : 0, 1, 1, 2, 3, 5, 8, 13, \) etc, starting from \( p_1 \).
3. The coefficient of \( \lambda_i^0 \) in \( p_i \) has the product term \( M_0 M_2 M_3 \ldots M_{i-2} \).
4. The coefficient of \( \lambda_i \) in \( p_i \) has the product term \( M_0 M_1 \ldots M_{i-2} \).

The exact formula for the \( n^\text{th} \) term of the Fibonacci sequence is well-known to be \( t_n = (1/\sqrt{5})((1+\sqrt{5})/2)^n - (1-\sqrt{5})/2)^n \). A well-known approximation formula for \( t_n = g^n/\sqrt{5} \), and the exact formula \( t_n = \text{round}(g^n/\sqrt{5}) \) to the nearest integer, is true for all integers \( \geq 2 \).

Plugging in the least possible values \( M_i = 1 \), for all positive integers \( i \), one possible solution that allows this would be that \( \lambda_i^0 = (\lambda_i/r) \), and \( p_{i+1} = (p_i / r) \) for all integers \( i \geq 0 \), where \( r \) is a real \( > 1 \). We are now in a position to find the value of \( r \) that allows this by writing these 3 equations:

\[ p_{i+2} = (p_{i+1} / r) \]
\[ p_{i+1} = (p_i / r) \]
\[ p_{i+2} + p_{i+1} = p_i \]

From the first two equations, we get \( p_{i+2} = (p_i / r^2) \). From the second equation, we get \( p_{i+1} = (p_i / r) \). Plugging these into the third equation, and eliminating \( p_i \), we get \( (1/r)(1+(1/r)) = 1 \), which means \( r = (1+(1/r)) \), and the only solution for \( r \) that satisfies this is the golden ratio \( g \). Hence, \( p_i = \lambda_i / g \) when \( \lambda_i = (\lambda_i/r) \), for integers \( i \geq 0 \). Since \( t_i = t_{i+1} + t_{i+2} \), we know that \( t_{i+2} = t_i \cdot t_{i+1} \), implying that after every \( t_i \) consecutive values of \( n \) in the algorithm, \( p_{i+1} = p/g \).

Another way of looking at why ensuring \( p_{i+1} = p/g \) for all integers \( i \geq 2 \) leads to the fastest possible reduction in \( p_i \) is to consider these cases of the value of \( K \), where \( p_{i+1} = (p/K) \) and \( p_{i+1} < p_i \).

**Case 1: \( K = 2 \)**. In this case, the coefficient of \( \lambda_i \) in \( p_i \) would be at least \( t_i (1 + M_0 M_1 \ldots M_{i-2}) \) that is \( (t_i - 1 + K^{i-1}) \) by putting each \( M_i = \text{floor}(K) \). Since our final aim is to look for \( n \) beyond which \( (\text{ratio})^n(t_n - 1 + K^{i-2}) > (K^n) \), where "ratio" in the LHS represents the ratio \( (K^2) \) between the largest root modulus to the next largest root modulus. As the LHS grows at a faster rate wrt \( n \), the inequality is satisfied after a much smaller value of \( n \), when \( K = 2 \). Note that the growth of the RHS represents the rate at which \( p_i \) decreases with respect to \( i \).

**Case 2: \( 1 < K < 2 \)**. In this case, \( p_{i+2} = p_i - p_{i+1} = p_i(K - 1) \). We can further consider three sub-cases:

- **Subcase 2.1:** \( g < K < 2 \). In this subcase, \( (g-1)(K-1) \), so \( p_{i+2} = p_{i+1}(K-1) < p_{i+1} \), so \( p_{i+2}/g < p_{i+2} < p_{i+1} \), showing that \( p_{i+2} \) has not reduced much compared to \( p_{i+1} \), and it also leads to Subcase 2.2.
- **Subcase 2.2:** \( 1 < K < g \). In this subcase, \( 0 < (K-1)(g-1) \), so \( 0 < p_{i+2} < p_{i+1}(g-1) \), which means \( 0 < p_{i+2} < p_{i+1}/g \). If \( 0 < p_{i+2} < p_{i+1}/g \), it would lead to Case 1. If \( p_{i+2}/g < p_{i+1}/g \), it would lead to Subcase 2.1.
- **Subcase 2.3:** \( K = g \). In this subcase, \( p_{i+2} = p_{i+1}(g-1) = p_{i+1}/g \), since it is known that \( 1/g = (g-1) \), and \( M_i = 1 \) for every integer \( i \geq 2 \), and the value of \( p_i \) has the fastest reduction for the RHS.

In summary, Case 1, subcase 2.1 and subcase 2.2 each have slower reduction of \( p_i \) than subcase 2.3, with respect to \( i \).

**Hence Proved Theorem 3.**

**Theorem 4:** Let \( \lambda \) be a given irrational constant. For every integer \( n \geq 0 \), let \( L_\lambda(n) \) be a lower bound on the values of
both:
1. minimum(absolute(cos(2mπ/λ)), over positive integers m ≤ n).
2. minimum(absolute(sin(2mπ/λ)), over positive integers m ≤ n).

Then one choice for L_µ(n) is given by L_µ(t_µ) = 4/(λγ^4), where t_µ = t_µ+1 + t_µ, and t_µ=0 and t_µ=1. This lower bound may be approximated by the formula L_µ(n) = (4/(sqrt(5) λ^2) n) for large n.

**Proof:** It is clear that the distance of the zeros of sin(4π/λ) from the nearest integer, decreases at a rate equal to or faster than that of both sin(2π/λ) or cos(2π/λ), as t increases continuously from a small positive quantity. This is because the zeros of sin(2π/λ) and the zeros of cos(2π/λ), are each separately subsets of the zeros of sin(4π/λ). The lower bound of the distance of the t_µ zero of sin(4π/λ) from the nearest integer, is directly available from Theorem 3, by substituting λ_0=1 and λ_0=frac(1,λ), which gives a lower bound of 1/g^4. A good choice for L_µ(t_µ) would therefore be the product of 1/g^4 and 4/λ, the slope of the line joining the origin (0,0) to the point (λ/4,1). So L_µ(t_µ) = 4/(λγ^4), where t_µ = t_µ+1 + t_µ, and t_µ=0 and t_µ=1. Using the well-known result that t_µ=round(g^n/sqrt(5)), or that t_µ = approximately (g^n/sqrt(5)) for large n, we can say that L_µ(t_µ)=4/(λγ^4)=approximately (4/(sqrt(5) λ^2) t_µ) for large n. So L_µ(n)=approximately (4/(sqrt(5) λ^2) g^n) for large n.

**Hence Proved Theorem 4.**

**Theorem 5:** Let λ be a given irrational constant. Let β be a real constant. For every integer n≥0, let L_c(n) be a lower bound on the non-zero absolute value of minimum(absolute(sin(2π(m/λ + 1/β))), over positive integers m ≤ n). Then one choice for L_c(n) is given by L_c(t_c) = 4/(λγ^4), where t_c = t_c+1 + t_c, and t_c=0 and t_c=1, and where c is a non-negative integer constant dependent on β and λ. This lower bound may be approximated by the formula L_c(n) = (4/(sqrt(5) λ^2) c^4) n for large n.

**Proof:** This Theorem is a more generalized version of Theorem 4. L_c(n) for sin(2π(m/λ + 1/β)), can be ≤ L_µ(n) for sin(2π/λ). This is because sin(2π/λ)=0 when m=0, but then is situated at a distance of remainder(1/λ) from the subsequent zero. Contrast this with the fact that sin(2π(m/λ + 1/β))>0 at m=0, but is situated at a distance of remainder((1+β)/λ) or remainder(1-β)/λ from the next closest 0, either of which can be lesser than remainder(1/λ). Note that in some cases, it can be greater too, however, we are interested only in a worst-case scenario. So absolute(sin(2π(m/λ + 1/β))) can get an initial constant advantage in being closer to a zero, over absolute(sin(2π/λ))) for the same value of integer m>0. This has an effect of L_c(t_c) = 4/(λγ^4), where integer constant c0 depends on the values of β and λ. Using t_c = approximately (g^n/sqrt(5)) for large n, we can say that L_c(t_c)=4/(λγ^4)=approximately (4/(sqrt(5) λ^2) g^n) for large n. So L_c(n)=approximately (4/(sqrt(5) λ^2) g^n) for large n.

**Hence Proved Theorem 5.**

Having established the lower bounds of the absolute non-zero values of the cosine functions in s_mλ, we are now in a position to use them to develop our deterministic algorithms.

**Theorem 6:** Let G(x) be such that, for every pair of complex roots of G(x), (r_k e^(iθ_k)) and (r_k e^(iθ_k)), this statement is true: (r_k=r_k)⇒(absolute(θ_k)=absolute(θ_k)). Then there exists a deterministic algorithm to find the largest non-periodic zero of f_k.

**Proof:** This Theorem basically states that if the absolute values of arguments of every pair of complex roots of G(x) of equal moduli, are equal, then there exists a deterministic algorithm for finding an upper bound of the largest non-periodic zero of f_k.

The algorithm is described below:
1. From Theorem 1, write f_k = (r_k^n (d_k,1 cos(nθ_1+β_1,1) + d_k,1 cos(nθ_1+β_1,1) n + d_k,2 cos(nθ_2+β_2,1) n^2 + ... + d_k,l-1,1 cos(nθ_l-1,1) n^{l-1}) + (r_k^n (d_k,2 cos(nθ_2+β_2,1) + d_k,2 cos(nθ_2+β_2,1) n + d_k,2 cos(nθ_2+β_2,1) n^2 + ... + d_k,l-2,1 cos(nθ_l-2,1) n^{l-2}) + ... + (r_k^n (d_k,l cos(nθ_l+β_l,1) + d_k,l cos(nθ_l+β_l,1) n + d_k,l cos(nθ_l+β_l,1) n^2 + ... + d_k,l,l-1 cos(nθ_l+l-1,1) n^{l-1}), such that r_1 > r_2 > ... > r_l.
2. Set k=1, w_rational=0, w_irrational=0.
3. If (θ_k is a rational multiple of π)
4. { Set w_rational = value of n, beyond which minimum non-zero absolute (r_k^n (d_k,1 cos(nθ_1+β_1,1) + d_k,1 cos(nθ_1+β_1,1) n + d_k,1 cos(nθ_1+β_1,1) n^2 + ... + d_k,l,k cos(nθ_l+k,1) n^{l,k}))/SUM((r_k^n (absolute(d_k,1)+absolute(d_k,1)n) n + absolute(d_k,k) n^2 + ... + absolute(d_k,l,k) n^{l,k})) over integers k in [k+1, L]).
5. While ((θ_k is a rational multiple of π) and (k<L))
6. { k = k+1. }
7. If (θ_k is an irrational multiple of π)
Let $G(x)$ be such that, for every pair of complex roots of $G(x)$, $(r_k, e^{i\theta_k})$ and $(r_{k'}, e^{i\theta_{k'}})$, this statement is true: $(r_k=r_{k'})$ → (There exist non-zero integers $q_k$ and $q_{k'}$ such that $(q_k\theta_k=q_{k'}\theta_{k'})$). Then there exists a deterministic algorithm to find an upper bound of the non-periodic zero of $f_r$. 

**Proof:** This theorem basically states that if the absolute values of arguments of every pair of complex roots of $G(x)$ of equal moduli, are commensurable, then there exists an algorithm for finding an upper bound of the largest non-periodic zero of $f_r$.

In such a situation, we proceed in a similar manner as in the previous proof, except that now from Theorem 1, we can write $f_r = \text{SUM}(r_{v_1}^n, s_{v_1}(V))$, over all integers $j$ in $[0, L-1]$, over all subsets $V'$ in $\mathcal{V}=[k_1, k_2, \ldots, k_l]$ of integers belonging to $[1, L]$ such that $k_1 = k_2 = \ldots = k_{l-1} = k$. Here, $v$ denotes the cardinality of subset $V'$. Also, the expression of each $s_{v_1}(V')$ will be more complicated and will be $= \text{SUM}(d_{j,k}^n \cos(n\theta_k+\beta_{j,k}))$, over integers $h$ in $[1, v]$, over integers $j$ in $[1, v]$. Since there exist non-zero integers $(q_1, q_2, \ldots, q_l)$ such that $(q_1\theta_1=q_2\theta_2=\ldots=q_l\theta_l)$, we can denote LCM($q_1$, $q_2$, ..., $q_l$)=$y$. Then let $\theta_r$ be such that $(q_1\theta_1=q_2\theta_2=\ldots=q_l\theta_l)=\theta_r$. Then write $\cos(n\theta_r)=\cos(n\theta_1/y)$, $\cos(n\theta_{r_2}/y)$, ..., $\cos(n\theta_r)=\cos(n\theta_2/y)$. Similarly, write $\sin(n\theta_r)=\sin(n\theta_1/y)$, $\sin(n\theta_{r_2}/y)$, ..., $\sin(n\theta_r)=\sin(n\theta_2/y)$. Next, write each $x(y)=\text{sqrt}((1-cos(y))^2)$. Note that each $y$ is independent, for each integer $i$ in $[1, v]$. Use the cosine expansion formula of expressing $\cos(mx)$ as a univariate polynomial of $\cos(x)$, where $m$ is an integer, to express each $s_{v_1}(V')$ as a univariate expression in $\cos(n\theta_r)$ and with algebraic coefficients, which is not yet a univariate polynomial in $\cos(n\theta_r)$ due to the presence of nested $\text{sqrt}$ functions of $\cos(n\theta_r)$. We shall now directly describe our deterministic algorithm below:

1. From Theorem 1, and as described in this theorem, we can write $f_r = 
   \begin{align*}
   (r_{v_1}^n s_{v_0}(V'_1, s_{v_1}(V'_1) n + s_{v_2}(V'_1) n^2 + \ldots + s_{v_{L-1}}(V'_1) n^{L-1})) + \\
   (r_{v_2}^n s_{v_0}(V'_2, s_{v_1}(V'_2) n + s_{v_2}(V'_2) n^2 + \ldots + s_{v_{L-1}}(V'_2) n^{L-1})) + \\
   \vdots \\
   (r_{v_{L-1}}^n s_{v_0}(V'_{L-1}, s_{v_1}(V'_{L-1}) n + s_{v_2}(V'_{L-1}) n^2 + \ldots + s_{v_{L-1}}(V'_{L-1}) n^{L-1})) 
   \end{align*}

2. Set $k=1, w_{\text{irrational}}=0, w_{\text{rational}}=0$.
3. If $(\theta_{v_k} \in [k+1, L])$ is a rational multiple of $\pi$:
4. \{ Set $w_{\text{rational}} = \text{value of } n$, beyond which minimum non-zero absolute $(r_{v_k}^n s_{v_0}(V'_k, s_{v_1}(V'_k) n + s_{v_2}(V'_k) n^2 + \ldots + s_{v_{L-1}}(V'_k) n^{L-1}) \geq \text{SUM}(r_{v_k}^n (\max(s_{v_0}(V'_k)) + \max(s_{v_1}(V'_k)) n + \max(s_{v_2}(V'_k)) n^2 + \ldots + \max(s_{v_{L-1}}(V'_k)) n^{L-1}))$, over integers $k'$ in $[k+1, L]$.
5. While $(\theta_{v_{k+1}})$ is a rational multiple of $\pi$ and $(k < L)$:
6. \{ $k = k + 1$
7. If $(\theta_{v_{k+1}}$ is an irrational multiple of $\pi$:
8. \{ Set $w_{\text{irrational}} = \text{value of } n$, beyond which minimum non-zero absolute $(r_{v_{k+1}}^n s_{v_0}(V'_{k+1}) n + s_{v_1}(V'_{k+1}) n^2 + \ldots + \max(s_{v_{L-1}}(V'_{k+1})) n^{L-1})$
Theorem 9

\[ s_{n,L-1,V^-k} n^{l-1}) \geq \text{SUM} ( r_{v^-k}^n ( \max(s_{n,0,V^-k}) + \max(s_{n,1,V^-k}) \ n + \max(s_{n,2,V^-k}) n^2 + \ldots + \max(s_{n,L-1,V^-k}) n^{L-1}), \text{over integers } k' \text{ in } [k+1, L]). \]

9. OUTPUT maximum(\w_{\text{rational}}, w_{\text{rational}}).

The existence of \( w_{\text{rational}} \) and \( w_{\text{rational}} \) in steps 4 and 8 respectively, of the above algorithm, is proved by dividing both sides of the inequality by \( (r_{v^-k}^n) \), and the RHS quickly tends to 0 due to the presence of \( n^6 \) powers of positive reals less than 1. We are only left to prove that the LHS \( \geq \) RHS eventually. For this, we look for the value of \( n \), beyond which the lower bound of the LHS \( \geq \) RHS. It is clear that as \( n \) increases, the easiest way to reduce the absolute value of the LHS will tend to focus on minimizing the multiplications of the most dominant powers of \( n \), i.e. from \( n^l \) to \( n \), which is to minimize \( s_{n,L-1,V^-k} \), since it has both an upper bound and lower bound being an expression of \( \cos(\theta_v^-) \) that is itself bounded. To obtain the roots of this expression, put this expression to 0, iteratively get rid of all the nested sqrt() terms inside by repeatedly moving the the sqrt() terms to one side and then squaring. We finally get a real-algebraic univariate polynomial in \( \cos(\theta_v^-) \). Factorize this polynomial to obtain all its roots. Assume that the factorized polynomial is \( K((\cos(\theta_v^-)-r_1)(\cos(\theta_v^-)-r_2)\ldots(\cos(\theta_v^-)-r_d)/(\cos(\theta_v^-)-c_1)(\cos(\theta_v^-)-c_2)\ldots(\cos(\theta_v^-)-c_{e_v})) \), where each of \( \{r_1, r_2, \ldots, r_d\} \) is a real-algebraic constant, and where each of \( \{c_1, c_2, \ldots, c_{e_v}\} \) is a complex-algebraic constant, and \( K \) is real-algebraic. Note that \( (\cos(\theta_v^-)-c_1)(\cos(\theta_v^-)-c_2)\ldots(\cos(\theta_v^-)-c_{e_v}) \) will itself evaluate to an \( e_v \) degree univariate polynomial in \( \cos(\theta_v^-) \) with real-algebraic coefficients, without a real root. The minimum absolute value of a rootless univariate polynomial with real-algebraic coefficients, will be a real-algebraic constant, hence we may denote the minimum absolute value of \( (\cos(\theta_v^-)-c_1)(\cos(\theta_v^-)-c_2)\ldots(\cos(\theta_v^-)-c_{e_v}) \) as \( K_v \), which is a real-algebraic constant. We can now say that \( \text{LowerBound}(s_{n,L-1,V^-}) \geq (K \ K_v \ \text{PRODUCT}(\text{LowerBound}(\cos(\theta_v^-)-r_i))) \), over integers \( i \in [1,d_v]. \) We know from Theorem 5, that a good choice for \( \text{LowerBound}(\cos(\theta_v^-)-r_i)) \) is \( (4/(\sqrt{5} \ g)^{e_v} n) \), where \( \gamma_v \) is a constant positive integer, and where \( \lambda_v=2\pi/\theta_v \). Therefore, \( \text{LowerBound}(s_{n,L-1,V^-}) \geq (K \ K_v \ \text{PRODUCT}((2\theta_v/((\sqrt{5}) \ g)^{e_v} n)), \) over integers \( i \in [1,d_v]) \), which can be written as \( (\Omega_{L-1}/n^{l-1}) \), where \( \Omega_{L-1} \) is a real-algebraic constant. Similarly, find the \( \text{LowerBound}(s_{n,0,V^-})=(\Omega/\Omega_{L-1}) \) for each integer \( j \) in \([1,L-2]\). Substitute all these lower bound expressions.

Now write our expression in \( n \) as \( (\Omega_{L-1}/n^{l-1}) \pm (\Omega_{L-1}/n^{l-1}) \pm \ldots \pm (\Omega_{L-1}/n^{l-1}) \). Notice the use of the \( \pm \) because we are not yet sure whether to use + or minus to get minimum value. Simplify this expression of \( n \), eliminating common factors or common powers of \( n \) and write as a real-algebraic univariate polynomial \((c_0 - n^2c_1 - n^3c_2 - n^4c_3 - \ldots + n^{l-2}c_{L-2} - n^{l-1}c_{L-1}) \). Notice that now the \( \pm \) has been replaced with alternate + and - signs for the coefficients of \( 1, n, n^2, \ldots n^{l-1} \) to achieve minimum value. It is well known that the zeros of a univariate polynomial equation can be isolated efficiently. So we basically look for the last zero, beyond which the absolute value of this polynomial \( \geq \) RHS.

Hence Proved Theorem 7.

For our algorithms to be truly deterministic, one important criterion is that they can run on finite precision computers, and also be made to consume only finite-sized input data. So it is worth noting here that each of the cosines of arguments described so far are actually algebraic, i.e. each of \{\cos(\theta_i), \cos(\pi r_{a,i}), \cos(\pi r_{b,i}), \cos(\pi r_{c,i}), \cos(\pi r_{d,i})\} \) is actually algebraic after factoring them from the characteristic polynomial \( G(x) \). However their respective arguments need not be algebraic, i.e., each of \{\theta_i, r_{a,i}, r_{b,i}, r_{c,i}, r_{d,i}\} need not be algebraic and can be only described as being real. And the formulae described in this paper for the lower bounds involve the arguments and not the cosines of the arguments. Still, one can approximate the argument to some desired accuracy, since our aim is to finally obtain a lower bound, and not an exact value. For example, in the lower bound formula of step 10 in Theorem 6, we have \( L_{b,k,n} = (40/((\sqrt{5}) \ pi)) \), where non-algebraic numerator terms need to be rounded down and denominator terms need to be rounded up. So if \( \theta=1.56784 \) and \( \pi=3.14159 \ldots \), simply choose a desired accuracy of 3 significant figures by approximating down \( \theta=1.56 \) and approximating up \( \pi=3.15 \) to get a final lower bound figure that is algebraic. The only thing to keep in mind is the more the number of significant figures considered, one is likely to develop a lower bound that is as high as possible, so that the upper bound to the largest non-periodic zero is as low as possible.

Now that we have described deterministic algorithms for the special cases of \( G(x) \), our next task will be to attempt to generalize the same for all cases of \( G(x) \). As should be evident now, the key concept on which our paper is based to obtain the upper bound of the largest zero in set \( Q \) of \( \Gamma_a \) is the determination of an effective lower bound for the absolute non-zero value of the summation of weighted cosines of arguments of roots, and then using the fact that the ratio (any ratio lesser than 1) of root moduli raised to the power of \( n \) eventually decreases at a much faster rate. In an attempt to get an effective lower bound of this weighted sum of \( m \) cosines \( (m>2) \), where the arguments of the functions may or may not be commensurable, the next Theorem 8 finds the lower bound of the distance between every \( x^k \)th zero.
Theorem 8: Let m be a given positive integer constant. Let $d_k$ and $\theta_k$ each be a given real constant, for each integer $k$ in $[1,m]$. Let $s_t = \text{SUM}(d_k \cos(\theta_k))$, over all integers $k$ in $[1,m]$, for real variable $t \geq 0$. Let $\theta_{\text{max}} = \text{maximum}(\theta_k)$, over all integers $k$ in $[1,m]$. There exists a lower bound $L_{\text{min}} = (\pi / (\theta_{\text{max}} \sqrt{5} t))$ between every $(2^{n+1})^\text{th}$ zero of $s_t$.

Proof: For each $(d_k \cos(\theta_k))$, divide the domain of $t \geq 0$ into alternating convex and concave spaces with period of $t_0$ equal to $2\pi$, for integers $n \geq 0$:

1. concave spaces in $t_0 \in [2n\pi - \pi/2, 2n\pi + \pi/2]$.
2. convex spaces in $t_0 \in [2n\pi + \pi/2, 2n\pi + 3\pi/2]$.

Now divide the domain of $t \geq 0$ into blocks, not necessarily periodic, such that each block is bounded by a zero of $\cos(\theta_k)$ for some integer $k$ in $[1,m]$. So the lower bound of the width of a block is the lower bound of the distance of a zero of $\cos(\theta_k)$ from a zero of $\cos(\theta_k')$, where integers $k$ and $k'$ are not equal. Denote $\theta_{\max} = \text{maximum}(\theta_k)$, over all integers $k$ in $[1,m]$, since $\cos(\theta_{\max})$ would have the least wavelength. Applying the result of Theorem 3, we get the lower bound of a block's width $L_{\text{min}}$, to approximately be $(2\pi / (2\theta_{\max} \sqrt{5} t)) = (\pi / (\theta_{\max} \sqrt{5} t))$.

It is obvious that each block will contain either a convex space or a concave space of $(d_k \cos(\theta_k))$, but not both, for each integer $k$ in $[1,m]$. This means that there are $2^n$ types of space combinations in each block.

An inductive argument will prove that there can be no more than $(2^{n+1})$ zeros inside a block. Consider the $\text{SUM}(d_k \cos(\theta_k))$, over all integers $k$ in $[1,1]$ within a block. It has a single piecewise function that is either concave or convex with not more than 2 zeros. Now consider the $\text{SUM}(d_k \cos(\theta_k))$, over all integers $k$ in $[1,2]$ within a block. If both $\cos(\theta_k)$ and $\cos(\theta_k')$ are convex, then the resulting sum is also convex, and the maximum number of zeros remains 2. Similarly, if both $\cos(\theta_k)$ and $\cos(\theta_k')$ are concave, then the resulting sum is also concave, and the maximum number of zeros remains 2. But when one is convex and the other is concave, then the maximum number of piecewise concave functions in the sum is doubled, and so is the maximum number of concave functions and so is the maximum number of zeros in the sum. Now consider the $\text{SUM}(d_k \cos(\theta_k))$, over all integers $k$ in $[1,4]$ where $q < m$ within a block, and assume that it has a maximum of $2^{n-1}$ piecewise functions, that are alternately convex and concave, with a maximum of $2^{n-1}$ zeros. When we now add a convex or concave part of $(d_{k+1} \cos(\theta_{k+1}))$ to the sum, each of {maximum number of piecewise concave functions, maximum number of piecewise concave functions, maximum number of zeros} is doubled. Therefore, the maximum number of zeros in $s_t$ within a block, becomes $2^{n+1}$. Since we have not established a lower bound on the distance between each of these zeros within a block, we can only safely say that $L_{\text{min}} = (\pi / (\theta_{\max} \sqrt{5} t))$ is the lower bound of the distance between every $(2^{n+1})^\text{th}$ zero of $s_t$.

Hence Proved Theorem 8.

We now discuss some areas for future work.

3. Future Work

Though Theorem 8 establishes the lower bound of the horizontal distance between every $x^\text{th}$ zero, where $x$ is an integer constant, we are yet to find the lower bound of the non-zero value of $(d_1 \cos(\theta_1) + d_2 \cos(\theta_2) + ... + d_m \cos(\theta_m))$, since:

1. We have not yet identified a lower bound of the distance between successive zeros of $s_t$.
2. We have not yet identified a lower bound to the absolute value of the slope of $s_t$ near a zero, unlike in the case of $m=1$ where the slope at a point close to a zero of $(d_1 \cos(\theta_1))$ tends to $(d_1 \theta_1)$.

Let us conveniently assume that $s_t$ is equivalent to a series of cosine waves, placed serially one after the other, each of progressively lower wavelength equal to the lower bound of Theorem 8. Though this assumption is obviously false, we can use this to make a conjecture. Applying the Theorem 3 to the current wavelength at $n$ yields $(\pi / (2\theta_{\max} g^6 \sqrt{5} n))$, which could be used as the lower bound of the absolute horizontal distance of $s_t$ from the nearest 0. Another idea would be to evaluate the $\text{SUM}(((\pi / (2\theta_{\max} g^6 \sqrt{5} n)))$, over integers $n$ in $[1,T])$, which is approximately $(\pi/(2\theta_{\max} \sqrt{5} \ln(T)))$, where $\ln(T)$ is the natural logarithm of $T$. The fraction of this value could be the absolute lower bound of the horizontal distance of $(d_1 \cos(\theta_1) + d_2 \cos(\theta_2) + ... + d_m \cos(\theta_m))$ from the nearest zero. That horizontal distance multiplied by the lower bound of the absolute non-zero slope would be a lower bound on the absolute non-zero value of $(d_1 \cos(\theta_1) + d_2 \cos(\theta_2) + ... + d_m \cos(\theta_m))$, using which an algorithm can be then developed on the similar lines of Theorems 6 and 7, to find the largest non-periodic zero of the generic $f_n$.

The study of zeros of a weighted sum of three or more trigonometric functions, is already a well known topic [8] under the study of zeros of the Riemann Zeta Function, so ideas from there could be applied here.

Finally, it has been highlighted by multiple authors [0] that all known proofs of the SML Theorem use a non-constructive
proof that we understand to be something like a proof by contradiction. One line of future work will be to therefore, re-examine the conditions on which those theorems or axioms on which the SML Theorem is based, especially when the arguments of roots of equal moduli, are incommensurable.

4. Conclusion

We presented a deterministic algorithm to determine an upper bound for the largest non-periodic zero for an integer linear homogeneous recurrence $f_n$, when the absolute values of arguments of every pair of complex roots of its characteristic polynomial $G(x)$ of equal moduli, are either equal or commensurable. Our algorithm is based on the concept that a ratio of reals raised to the power of $n$, as long as the ratio is lesser than 1, decreases at a faster rate than the rate at which the eventual lower bound decreases for the absolute non-zero value of $\cos(\pi n)$ where $a$ is an irrational constant. We then discussed the difficulty in extending this concept to the generic case of $G(x)$ since one needs to calculate the lower bound of the absolute value of a weighted sum of $m$ cosine terms, which is more challenging. We were able to only develop a lower bound between a zero and the $x^{th}$ successive zero for the weighted sum of $m$ cosines. We finally presented some directions for future work.

References


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