Gravitational Time Dilation, Relativistic Gravity Theory, Schwarzschild's Physically Sound Original Metric and the Consequences for Cosmology

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Abstract It is natural to assume that the expanding universe was arbitrarily compact in the sufficiently remote past, in which state gravitational time dilation strongly affected its behavior. We first regard gravitational time dilation as the speed time dilation of a clock falling gravitationally from rest. Energy conservation implies that this depends solely on the the Newtonian gravitational potential difference of the clock trajectory's ends. To extend this to the relativistic domain we work out relativistic gravity theory. The metric result it yields for gravitational time dilation is consistent with our Newtonian gravitational potential result in the Newtonian limit. However the Robertson-Walker metric form for the universe implies complete absence of gravitational time dilation. Since we assume the universe was once arbitrarily compact, we turn instead to the metric for a static gravitational point source, but find that its textbook form puts a sufficiently compact universe inside an event horizon. This is due to transformation of the three radial functions which describe a static, spherically-symmetric metric into only two before inserting that now damaged metric form into the Einstein equation. Schwarzschild's original metric solution, which isn't in textbooks, involved no such transformation and therefore is physically sound; we obtain from it a picture of a universe which had an outburst of star and galaxy formation in the wake of its inflation.

1. Gravitational time dilation equals the speed time dilation of a gravitationally falling clock

The gravitational potential energy of a test particle of mass m located at height h above the earth's surface can be taken as mgh, where $g = 9.8\,\mathrm{m/s^2}$, the acceleration of gravity at the earth's surface, *provided that the height h is very much less than the earth's radius r_E* $\approx (40,000/(2\pi))\,\mathrm{km} = 6.37 \times 10^6\,\mathrm{m}$.

Thus a test particle *at rest at height h_1 has energy mgh_1*, and if it falls from height h_1 *to a lesser height* h_2 ($h_2 < h_1$), it acquires kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2$, which, by conservation of total energy, satisfies,

$$mgh_1 = mgh_2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 \text{ so, } \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 = mgh_1 - mgh_2, \tag{1.1a}$$

and therefore,

$$|\dot{\mathbf{r}}|^2 = 2(gh_1 - gh_2). \tag{1.1b}$$

If a clock is embedded in this gravitationally falling test particle, then that clock is observed to tick at a slightly lesser rate at $h_2 < h_1$, where it has squared speed $|\dot{\mathbf{r}}|^2$, than the rate at which it ticked at h_1, where it was *at rest,* because of the well-known squared-speed Lorentz time dilation tick-rate reduction factor,

$$\sqrt{1 - \left(|\dot{\mathbf{r}}|^2/c^2\right)} = \sqrt{1 - \left((2(gh_1 - gh_2))/c^2\right)} \approx \Big[1 - \left((gh_1 - gh_2)/c^2\right)\Big]. \tag{1.2a}$$

Because the test particle's gravitational mass m, which occurs in its gravitational potential energy mgh, is equal to its inertial mass m, which occurs in its kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2$, the test particle's mass m *doesn't appear at all* in the Eq. (1.2a) factor for the change in the tick rate of the test particle's embedded clock when it falls gravitationally from rest at height h_1 to the lesser height h_2. Thus it is very convenient to define a gravitational potential function $\phi(h) \stackrel{\text{def}}{=} gh$, which is independent of the test particle's mass m and gives the test particle's gravitational potential energy mgh when it is multiplied by the test particle's mass m (this is analogous to the electrical potential function $\phi(\mathbf{r})$, which is independent of the test particle's charge q and gives the test particle's electrical potential energy $q\phi(\mathbf{r})$ when it is multiplied by the test particle's charge q). In terms of the gravitational potential function values $\phi(h_2)$ and $\phi(h_1)$, the dimensionless Eq. (1.2a) gravitational tick-rate change factor (GTRCHF) for the clock which falls gravitationally from rest at height h_1 to the lesser height h_2 is,

$$\mathrm{GTRCHF}(h_2; h_1) = \sqrt{1 - \left((2(\phi(h_1) - \phi(h_2)))/c^2\right)}, \tag{1.2b}$$

which, aside from the universal constant c, depends *solely* on the difference between the gravitational potential function values at heights h_1 and h_2. This *very strongly suggests* that a clock's gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ *doesn't depend on the details of the process whereby the clock changes its height from h_1, where it is at rest, to the lesser height h_2.*

Consider, for example, the case where the mass m test particle with embedded clock is lowered from height h_1 to the lesser height h_2 *arbitrarily slowly by a correspondingly arbitrarily-slow lowering device.* In

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that case *work* $mg(h_1 - h_2) = mgh_1 - mgh_2$ *is done on the arbitrarily-slow lowering device, so energy* $(mgh_1 - mgh_2)$ *is removed from the test particle by that lowering device,* resulting in the test particle being virtually *motionless* at the lesser height h_2 (which of course *is the reason* for *using* the arbitrarily-slow lowering device) *instead* of having the kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2 = mgh_1 - mgh_2$ at the lesser height h_2 that *it would have* if the arbitrarily-slow lowering device *wasn't used* (see Eq. (1.1a)). Eq. (1.2b), however, says that the gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ for the mass-m test particle's embedded clock *depends solely on* $(\phi(h_1) - \phi(h_2)) = ((mgh_1 - mgh_2)/m)$ regardless of whether the mass-m test particle *sheds the gravitational potential energy* $(mgh_1 - mgh_2)$ by converting it to kinetic energy or by doing that amount of work on the arbitrarily-slow lowering device. In other words, the Eq. (1.2b) gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ for the mass-m test particle's embedded clock *is independent* of whether the test particle *falls gravitationally* from height h_1 to the lesser height h_2 or *is lowered arbitrarily slowly* from height h_1 to the lesser height h_2.

In a celebrated experiment two hyper-accurate atomic clocks were attached to a wall, one clock 33 cm above the other. According to Eq. (1.2b), the clock below runs slower than the clock above by the factor $\sqrt{1 - \big((2(\phi(h_1) - \phi(h_2)))/c^2\big)} \approx \big[1 - \big((g(h_1 - h_2))/c^2\big)\big]$, which is equal to $\big[1 - \big((9.8\ \mathrm{m/s^2})(0.33\,\mathrm{m})/(3 \times 10^8\ \mathrm{m/s})^2\big)\big] = \big[1 - 3.59 \times 10^{-17}\big]$. After five days (120 hours, or 4.32×10^5 seconds), the clock below is therefore supposed to record 15.5×10^{-12} seconds (15.5 picoseconds) less than the clock above records. Remarkably, certain atomic clocks are actually accurate for such fantastically short time intervals. As a precaution against systematic errors, the positions of the two clocks were swapped and the experiment was repeated.

We next consider the gravitational time dilation of a clock on the earth's surface at the equator relative to a clock in a satellite directly overhead *whose circular orbit lies in the equator's plane, has a period of 24-hours and travels in the direction of the earth's rotation.* If such a satellite is directly overhead the clock on the earth's surface at the equator, it of course *remains fixed directly overhead that clock,* just as the upper clock on the wall in the experiment described above *remains fixed directly above the lower clock.* Most satellites *contrariwise move at high speed relative to any clock on the earth's surface,* so for most satellites it is *necessary* to take into account squared-speed Lorentz time dilation *in addition* to gravitational time dilation. Even the famed GPS satellites have 12-hour orbit periods instead of 24-hour orbit periods, so squared-speed Lorentz time dilation *must be taken into account* for GPS satellites *in addition* to gravitational time dilation.

Since the height h_{GS} above the earth's surface of a geosynchronous 24-hour circular orbit in the equator's plane is *several times* the earth's radius $r_E \approx 6.37 \times 10^6$ m, we must abandon the approximation implicit in Eq. (1.1a) that the acceleration of gravity *is fixed at* $g = 9.8\ \mathrm{m/s^2}$, and switch to full Newtonian gravity.

The earth's Newtonian gravitational force $\mathbf{F}$ on a mass-m test particle is $-GmM\mathbf{r}/|\mathbf{r}|^3$, where G is the universal gravitational constant, M is the earth's mass and $\mathbf{r}$ is the vector from the earth's center to the test particle, provided that $|\mathbf{r}| \geq r_E$. Furthermore, Newton's Second Law states that $\mathbf{F} = m\ddot{\mathbf{r}}$, so,

$$\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3, \tag{1.3a}$$

regardless of the specific nonzero value m of the test particle's mass. The result of taking the norms of the vectors on both sides of Eq. (1.3a) is,

$$|\ddot{\mathbf{r}}| = GM/|\mathbf{r}|^2. \tag{1.3b}$$

In the special case that $\mathbf{r}$ lies on the earth's surface, so that $|\mathbf{r}| = r_E$, Eq. (1.3b) becomes,

$$|\ddot{\mathbf{r}}|_{|\mathbf{r}|=r_E} = GM/(r_E)^2. \tag{1.3c}$$

The entity $|\ddot{\mathbf{r}}|_{|\mathbf{r}|=r_E}$ is the norm of the acceleration vector of a gravitational test particle at the earth's surface, which of course is equal to $g = 9.8\,\mathrm{m/s^2}$. Eq. (1.3c) thus implies that,

$$GM = g(r_E)^2, \tag{1.3d}$$

so we can rewrite Eq. (1.3b) as,

$$|\ddot{\mathbf{r}}| = g(r_E/|\mathbf{r}|)^2, \tag{1.3e}$$

and we can likewise rewrite Eq. (1.3a) as,

$$\ddot{\mathbf{r}} = -g(r_E)^2\big(\mathbf{r}/|\mathbf{r}|^3\big). \tag{1.3f}$$

We now reexpress Eq. (1.3f) as, $\ddot{\mathbf{r}} + g(r_E)^2\big(\mathbf{r}/|\mathbf{r}|^3\big) = \mathbf{0}$, and we then take the dot product of both sides of that equation with $\dot{\mathbf{r}}$ to obtain $(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}) + g(r_E)^2\big((\dot{\mathbf{r}}\cdot\mathbf{r})/(\mathbf{r}\cdot\mathbf{r})^{3/2}\big) = 0$. This last equation can be rewritten, $\frac{d}{dt}\big(\frac{1}{2}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) - g(r_E)^2\big(1/(\mathbf{r}\cdot\mathbf{r})^{1/2}\big)\big) = 0$, which expresses a conservation law that is more neatly written as,

$$\tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/|\mathbf{r}|)\big) = 0. \tag{1.4a}$$

We digress to derive the very useful approximation to Eq. (1.4a) for those cases where the mass-m test particle's distance h above the surface of the earth, $h = \big(|\mathbf{r}| - r_E\big)$, satisfies $0 \le h \ll r_E$,

$$0 = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/|\mathbf{r}|)\big) = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/(r_E+h))\big) = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(1/(1+(h/r_E)))\big) =$$
$$\tfrac{d}{dt}\Big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E + gh + g\,r_E\,O\big((h/r_E)^2\big)\Big) = \tfrac{d}{dt}\Big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 + gh\big(1 + O\big((h/r_E)\big)\big)\Big) \approx \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 + gh\big), \tag{1.4b}$$

where we have used the fact that $\frac{d}{dt}(-g\,r_E) = 0$, since g and r_E are constants. The Eq. (1.4b) result, which holds when $0 \le h \ll r_E$, as well implies that $\frac{d}{dt}\big(\frac{1}{2}m|\dot{\mathbf{r}}|^2 + mgh\big) \approx 0$, which of course *underlies* Eq. (1.1a).

Returning now to Eq. (1.4a), we *deduce* from it that *the squared speed* $|\dot{\mathbf{r}}|^2$ of a test particle with an embedded clock *which has fallen gravitationally to the earth's surface* (namely to the radius $|\mathbf{r}| = r_E$) *from initial rest* (namely the initial speed $|\dot{\mathbf{r}}| = 0$) *and the initial radius* $|\mathbf{r}| = r_{\mathrm{GS}}$*, where* r_{GS} *is the orbit radius of the geosynchronous satellite, satisfies,*

$$-g\,r_E(r_E/r_{\mathrm{GS}}) = \tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/r_E) \text{ so, } |\dot{\mathbf{r}}|^2 = 2g\,r_E(1 - (r_E/r_{\mathrm{GS}})). \tag{1.5}$$

Due to the Eq. (1.5) squared speed $|\dot{\mathbf{r}}|^2$ which the embedded clock gains as a result of its embedding test particle's gravitational fall from rest at the radius r_{GS} to the earth's radius r_E, the embedded clock's tick rate is slowed at the end of that fall by the Lorentz time dilation tick-rate reduction factor,

$$\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = \sqrt{1 - \big(\big(2g\,r_E(1 - (r_E/r_{\mathrm{GS}}))\big)/c^2\big)} \approx \Big[1 - \big(\big(g\,r_E(1 - (r_E/r_{\mathrm{GS}}))\big)/c^2\big)\Big]. \tag{1.6}$$

To evaluate the Eq. (1.6) gravitational reduction factor for the tick rate of a clock on the earth's surface at the equator relative to the tick rate of an identical clock directly overhead in a geosynchronous satellite, *we need the value of* r_{GS}, the radius of the geosynchronous orbit. If we take the equator's plane to be the x-y plane, so the earth's axis of rotation is the z-axis and the earth's center is located at $x = y = z = 0$, a circular test-particle orbit in the equator's plane, whose center is the earth's center, and which has fixed radius $|\mathbf{r}|$ and period T has the form,

$$\mathbf{r} = |\mathbf{r}|\Big(\cos\big(2\pi(t/T) + \delta\big),\ \pm\sin\big(2\pi(t/T) + \delta\big),\ 0\Big), \tag{1.7}$$

where the $\pm$ sign determines whether the test particle travels in the direction of the earth's rotation or in the opposite direction. *Two crucial properties of the* Eq. (1.7) *circular orbit* $\mathbf{r}$ *are that* $(\dot{\mathbf{r}}\cdot\mathbf{r}) = 0$, *which implies that* $|\mathbf{r}|^2$ *is independent of time, and that* $\ddot{\mathbf{r}} = -(2\pi/T)^2\,\mathbf{r}$, which implies that $\mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{0}$, an attribute shared by Eq. (1.3f), which is the property of *orbital angular momentum conservation* because $d\mathbf{L}/dt = d(\mathbf{r}\times(m\dot{\mathbf{r}}))/dt = m(\mathbf{r}\times\ddot{\mathbf{r}})$. Insertion of $\ddot{\mathbf{r}} = -(2\pi/T)^2\,\mathbf{r}$ into Eq. (1.3f) and application of the fact that $|\mathbf{r}|$ is independent of time *completely determines* $|\mathbf{r}|$ *in terms of* T *as,*

$$|\mathbf{r}| = \Big(g(r_E)^2\big(T/(2\pi)\big)^2\Big)^{\frac{1}{3}} \text{ or equivalently, } |\mathbf{r}| = \Big((g/r_E)\big(T/(2\pi)\big)^2\Big)^{\frac{1}{3}}\,r_E. \tag{1.8}$$

Inserting the geosynchronous period $T = 24$ hours $= 8.64\times10^4\,\mathrm{s}$ into Eq. (1.8) together with $(g/r_E) = 1.54\times10^{-6}\,\mathrm{s}^{-2}$ yields that the radius r_{GS} of the geosynchronous orbit equals $6.63\,r_E$. Therefore the height above the earth's surface of the geosynchronous orbit is $h_{\mathrm{GS}} = \big(r_{\mathrm{GS}} - r_E\big) = 5.63\,r_E \approx 35{,}800\,\mathrm{km}$.

Inserting $(r_E/r_{\mathrm{GS}}) = (1/6.63)$ together with $\big(g\,r_E/c^2\big) = 6.93\times10^{-10}$ into Eq. (1.6) yields that a clock on the earth's surface at the equator ticks at a rate which is slower by a factor of $\big[1 - 5.89\times10^{-10}\big]$ than an identical clock in a geosynchronous satellite directly overhead. This minute deviation from equality of the two clocks' tick rates is vastly greater however than the deviation described by the tick-rate reduction factor $\big[1 - 3.59\times10^{-17}\big]$ we previously noted for a clock on a wall 33 cm below an identical clock on that wall. These examples show that although terrestrial gravitational time dilation is exceedingly small, atomic-clock technology has nevertheless confirmed its existence and physical systematics beyond any reasonable doubt.

If we reexpress Eq. (1.4a) as $\frac{d}{dt}\big(\frac{1}{2}|\dot{\mathbf{r}}|^2 + \phi(|\mathbf{r}|)\big) = 0$, where $\phi(|\mathbf{r}|) = -g\,r_E(r_E/|\mathbf{r}|)$ is the Newtonian gravitational potential function for the earth, which is valid when $|\mathbf{r}| \geq r_E$, then *the squared speed* $|\dot{\mathbf{r}}|^2$ of a clock which falls gravitationally from rest at $|\mathbf{r}|_>$ to $|\mathbf{r}|_<$, where $|\mathbf{r}|_> > |\mathbf{r}|_< \geq r_E$, *follows from the conservation relation* $\phi(|\mathbf{r}|_>) = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \phi(|\mathbf{r}|_<)$, *which yields that the clock's tick-rate reduction factor is,*

$$\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = \sqrt{1 - \big(2\big(\phi(|\mathbf{r}|_>) - \phi(|\mathbf{r}|_<)\big)/c^2\big)} \approx \Big[1 - \big(\big(\phi(|\mathbf{r}|_>) - \phi(|\mathbf{r}|_<)\big)/c^2\big)\Big]. \tag{1.9}$$

Eq. (1.9) *captures the essence* of the foregoing Eqs. (1.6) and (1.2a).

We next *extend our understanding of gravitational time dilation to relativistic gravity theory,* which we now partially develop in the standard textbook manner from the principles of Lorentz covariance and equivalence, and the validity of Newtonian gravity as its limiting nonrelativistic, static weak-field case.

2. Relativistic gravity from Lorentz covariance, equivalence and its Newtonian precursor

Lorentz covariance asserts that for every given gravitational field a test particle of positive rest mass m is subject to *a Lorentz-covariant version of Newton's Second Law,*

$$\frac{d^2x^\lambda}{d\tau^2} = F^\lambda/m, \tag{2.1}$$

where τ is a Lorentz invariant and *both* the trajectory $x^\lambda(\tau)$ and the four-force F^λ *are Lorentz four-vectors.*

The equivalence principle asserts that for every given gravitational field and test-particle space-time trajectory $x^\mu(\tau)$ *in that gravitational field, there exists a one-to-one transformation* $r^\alpha(x^\mu)$ *of space-time onto itself such that the transformed trajectory* $R^\alpha(\tau) \stackrel{\text{def}}{=} r^\alpha(x^\mu(\tau))$ *manifests zero proper acceleration, i.e.,*

$$\frac{d^2R^\alpha}{d\tau^2} = 0, \tag{2.2a}$$

where,

$$d\tau \stackrel{\text{def}}{=} (dR^0/c)\sqrt{1 - |d\mathbf{R}/dR^0|^2} = \big(\sqrt{(dR^0)^2 - |d\mathbf{R}|^2}\,\big)/c = \big(\sqrt{\eta_{\alpha\beta}dR^\alpha dR^\beta}\,\big)/c, \tag{2.2b}$$

where, of course,

$$\eta_{00} = +1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -1 \quad \text{and} \quad \eta_{\alpha\beta} = 0 \;\text{ if }\; \alpha \neq \beta. \tag{2.2c}$$

Eq. (2.2b) implies that,

$$\eta_{\alpha\beta}\frac{dR^\alpha}{d\tau}\frac{dR^\beta}{d\tau} = c^2, \tag{2.2d}$$

and it also implies that,

$$(c\,d\tau)^2 = \eta_{\alpha\beta}dR^\alpha dR^\beta. \tag{2.2e}$$

The insertion of $R^\alpha(\tau) \stackrel{\text{def}}{=} r^\alpha(x^\mu(\tau))$ into Eq. (2.2a) enables us to extract an equation of motion for $x^\lambda(\tau)$, the trajectory of the test particle in the gravitational field, that is of the form of Eq. (2.1), where the four-acceleration F^λ/m is independent of the test particle's mass m, but depends on its proper velocity $dx^\mu/d\tau$ and on partial derivatives of the transformation $r^\alpha(x^\mu)$ and its inverse $x^\lambda(r^\alpha)$,

$$0 = \frac{d^2R^\alpha}{d\tau^2} = \frac{d}{d\tau}\Big(\frac{d}{d\tau}\big(r^\alpha(x^\mu(\tau))\big)\Big) = \frac{d}{d\tau}\Big(\frac{\partial r^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\Big) = \frac{\partial r^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}. \tag{2.3a}$$

Because the transformation $r^\alpha(x^\mu)$ maps space-time one-to-one onto itself, it has the unique inverse $x^\lambda(r^\alpha)$ (with the same property), and the partial derivatives of $x^\lambda(r^\alpha)$ and $r^\alpha(x^\mu)$ satisfy the identity,

$$\frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial r^\alpha}{\partial x^\mu} = \frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu, \tag{2.3b}$$

where the repeated index α is summed over. We now multiply the expression in Eq. (2.3a) which follows its rightmost equal sign by $\frac{\partial x^\lambda}{\partial r^\alpha}$, sum over the index α and then apply the Eq. (2.3b) identity to obtain,

$$0 = \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial r^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2x^\lambda}{d\tau^2} + \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{2.3c}$$

Therefore,

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{2.3d}$$

where,

$$\Gamma^{\lambda}_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial x^{\lambda}}{\partial r^{\alpha}} \frac{\partial^2 r^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}, \tag{2.3e}$$

is called *the affine connection.* Eqs. (2.3d) and (2.3e) show that the four-acceleration F^{λ}/m of Eq. (2.1) is equal to $-\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$, which is independent of the test particle's mass m, but depends on its proper velocity $dx^{\mu}/d\tau$ and on partial derivatives of the transformation $r^{\alpha}(x^{\mu})$ *and its inverse* $x^{\lambda}(r^{\alpha})$. *A critically important point here* is that the Eq. (2.3e) affine connection $\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial r^{\alpha}} \frac{\partial^2 r^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$ *is ill-defined if the transformation* $r^{\alpha}(x^{\mu})$ *doesn't have the well-defined inverse* $x^{\lambda}(r^{\alpha})$.

Just as we inserted $R^{\alpha}(\tau) \stackrel{\text{def}}{=} r^{\alpha}(x^{\mu}(\tau))$ into Eq. (2.2a), which is the equation of the non-accelerating motion of R^{α}, to obtain Eqs. (2.3d) and (2.3e), the equations of the gravitationally-accelerating motion of x^{λ}, we shall now insert $R^{\alpha}(\tau) \stackrel{\text{def}}{=} r^{\alpha}(x^{\mu}(\tau))$ into Eq. (2.2d) to obtain the generalization of $\eta_{\alpha\beta}$ which corresponds to the gravitationally-accelerating motion of x^{μ},

$$c^2 = \eta_{\alpha\beta}\frac{dR^{\alpha}}{d\tau}\frac{dR^{\beta}}{d\tau} = \eta_{\alpha\beta}\frac{dr^{\alpha}(x^{\mu}(\tau))}{d\tau}\frac{dr^{\beta}(x^{\nu}(\tau))}{d\tau} = \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{dx^{\mu}}{d\tau}\frac{\partial r^{\beta}}{\partial x^{\nu}}\frac{dx^{\nu}}{d\tau} = \left(\eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\nu}}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}. \tag{2.4a}$$

Therefore,

$$g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = c^2, \tag{2.4b}$$

where,

$$g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\nu}}, \tag{2.4c}$$

is called *the metric tensor field.* Eq. (2.4b) is the generalization of Eq. (2.2d) to the gravitationally-accelerating motion of x^{μ}, and it obviously implies that,

$$(c\,d\tau)^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \quad \text{and} \quad d\tau = \left(\sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}}\,\right)/c, \tag{2.4d}$$

which are the generalizations of Eqs. (2.2e) and (2.2b) to the gravitationally-accelerating motion of x^{μ}.

We have seen that *it is critically important* for the inverse $x^{\lambda}(r^{\alpha})$ of the transformation $r^{\alpha}(x^{\mu})$ *to be well-defined,* otherwise *the affine connection is ill-defined.* Therefore we can be sure that the partial-derivative identities such as $\frac{\partial x^{\lambda}}{\partial r^{\alpha}}\frac{\partial r^{\alpha}}{\partial x^{\mu}} = \delta^{\lambda}_{\mu}$ of Eq. (2.3b) hold, so the 4×4 partial-derivative matrix $\frac{\partial r^{\alpha}}{\partial x^{\mu}}$ *definitely has a matrix inverse.* Therefore the Eq. (2.4c) *definition* of the metric tensor, $g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\nu}}$, *guarantees that it has a* 4×4 *matrix inverse* (the 4×4 matrix $\eta_{\alpha\beta}$ clearly is its own inverse).

Besides having a matrix inverse, the metric tensor is a symmetric matrix, so it has four eigenvalues, none of which can equal zero (or it wouldn't have an inverse). The four eigenvalues of the diagonal matrix $\eta_{\alpha\beta}$ are +1, -1, -1, -1, and the matrix form of the metric tensor $g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\nu}}$ is $g = D\eta D^T$, where $D_{\mu\alpha} \stackrel{\text{def}}{=} \frac{\partial r^{\alpha}}{\partial x^{\mu}}$; this matrix D *definitely has an inverse.* Therefore the symmetric metric tensor g *is a congruence transformation* of the diagonal matrix η, so *by the Sylvester's law of inertia theorem,* the *signs* of the eigenvalues of the metric tensor g are the *same* as the *signs* of the eigenvalues of η, namely +, -, -, -. *No exception to the rule that the signature of the metric tensor is +, -, -, - can be tolerated.*

The affine connection can be expressed as a linear combination of three partial derivatives of the metric tensor contracted into the metric tensor's *inverse.* Since $g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\nu}}$,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \eta_{\alpha\beta}\frac{\partial^2 r^{\alpha}}{\partial x^{\mu}\partial x^{\lambda}}\frac{\partial r^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial^2 r^{\beta}}{\partial x^{\nu}\partial x^{\lambda}} = \eta_{\alpha\beta}\frac{\partial x^{\kappa}}{\partial r^{\gamma}}\frac{\partial^2 r^{\gamma}}{\partial x^{\mu}\partial x^{\lambda}}\frac{\partial r^{\alpha}}{\partial x^{\kappa}}\frac{\partial r^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta}\frac{\partial r^{\alpha}}{\partial x^{\mu}}\frac{\partial r^{\beta}}{\partial x^{\kappa}}\frac{\partial x^{\kappa}}{\partial r^{\gamma}}\frac{\partial^2 r^{\gamma}}{\partial x^{\nu}\partial x^{\lambda}} =$$
$$\Gamma^{\kappa}_{\mu\lambda}g_{\kappa\nu} + g_{\mu\kappa}\Gamma^{\kappa}_{\nu\lambda} = g_{\mu\kappa}\Gamma^{\kappa}_{\nu\lambda} + g_{\nu\kappa}\Gamma^{\kappa}_{\mu\lambda}. \tag{2.5a}$$

Therefore,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = g_{\mu\kappa}\Gamma^{\kappa}_{\nu\lambda} + g_{\nu\kappa}\Gamma^{\kappa}_{\mu\lambda} + g_{\lambda\kappa}\Gamma^{\kappa}_{\nu\mu} + g_{\nu\kappa}\Gamma^{\kappa}_{\lambda\mu} - g_{\mu\kappa}\Gamma^{\kappa}_{\lambda\nu} - g_{\lambda\kappa}\Gamma^{\kappa}_{\mu\nu} = 2g_{\nu\kappa}\Gamma^{\kappa}_{\lambda\mu}, \tag{2.5b}$$

where we have used the facts that $g_{\mu\nu} = g_{\nu\mu}$ and that $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$.

We know that the metric tensor *always has the signature* +, -, -, -, *and always has an inverse.* To extract the affine connection $\Gamma^{\sigma}_{\lambda\mu}$ from the Eq. (2.5b) result, *we need the inverse of the metric tensor* $g_{\nu\kappa} = \frac{\partial r^{\alpha}}{\partial x^{\nu}}\eta_{\alpha\beta}\frac{\partial r^{\beta}}{\partial x^{\kappa}}$, which in fact is $g^{\sigma\nu} = \frac{\partial x^{\sigma}}{\partial r^{\gamma}}\eta^{\gamma\lambda}\frac{\partial x^{\nu}}{\partial r^{\lambda}}$, where *the inverse* $\eta^{\gamma\lambda}$ *of* $\eta_{\gamma\lambda}$ *is* $\eta_{\gamma\lambda}$ *itself,* since $\eta_{\gamma\lambda}$ is a diagonal matrix whose diagonal matrix elements are all either $+1$ or -1. To show *explicitly* that the $g^{\sigma\nu}$ written above *is indeed the inverse of the metric tensor* $g_{\nu\kappa}$, which is also written above, we now calculate,

$$g^{\sigma\nu} g_{\nu\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\lambda}\frac{\partial x^\nu}{\partial r^\lambda}\,\frac{\partial r^\alpha}{\partial x^\nu}\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\lambda}\delta^\alpha_\lambda\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\alpha}\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\delta^\gamma_\beta\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \delta^\sigma_\kappa. \quad (2.5c)$$

Thus multiplying both sides of Eq. (2.5b) by $\frac{1}{2}g^{\sigma\nu}$ and summing over ν yields the affine connection $\Gamma^\sigma_{\lambda\mu}$ as,

$$\Gamma^\sigma_{\lambda\mu} = \tfrac{1}{2}g^{\sigma\nu}\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right]. \quad (2.5d)$$

We next insert the Eq. (2.5d) result for the affine connection in terms of the metric tensor into the Eq. (2.3d) dynamical equation for a test particle in a gravitational field. We relate that to Newtonian gravity by assuming *that the speed* $|\dot{\mathbf{x}}|$ *of the test particle is much less than c, and that the metric tensor* $g_{\mu\nu}$ *is static, with a deviation* $h_{\mu\nu} \stackrel{\text{def}}{=} (g_{\mu\nu} - \eta_{\mu\nu})$ *from* $\eta_{\mu\nu}$ *that is much smaller in norm than unity.*

For $|\dot{\mathbf{x}}| \ll c$, $\frac{dx^\lambda}{d\tau} = \left(\frac{dt}{d\tau}\right)\left(\frac{dx^\lambda}{dt}\right) = \left(\frac{dt}{d\tau}\right)(c, \dot{\mathbf{x}}) = c\left(\frac{dt}{d\tau}\right)(1, (\dot{\mathbf{x}}/c)) \approx c\left(\frac{dt}{d\tau}\right)(1, \mathbf{0}) = c\left(\frac{dt}{d\tau}\right)\delta^\lambda_0$. Therefore, for $|\dot{\mathbf{x}}| \ll c$, the Eq. (2.3d) dynamical equation for a test particle in a gravitational field, namely,

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\lambda\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\mu}{d\tau} = 0, \quad (2.6a)$$

is well approximated by,

$$\frac{d^2x^\sigma}{d\tau^2} + c^2\left(\frac{dt}{d\tau}\right)^2\Gamma^\sigma_{00} = 0. \quad (2.6b)$$

We next insert into Γ^σ_{00}, as it is given by Eq. (2.5d), the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, *where* $h_{\mu\nu}$ *is assumed to be static, and contributions to* Γ^σ_{00} *which are second-order or higher in* $h_{\mu\nu}$ *are discarded,*

$$\Gamma^\sigma_{00} = \tfrac{1}{2}g^{\sigma\nu}\left[\frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\right] \approx -\tfrac{1}{2}\eta^{\sigma\nu}\frac{\partial h_{00}}{\partial x^\nu}, \quad (2.6c)$$

which, upon insertion into Eq. (2.6b), yields,

$$\frac{d^2x^\sigma}{d\tau^2} = \tfrac{1}{2}\,c^2\left(\frac{dt}{d\tau}\right)^2\eta^{\sigma\nu}\frac{\partial h_{00}}{\partial x^\nu}. \quad (2.6d)$$

Because h_{00} *is static,* $\frac{\partial h_{00}}{\partial x^0} = 0$, so the $\sigma = 0$ component of Eq. (2.6d) yields,

$$\frac{d^2(ct)}{d\tau^2} = 0, \quad (2.6e)$$

which implies that,

$$\left(\frac{dt}{d\tau}\right) \text{ is } \textit{constant.} \quad (2.6f)$$

The *static nature of* h_{00} *and the* $\sigma = 1$, 2 *and* 3 *components of* Eq. (2.6d) yield the *three-vector* equation,

$$\frac{d^2\mathbf{x}}{d\tau^2}\Big/\left(\frac{dt}{d\tau}\right)^2 = -\nabla_{\mathbf{x}}\left(\tfrac{1}{2}\,c^2\,h_{00}(\mathbf{x})\right), \quad (2.6g)$$

which, together with $\frac{dt}{d\tau}$ being constant, implies that,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla_{\mathbf{x}}\left(\tfrac{1}{2}\,c^2\,h_{00}(\mathbf{x})\right). \quad (2.6h)$$

The corresponding *Newtonian* gravitational acceleration equation of course is,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla_{\mathbf{x}}\,\phi(\mathbf{x}), \quad (2.6i)$$

where *a typical example* of such a Newtonian gravitational potential $\phi(\mathbf{x})$ is,

$$\phi(\mathbf{x}) = -GM/|\mathbf{x}|, \quad (2.6j)$$

which applies specifically *when the gravitational source is a static point mass* M *located at* $\mathbf{x} = \mathbf{0}$. Comparison of Eq. (2.6h) with Eq. (2.6i) shows that,

$$h_{00}(\mathbf{x}) = 2\phi(\mathbf{x})/c^2. \quad (2.6k)$$

Thus, *in the Newtonian limit, wherein* $|h_{00}(\mathbf{x})| \ll 1$ *was noted below* Eq. (2.5d), Eq. (2.6k) *implies that,*

$$g_{00}(x) \approx g_{00}(\mathbf{x}) = \eta_{00} + h_{00}(\mathbf{x}) = 1 + 2\phi(\mathbf{x})/c^2, \text{ where } |\phi(\mathbf{x})| \ll \tfrac{1}{2}c^2. \quad (2.6l)$$

We next consider the *ratio* of *the differential time* $d\tau$ *recorded by a clock at rest in zero gravitational field,* which, for the gravitational field described by the metric tensor field $g_{\mu\nu}(x)$, is given by Eq. (2.4d) as $d\tau = \left(\sqrt{g_{\mu\nu}(x)\,dx^\mu dx^\nu}\,\right)/c$, *to the differential time* $\frac{dx^0}{c}$ *recorded by a clock embedded in a test particle that is moving arbitrarily in the gravitational field described by* $g_{\mu\nu}(x)$,

$$\left[d\tau \Big/ \tfrac{dx^0}{c}\right] = \tfrac{d(c\tau)}{dx^0} = \sqrt{g_{\mu\nu}(x)\tfrac{dx^\mu}{dx^0}\tfrac{dx^\nu}{dx^0}}\,. \tag{2.7a}$$

Therefore if the test particle and its embedded clock *are at rest* in the gravitational field described by $g_{\mu\nu}(x)$,

$$\tfrac{dx^\mu}{dx^0} = \delta^\mu_0,\ \textit{which reduces } \text{Eq. (2.7a)} \textit{ to the special case, } \tfrac{d(c\tau)}{dx^0} = \sqrt{g_{00}(x)}\,. \tag{2.7b}$$

We use Eq. (2.7b) to obtain *the ratio* $\frac{dx_1^0}{dx_2^0}$ *of the two differential times* $\frac{dx_1^0}{c}$ *and* $\frac{dx_2^0}{c}$ *recorded by two different clocks at rest at two different space-time points* x_1 *and* x_2 in the gravitational field described by $g_{\mu\nu}(x)$,

$$\tfrac{d(c\tau)}{dx_1^0} = \sqrt{g_{00}(x_1)}\ \ \&\ \ \tfrac{d(c\tau)}{dx_2^0} = \sqrt{g_{00}(x_2)}\ \ \textit{together yield, }\ \tfrac{dx_1^0}{dx_2^0} = \tfrac{d(c\tau)}{dx_2^0}\Big/\tfrac{d(c\tau)}{dx_1^0} = \sqrt{g_{00}(x_2)/g_{00}(x_1)} \tag{2.7c}$$

Eq. (2.7c) in turn implies that,

$$\big[(\text{tick rate of the clock at } x_2)/(\text{tick rate of the clock at } x_1)\big] \;=\; \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,. \tag{2.7d}$$

In the Newtonian limit $g_{00}(x)$ *is static and very close to unity, i.e.,* $g_{00}(x) \approx g_{00}(\mathbf{x}) = 1 + 2\phi(\mathbf{x})/c^2$, where $|\phi(\mathbf{x})| \ll \frac{1}{2}c^2$ (see Eq. (2.6l)). Therefore, *in the Newtonian limit* Eq. (2.7d) *yields that,*

$$\big[(\text{tick rate of the clock at } \mathbf{x}_2)/(\text{tick rate of the clock at } \mathbf{x}_1)\big] \;\approx\; \sqrt{(1 + 2\phi(\mathbf{x}_2)/c^2)/(1 + 2\phi(\mathbf{x}_1)/c^2)}$$

$$\approx\; \sqrt{1 - \big(2\big(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\big)/c^2\big)} \;\approx\; \Big[1 - \big(\big(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\big)/c^2\big)\Big]. \tag{2.7e}$$

When $|\mathbf{x}_1| > |\mathbf{x}_2|$ and $\phi(\mathbf{x}_1) > \phi(\mathbf{x}_2)$, Eq. (1.9) *follows from* Eq. (2.7e) (as do Eqs. (1.6) and (1.2a)).

Although gravitational time dilation is perforce an extremely small effect in the Newtonian limit, *there is no reason* that its Eq. (2.7d) ratio form $\sqrt{g_{00}(x_2)/g_{00}(x_1)}$ *for the corresponding ratio of the clock tick rates* should *not* have deviated very strongly from unity in an early universe *which was sufficiently compact and dense.* The discovery in the late 1920's of *the cosmological red shift* very, very strongly suggests *that the universe is expanding,* which would make it a travesty of common sense to *not* suppose that the universe was arbitrarily compact and dense *in the sufficiently remote past* and, *because of that degree of compactness and denseness at that time,* manifested *very strong gravitational time dilation* at that time.

The part of the universe *which astronomers have been able to see,* however, gives a strong impression of large-scale homogeneity and isotropy, which in the mid-1930's led to a concerted effort to produce *a metric tensor form* that is consistent with a universe *which always has been, and always will be, homogeneous and isotropic,* a hypothesis that was dubbed the Cosmological Principle. A very prominent and quite astonishing feature of the Robertson-Walker metric form which emerged from those mid-1930's efforts is the requirement that $g_{00} = 1$ *at every point of space-time.* The requirement that $g_{00} = 1$ everywhere in space-time is consistent with the *absence* of a gravitational field, i.e., when $g_{\mu\nu} = \eta_{\mu\nu}$, but in the *presence* of a gravitational field, i.e., when $g_{\mu\nu}$ *is a tensor field,* the requirement that $g_{00} = 1$ *at every point of space-time* is *inconsistent* with $g_{\mu\nu}$ being a *Lorentz-covariant* tensor field.

At the beginning of this section we stipulated that gravity theory *is Lorentz-covariant,* just as electrodynamics *or any other sensible physical theory is required to be Lorentz-covariant.* The Einstein equation for the metric tensor $g_{\mu\nu}$ in terms of its stress-energy source $T_{\mu\nu}$ is actually *generally covariant* under *arbitrary* one-to-one transformations of space-time onto itself, *not just Lorentz transformations,* which *ensures that the equivalence principle is honored.* The Einstein equation, however, *by itself determines only six of the ten independent components of the metric tensor field* $g_{\mu\nu}$, so the Einstein equation by itself *definitely isn't the complete theory of the gravitational metric tensor* $g_{\mu\nu}$. (This is *entirely different* from the situation in classical electromagnetic theory, where six independent first-order Heaviside-Maxwell electromagnetic field equations *completely determine* the three components of the electric field **E** and the three components of the magnetic field **B**.) To *complete the determination of the gravitational metric tensor field* $g_{\mu\nu}$, the Einstein equation must be supplemented *by four additional physically cogent equations;* the equation $g_{00} = 1$, *which is incompatible with the Lorentz-covariance of* $g_{\mu\nu}$ *when a gravitational field is present,* is *the opposite* of physically cogent. One *immensely anomalous consequence* of requiring that $g_{00} = 1$ everywhere in space-time is that, according to Eq. (2.7d), *gravitational time dilation fails to exist at all.* In order for gravitational time dilation *not to exist at all,* Eq. (2.7e) for the weak version of gravitational time dilation in the Newtonian gravitational limit *tells us that the constant c must go to infinity,* i.e., *that* g_{00} *cannot be fixed to unity everywhere in space-time in the presence of a gravitational field unless c is driven to infinity.* That this *is* *indeed* the consequence of fixing g_{00} to unity everywhere in space-time in the presence of a gravitational field was discovered by the engineer and amateur Riemann geometer A. Friedmann in 1922. Friedmann was thrilled when he found that the Einstein equation became solvable in closed form when he fixed g_{00} to unity everywhere in space-time, but by and by it became clear that *that maneuver forces the Einstein equation to exactly describe Newtonian gravity,* which is the $c \to \infty$ *limit* of relativistic gravitational theory. It is therefore *obvious* that the Robertson-Walker metric form, *which has* $g_{00} = 1$ *everywhere in space-time,* is *incompatible* with a physically-cogent *relativistic* theory of gravity.

Requirements for *relativistic* gravity theory *which in fact are physically cogent* are that $g_{\mu\nu}$ *must be a Lorentz-covariant tensor field, must have a matrix inverse and must have the signature* +, -, -, -. Thus *the Lorentz-invariant stipulation that* $\det(g_{\mu\nu}) \neq 0$, for example, is a physically cogent one for *relativistic* gravity theory. *However we need four physically-cogent Lorentz-covariant equations to complete the intrinsically incomplete Einstein equation.* Four *Lorentz-covariant equations in fact follow from the stipulation that,*

$$\det(g_{\mu\nu}) = -1, \tag{2.8a}$$

which, *besides being Lorentz-invariant and ensuring that* $g_{\mu\nu}$ *has a matrix inverse, is consistent with the case of no gravitational field,* $g_{\mu\nu} = \eta_{\mu\nu}$. The *four Lorentz-covariant equations which* Eq. (2.8a) *implies are, of course,* $\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 0$, which are more neatly expressed in terms of the affine connection as,

$$\Gamma^\sigma_{\sigma\lambda} = 0, \tag{2.8b}$$

because it is the case that,

$$\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 2\det(g_{\mu\nu})\,\Gamma^\sigma_{\sigma\lambda}. \tag{2.8c}$$

Proving Eq. (2.8c) is lengthy and delicate; we divide it into proving two lemmas, (1) $\Gamma^\sigma_{\sigma\lambda} = \frac{1}{2}g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda)$ and (2) $\partial(\det(g_{\mu\nu}))/\partial x^\lambda = \det(g_{\mu\nu})\,\mathrm{Tr}\big[g^{\mu\nu}\big(\partial g_{\mu\nu}/\partial x^\lambda\big)\big] = \det(g_{\mu\nu})\, g^{\sigma\beta}(\partial g_{\beta\sigma}/\partial x^\lambda) = 2\det(g_{\mu\nu})\,\Gamma^\sigma_{\sigma\lambda}$.

The first lemma follows from Eq. (2.5d),

$$\Gamma^\sigma_{\sigma\lambda} = \tfrac{1}{2}g^{\sigma\nu}\Big[\tfrac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \tfrac{\partial g_{\lambda\nu}}{\partial x^\sigma} - \tfrac{\partial g_{\sigma\lambda}}{\partial x^\nu}\Big] = \tfrac{1}{2}g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda), \tag{2.8d}$$

because $\big(\frac{\partial g_{\lambda\nu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\lambda}}{\partial x^\nu}\big)$ is antisymmetric under interchange of σ and ν.

The second lemma calculates $\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda$ in two different ways; the first way is very lengthy,

$$\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda = (1/\delta x^\lambda)\big[\ln(\det(g_{\mu\nu} + \delta x^\lambda(\partial g_{\mu\nu}/\partial x^\lambda))) - \ln(\det(g_{\mu\nu}))\big] =$$
$$(1/\delta x^\lambda)\ln\big(\det(g_{\mu\nu} + \delta x^\lambda(\partial g_{\mu\nu}/\partial x^\lambda))/\det(g_{\mu\nu})\big) = (1/\delta x^\lambda)\ln\big(\det\big(\mathrm{I} + \delta x^\lambda g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big)\big) =$$
$$(1/\delta x^\lambda)\ln\big(1 + \delta x^\lambda \mathrm{Tr}\big[g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big]\big) = \mathrm{Tr}\big[g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big] = g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda) = 2\,\Gamma^\sigma_{\sigma\lambda}, \tag{2.8e}$$

where the first lemma was applied in the final step. We next note that $\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda$ *also yields,*

$$\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda = \big(\partial(\det(g_{\mu\nu}))/\partial x^\lambda\big)/(\det(g_{\mu\nu})), \tag{2.8f}$$

which, combined with the result of Eq. (2.8e), implies that,

$$\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 2\,(\det(g_{\mu\nu}))\,\Gamma^\sigma_{\sigma\lambda}, \tag{2.8g}$$

which completes the proof of Eq. (2.8c).

The Eq. (2.8b) *consequence* $\Gamma^\sigma_{\sigma\lambda} = 0$ of the Eq. (2.8a) physically-cogent Lorentz-covariant stipulation $\det(g_{\mu\nu}) = -1$ is somewhat reminiscent of the far better known *harmonic* stipulation,

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0, \tag{2.8h}$$

which is *also* Lorentz covariant. However, *unlike* the Eq. (2.8a) stipulation $\det(g_{\mu\nu}) = -1$, the Eq. (2.8h) *harmonic* stipulation *doesn't ensure that* $g_{\mu\nu}$ *has a matrix inverse.*

The Eq. (2.8a) stipulation $\det(g_{\mu\nu}) = -1$ *also brings some welcome simplifications to gravity theory.* Since $\sqrt{-\det(g_{\mu\nu})} = 1$, the four-volume element becomes just d^4x for example. The first-order Heaviside-Maxwell equations for the electric and magnetic fields for this reason *also keep the same form in the presence of a gravitational field as they have in its absence* (gravity of course *changes those fields' four-current source*).

Since an expanding universe presumably would have been arbitrarily compact in the sufficiently remote past, we next examine the gravitational time dilation implications of a gravitational point source.

3. The metric tensor field of a static gravitational point source

The metric tensor field $g_{\mu\nu}(x)$ of a static gravitational point source located at the origin $\mathbf{x} = \mathbf{0}$ *must reflect spherical symmetry about the origin and insensitivity to time reversal,* so $(c\,d\tau)^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ must perforce have the form,

$$(c\,d\tau)^2 = D(|\mathbf{x}|)(dx^0)^2 - G(|\mathbf{x}|)((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)^2 - H(|\mathbf{x}|)|d\mathbf{x}|^2. \tag{3.1a}$$

A term of the form $E(|\mathbf{x}|)(dx^0)((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)$ is excluded because it is sensitive to time reversal $x^0 \to -x^0$.

It is of course very useful to write the Eq. (3.1a) spherically-symmetric invariant $(c\,d\tau)^2$ in terms of spherical polar coordinates, $\mathbf{x} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$, which yields that, $|\mathbf{x}| = r$, $((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)^2 = (dr)^2$ and $|d\mathbf{x}|^2 = (dr)^2 + r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2$. Thereby Eq. (3.1a) becomes,

$$(c\,d\tau)^2 = D(r)(dx^0)^2 - F(r)(dr)^2 - H(r)\big(r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2\big). \tag{3.1b}$$

where $F(r) = (G(r) + H(r))$.

The *determinant* of the Eq. (3.1b) metric form is of course $-D(r)F(r)(H(r))^2$, so for $r > 0$, the empty-space version of the Einstein equation should be solved using the Eq. (3.1b) metric form *subject to the additional stipulation that* $D(r)F(r)(H(r))^2 = 1$. A. Einstein posed precisely this mathematical problem to K. Schwarzschild in 1915. Schwarzschild was a consummate master of sophisticated problem-solving techniques, so he did work out the metric solution Einstein requested; it was published in January 1916.

A lamentable intervention by D. Hilbert in 1918 resulted in Schwarzschild's impeccable 1916 solution metric *not being shown in gravitational-theory textbooks,* which *instead* show solution metrics for the static point source *which have an unphysical signature at* $r > 0$ *that doesn't exist in Schwarzschild's 1916 solution.*

The *root cause of the unphysical signature at* $r > 0$ is Hilbert's *application of a radial transformation which reduces the number of functions of the radial coordinate* r *in the* Eq. (3.1b) *metric form from three to two before it is inserted into the empty-space Einstein equation.* Radial transformations of solution metrics of the Einstein equation are, given its general covariance, *also* solution metrics of the Einstein equation, *but they may not be physical metrics which have the signature* +, -, -, - *everywhere.* Radial transformations *which reduce from three to two* the number of functions of the radial coordinate in the Eq. (3.1b) metric form *assuredly damage the physics when it requires all three* Eq. (3.1b) *functions of the radial coordinate.*

A specific transformation $r'(r)$ of the radial coordinate r *can be exhibited* which changes Eq. (3.1b) to,

$$(c\,d\tau)^2 = C(r')(dx^0)^2 - J(r')\big((dr')^2 + (r')^2(d\theta)^2 + (r')^2(\sin\theta\,d\phi)^2\big), \tag{3.2a}$$

which has only two functions of r' *in its* Eq. (3.2a) *metric form;* it is dubbed the "isotropic" form of the static, spherically symmetric metric. *Another specific transformation* $R(r)$ of the radial coordinate r *can be exhibited* which changes Eq. (3.1b) to,

$$(c\,d\tau)^2 = B(R)(dx^0)^2 - A(R)(dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\,d\phi)^2, \tag{3.2b}$$

which also has only two functions of R *in its* Eq. (3.2b) *metric form;* it is dubbed the "standard" form of the static, spherically symmetric metric. These "isotropic" and "standard" *two-function forms* of the Eq. (3.1b) *three-function general form* of the static, spherically symmetric metric *could well be physically damaged.*

Although *it could well be physically damaged,* practically *all* gravity theory textbooks *nevertheless follow* D. Hilbert *in inserting the* Eq. (3.2b) *"standard" metric form into the empty-space Einstein equation,* which yields that $A(R)B(R) = K$, where K is a dimensionless constant, and that $B(R) = 1 - (r_0/R)$, where r_0 is a constant with the dimension of length. *Both of the constants* K *and* r_0 *are determined by properties of the metric tensor* $g_{\mu\nu}(R)$ *at sufficiently large values of* R.

For the static point source of mass M located at $R = 0$, we note from Eq. (2.6l) that as $R \to \infty$ (which makes its gravitational field arbitrarily weak), $g_{00}(R) \simeq 1 + 2\phi(R)/c^2$, *where* $\phi(R)$ *is the corresponding Newtonian gravitational potential,* which we see from Eq. (2.6j) is $\phi(R) = -GM/R$. Thus,

$$g_{00}(R) \simeq 1 + 2\phi(R)/c^2 \text{ as } R \to \infty, \text{ with } \phi(R) = -GM/R \;\Rightarrow\; g_{00}(R) \simeq 1 - (2GM/c^2)/R \text{ as } R \to \infty. \tag{3.2c}$$

Of course $g_{00}(R)$ is the same as the coefficient of $(dx^0)^2$ in Eq. (3.2b), which is $B(R)$. Therefore,

$$B(R) \simeq 1 - (2GM/c^2)/R \text{ as } R \to \infty. \tag{3.2d}$$

Since the empty-space Einstein equation applied to the Eq. (3.2b) "standard" form of the static, spherically symmetric metric yields that $B(R) = 1 - (r_0/R)$, where the constant r_0 is determined by properties of the metric tensor $g_{\mu\nu}(R)$ at sufficiently large values of R, Eq. (3.2d) implies that $r_0 = (2GM/c^2)$, so,

$$B(R) = 1 - (2GM/c^2)/R. \tag{3.2e}$$

We next obtain the value of the dimensionless constant K in the relation $A(R)B(R) = K$, which is a consequence of applying the empty-space Einstein equation to the Eq. (3.2b) "standard" form of the static, spherically symmetric metric. As $R \to \infty$, of course $g_{\mu\nu}(R) \to \eta_{\mu\nu}$, which implies that,

$$\text{as } R \to \infty,\ \ (c\,d\tau)^2 \to (dx^0)^2 - |d\mathbf{x}|^2 = (dx^0)^2 - (dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\,d\phi)^2, \tag{3.2f}$$

which, in conjunction with Eq. (3.2b), implies that as $R \to \infty$, $B(R) \to 1$ and $A(R) \to 1$. (Of course the fact that $B(R) \to 1$ as $R \to \infty$ follows as well from Eq. (3.2e).) Therefore, since the constant K satisfies $K = A(R)B(R)$, the value of the constant K is the limit of $A(R)B(R)$ as $R \to \infty$, so $K = 1$. Since, therefore, $A(R)B(R) = 1$, it follows that $A(R) = 1/B(R)$, which, in conjunction with $B(R) = 1 - (2GM/c^2)/R$ from Eq. (3.2e) yields that $A(R) = [1/(1 - (2GM/c^2)/R)]$. Inserting these values of $B(R)$ and $A(R)$ into the Eq. (3.2b) "standard" form of the static, spherically symmetric metric yields *the "standard" form of the solution metric for a static point source of mass M located at $R = 0$,*

$$(c\,d\tau)^2 = (1 - (r_0/R))(dx^0)^2 - [1/(1 - (r_0/R))](dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\,d\phi)^2, \tag{3.2g}$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. This metric's four eigenvalues are $(1 - (r_0/R))$, $-[1/(1 - (r_0/R))]$, -1 and -1, so the physical requirement that a metric's signature must be +, -, -, - *clearly fails at* $R = r_0 > 0$. This metric's determinant *is also undefined at* $R = r_0 > 0$. Very closely related, but more striking, is that since $\sqrt{g_{00}(R)} = \sqrt{1 - (r_0/R)}$ for this metric, clock tick rates go to *zero* at $R = r_0 > 0$, so an object approaching r_0 *has its approach speed forced toward zero,* an anomaly called an "event horizon". Due to D. Hilbert's *influence,* practically *all* textbooks refer to the *unphysical* Eq. (3.2g) *solution metric* as "the solution which K. Schwarzschild found in 1916", but *in actuality* the solution metric in Schwarzschild's January 1916 paper *has no unphysical signature other than at the origin,* and its mathematical form *isn't that of* Eq. (3.2g).

The "isotropic" form of the solution metric for a static point source *fares no better than the* Eq. (3.2g) *"standard" form* of the solution metric; it *also* has an unphysical signature at $r' > 0$. Some texts exhibit a harmonic form of the solution metric for a static point source *which is constructed directly from the two matrix elements $B(R)$ and $A(R)$ of the "standard" form of the solution metric,* and thus, unsurprisingly, *also has an unphysical signature which isn't at the origin.* We have earlier pointed out that the harmonic coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$ *provides no inherent protection against the occurrence of unphysical anomalies in the metric tensor* $g_{\mu\nu}$, whereas the condition $\det(g_{\mu\nu}) = -1$ (which was applied by K. Schwarzschild at A. Einstein's request) *explicitly provides such protection to* $g_{\mu\nu}$, and it *as well* implies the four equations $\Gamma^{\mu}_{\mu\lambda} = 0$, which are similar to the harmonic coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$.

Since D. Hilbert *induced* the unphysical signature located at $R = r_0 > 0$ in the Eq. (3.2g) "standard" form of the solution metric for the static point source *by making a damaging radial transformation to reduce from three to two the number of functions of the radial coordinate present in the* Eq. (3.1b) *metric form,* a radial transformation of the *unphysical* Eq. (3.2g) "standard" form of the solution metric *which removes its unphysical signature at $R = r_0 > 0$ is guaranteed to exist.* The unphysical-signature "event horizon" at $R = r_0 > 0$ *apparently belongs at the origin instead, where the clearly unphysical idealized point mass resides.* Moving $R = r_0$ to $\rho = 0$ is accomplished *by the simple radial transformation* $\rho(R) = R - r_0$,

$$\rho(R) = R - r_0 \ \Rightarrow\ R(\rho) = \rho + r_0 \ \Rightarrow\ (1 - (r_0/R(\rho))) = (\rho/(\rho + r_0)) = [1/(1 + (r_0/\rho))]. \tag{3.3a}$$

Insertion of $R(\rho) = \rho + r_0$ into the Eq. (3.2g) "standard" form of the solution metric, which is $(c\,d\tau)^2 = (1 - (r_0/R))(dx^0)^2 - [1/(1 - (r_0/R))](dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\,d\phi)^2$, produces,

$$(c\,d\tau)^2 = [1/(1 + (r_0/\rho))](dx^0)^2 - (1 + (r_0/\rho))(d\rho)^2 - (1 + (r_0/\rho))^2\big(\rho^2(d\theta)^2 + \rho^2(\sin\theta\,d\phi)^2\big), \tag{3.3b}$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. The Eq. (3.3b) solution metric is free of unphysical signatures when $\rho > 0$; it has an "event horizon" at $\rho = 0$ because of the presence there of the unphysical idealized point mass. Note that the "price" which is paid for this physically reasonable behavior of the Eq. (3.3b) solution metric is that *it contains three different powers of the entity* $(1 + (r_0/\rho))$, whereas the Eq. (3.2g) *physically unjustifiable* "standard" form of the solution metric *contains only two different powers of the entity* $(1-(r_0/R))$. It is now *transparent* that D. Hilbert's *insistence on making a damaging radial transformation which reduces from three to two the number of functions of the radius variable present in the* Eq. (3.1b) *metric form is the root cause of the unphysical signature that isn't at the origin which occurs in the* Eq. (3.2g) *solution metric.* It is distressing that D. Hilbert was able to *ensure* that his favored solution metrics with an unphysical signature that isn't at the origin *are the only ones shown in gravity-theory textbooks,* and that the *physically sensible* solution metric in K. Schwarzschild's January 1916 paper *is never shown in gravity-theory textbooks.*

Returning to the physically reasonable solution metric of Eq. (3.3b), we note that its determinant has the value $-(1+(r_0/\rho))^4$, whereas we, like K. Schwarzschild in 1915, *seek the solution metric whose determinant is equal to* -1. We achieve that through a further radial transformation of the Eq. (3.3b) solution metric. Inserting a *general* radial transformation from ρ to r into Eq. (3.3b) produces,

$$(c\,d\tau)^2 = [1/(1+(r_0/\rho(r)))](dx^0)^2 - (1+(r_0/\rho(r)))(d\rho(r)/dr)^2(dr)^2 - \\ ((\rho(r)+r_0)/r)^2\big(r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2\big), \tag{3.3c}$$

whose determinant we equate to -1,

$$-(d\rho(r)/dr)^2((\rho(r)+r_0)/r)^4 = -1. \tag{3.3d}$$

We require $\rho(r)$ to *increase* monotonically with r, which implies that,

$$(d\rho(r)/dr)((\rho(r)+r_0)/r)^2 = 1. \tag{3.3e}$$

Separation of variables yields,

$$(\rho+r_0)^2\,d\rho = r^2\,dr, \tag{3.3f}$$

and integration yields,

$$(\rho(r)+r_0)^3 = r^3 + r_0^3 k, \tag{3.3g}$$

where k is a dimensionless integration constant. We require that $\rho(r=0)=0$, which implies that $k=1$, so,

$$(\rho(r)+r_0) = (r^3+r_0^3)^{\frac{1}{3}}, \tag{3.3h}$$

which implies that,

$$((\rho(r)+r_0)/r) = ((r^3+r_0^3)^{\frac{1}{3}}/r), \tag{3.3i}$$

and also implies that,

$$(d\rho(r)/dr) = r^2/(r^3+r_0^3)^{\frac{2}{3}} = (r/(r^3+r_0^3)^{\frac{1}{3}})^2, \tag{3.3j}$$

and as well implies that,

$$(\rho(r)/r_0) = ((r^3+r_0^3)^{\frac{1}{3}}/r_0) - 1, \tag{3.3k}$$

from which we obtain,

$$(1+(r_0/\rho(r))) = \big(1+\big(1/(((r^3+r_0^3)^{\frac{1}{3}}/r_0)-1)\big)\big) = \big(((r^3+r_0^3)^{\frac{1}{3}}/r_0)/(((r^3+r_0^3)^{\frac{1}{3}}/r_0)-1)\big), \tag{3.3l}$$

which in turn yields that,

$$[1/(1+(r_0/\rho(r)))] = \big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big). \tag{3.3m}$$

We now insert Eqs. (3.3m), (3.3j) and (3.3i) into the Eq. (3.3c) solution metric to obtain its det $=-1$ form,

$$(c\,d\tau)^2 = \big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big)(dx^0)^2 - \big[1/\big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big)\big]\big(r/(r^3+r_0^3)^{\frac{1}{3}}\big)^4(dr)^2 - \\ \big((r^3+r_0^3)^{\frac{1}{3}}/r\big)^2\big(r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2\big), \tag{3.3n}$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. It is easy to verify that the determinant of the Eq. (3.3n) solution metric is equal to -1, and that it is free of unphysical signatures when $r>0$. It has an event horizon at $r=0$, the location of the unphysical idealized point mass. *This is the solution metric in K. Schwarzschild's January 1916 paper.* We obtained it here *by transformation of the radial variable of the unphysical* Eq. (3.2g) *"standard" form* *of the solution metric.* Schwarzschild of course obtained it *directly from the* Eq. (3.1b) *three-radial-function general form for a static, spherically-symmetric metric, the empty-space Einstein equation and the stipulation that the determinant of the solution metric must be equal to* -1.

It is extremely harmful to proper understanding of gravity theory that gravity-theory textbooks *show only "standard", "isotropic" and harmonic forms of the solution metric for a static point source which have an unphysical signature that isn't at the origin, and completely ignore the physically sound* Eq. (3.3n) *solution metric for a static point source* published by K. Schwarzschild in January 1916, *whose unphysical signature occurs at the origin.* It is immensely regrettable that the misleading impact of the gravity-theory textbooks *motivated a vast amount of wholly-wasted research effort over the course of more than a century.* The "discipline" of "geometrodynamics" was based on the risibly unphysical "wormhole" present in the Eq. (3.2g) unphysical matrix element $g_{00}(R) = (1 - (r_0/R))$ in the region $0 < R < r_0$ *beyond its unphysical "event horizon" at* $R = r_0 > 0$. The unphysical "event horizons" *themselves* that aren't at the origin *motivated a veritable avalanche of wasted research effort on the thermodynamics and even particle physics of those physically nonexistent entities.* D. Hilbert advocated *the radial transformations which produce damaged two-function variants* of the Eq. (3.1b) three-function general form of the static, spherically-symmetric metric, and *persuaded* textbook authors *to show only solution metrics for a static point source which have an unphysical signature that isn't at the origin, which is the product of those damaging radial transformations.*

Fortunately we have in hand the physically sound matrix element $g_{00}(r) = \big(1 - \big(r_0/(r^3 + r_0^3)^{\frac{1}{3}}\big)\big)$ of the Eq. (3.3n) solution metric for a static point source published by K. Schwarzschild in January 1916. At $r = r_0$ *there is no event horizon,* but even so,

$$\sqrt{g_{00}(r_0)} = \sqrt{1 - \big(r_0/(r_0^3 + r_0^3)^{\frac{1}{3}}\big)} = \sqrt{1 - \big(1/2^{\frac{1}{3}}\big)} = .454. \tag{3.4a}$$

Therefore the tick rate of a clock at $r = r_0$ *is less than half that of a clock at a very large value of* r *since,*

$$\sqrt{g_{00}(r)} = \sqrt{1 - \big(1/((r/r_0)^3 + 1)^{\frac{1}{3}}\big)} \approx 1 - \tfrac{1}{2(r/r_0)} \quad \text{when} \quad (r/r_0) \gg 1, \tag{3.4b}$$

but the tick rate of a clock at $r = r_0$ *is much greater than that of a clock much closer to* $r = 0$ *because,*

$$\sqrt{g_{00}(r)} = \sqrt{1 - \big(1/(1 + (r/r_0)^3)^{\frac{1}{3}}\big)} \approx \big((r/r_0)^{\frac{3}{2}}/\sqrt{3}\,\big) \quad \text{when} \quad (r/r_0) \ll 1. \tag{3.4c}$$

Although there is no event horizon at $r = r_0$, *that point, where* $\sqrt{g_{00}(r)} \approx \frac{1}{2}$, *marks a transition in the behavior of* $\sqrt{g_{00}(r)}$ *from lingering near unity when* $r \gg r_0$ *to briskly proceeding toward zero as* $r \to 0$.

Another entity of interest implicit in $g_{00}(r)$ is $\phi(r) \stackrel{\text{def}}{=} \frac{1}{2}c^2(g_{00}(r) - 1)$, a conceptual *extension of the Newtonian gravitational potential.* Since $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$,

$$\begin{aligned}\phi(r) \stackrel{\text{def}}{=} \tfrac{1}{2}c^2(g_{00}(r) - 1) = -\tfrac{1}{2}c^2 r_0/(r^3 + r_0^3)^{\frac{1}{3}} = -GM/(r^3 + r_0^3)^{\frac{1}{3}} = \\ (-GM/r)\big(1/(1 + (r_0/r)^3)^{\frac{1}{3}}\big) = -\tfrac{1}{2}c^2\big(1/(1 + (r/r_0)^3)^{\frac{1}{3}}\big),\end{aligned} \tag{3.4d}$$

which implies that,

$$\phi(r_0) = \big(-\tfrac{1}{2}c^2\big)/2^{\frac{1}{3}} = (-GM/r_0)/2^{\frac{1}{3}}. \tag{3.4e}$$

Eq. (3.4d) also implies that,

$$\phi(r) \approx -GM/r \quad \text{when} \quad (r/r_0) \gg 1, \tag{3.4f}$$

and Eq. (3.4d) as well implies that,

$$\phi(r) \approx -\tfrac{1}{2}c^2\big(1 - \tfrac{1}{3}(r/r_0)^3\big) \quad \text{when} \quad (r/r_0) \ll 1. \tag{3.4g}$$

Unlike the unbounded-below Newtonian gravitational potential $\phi(r) = -GM/r$, *its* Eq. (3.4d) *extension* $\phi(r) = -GM/(r^3 + r_0^3)^{\frac{1}{3}}$ *cannot become more negative than* $-\frac{1}{2}c^2$, just as the speed of a relativistic particle cannot exceed c. Again, the point $r = r_0$ *marks a transition in the behavior of* $\phi(r) = -GM/(r^3 + r_0^3)^{\frac{1}{3}}$ *from being close to* $-GM/r$ *when* $r \gg r_0$ *to briskly leveling off at* $-\frac{1}{2}c^2$ *as* $r \to 0$.

In the case of *the Newtonian gravitational potential,* $\phi(r) = -GM/r$, a test particle *placed at rest at* $r = d > 0$ *at time* $t = 0$ accelerates toward $r = 0$, and as it closes in on $r = 0$, *its acceleration* $-d\phi(r)/dr = -GM/r^2$ *strengthens without bound.* At the finite time $t = (\pi/2)\sqrt{d^3/(2GM)}$ *the test particle* *reaches* $r = 0$*, and there its speed is infinite.* If the test particle *initially has zero angular momentum, i.e.,* $\big(\dot{\mathbf{r}}(t = 0) \times \mathbf{r}(t = 0)\big) = \mathbf{0}$*, and its initial speed is less than that of gravitational escape, i.e.,* $|\dot{\mathbf{r}}(t = 0)| < \sqrt{2GM/|\mathbf{r}(t = 0)|}$, it will *still* reach $r = 0$ *in a finite time, and its speed there will still be infinite.* This simple picture *is a time-reversed microcosm of the ostensible Big Bang.* Details to be added include many particles instead of one, which permits the static point mass M at $r = 0$ to be removed in favor of the mutual gravitational attraction of those particles, and also that when those particles smash into each other at infinite speed *the consequence is an infinite temperature.*

In the case of *the extended gravitational potential,* $\phi(r) = -GM/(r^3+r_0^3)^{\frac{1}{3}}$, the test particle's acceleration $-d\phi(r)/dr = -GMr^2/(r^3 + r_0^3)^{\frac{4}{3}}$, *actually goes to zero as* $r \to 0$ *instead of going to* $-\infty$ *as* $r \to 0$, as in the Newtonian case. At the same time, *the clock tick-rate factor* $\sqrt{g_{00}(r)} = (1 - (r_0/(r^3 + r_0^3)^{\frac{1}{3}}))^{\frac{1}{2}}$ *goes to zero as* $r \to 0$*, which forces the test particle's speed toward zero as it approaches* $r = 0$ (this of course is aided by the fact that the test particle's acceleration goes to zero as $r \to 0$). Thus *the situation in relativistic gravity is the opposite of that in Newtonian gravity: instead of the test particle reaching* $r = 0$ *in a finite time with infinite speed,* the clock tick-rate factor *forces its speed toward zero as it approaches* $r = 0$*, so it takes forever to reach* $r = 0$. At sufficiently large values of r, however, *the relativistic test particle's behavior obviously must be almost the same as the Newtonian test particle's behavior.* As we have twice noted above, *the transition in behavior occurs around* $r = r_0$.

In light of *these basic lessons in relativistic gravitational physics* which follow from the Eq. (3.3n) solution metric for the static point source published by K. Schwarzschild in January 1916, *we are obliged to rule out Big Bang cosmology,* but can present a very rough sketch of the universe's likely early evolution.

4. The universe's likely early evolution, a very rough sketch based on relativistic gravity

As we have said previously, a key fact about the universe is that it apparently is expanding, and therefore it must have been arbitrarily compact and dense at a sufficiently remote time in the past. We assume that in the sufficiently remote past the universe was compact enough *to be well inside the radius* $r_0 = (2GM/c^2)$, where M is the universe's mass. At such times the universe's behavior *would be dominated by gravitational time dilation* (see Eq. (3.4c)).

In a universe dominated by gravitational time dilation all physical processes would be greatly slowed and radiation frequencies would be greatly reduced; it would be a dark, cold universe with almost no discernible physical processes. *Even the expansion rate of such a universe would have been reduced. Going still further back in time only accentuates these features of the very early universe.* Going *forward* in time *eventually* brings us to a universe whose radius is approximately r_0. The accompanying *decrease in gravitational time dilation* allows the universe's *expansion rate to increase,* which *still further reduces gravitational time dilation,* causing the universe's expansion rate *to increase still further,* etc.

Thus a universe whose radius is approximately r_0 is one on the cusp *of rapid acceleration of its expansion,* which is termed inflation. Physical process rates in such a universe would greatly increase as the dead hand of extreme gravitational time dilation rapidly falls away. *Notwithstanding its inflationary expansion,* such a universe *would still be vastly more compact and dense than today's universe is,* with its billions of years *of additional expansion.* So *dense* a universe, *which was liberated from extreme gravitational time dilation,* would have been able to give birth to every conceivable kind of young star at an utterly enormous rate, with particular emphasis on immensely massive, extremely short-lived giants, but *considering how much even denser than that the universe was when it neared its liberating radius* r_0, only a small fraction of its matter *would have been able to participate directly in those fireworks; by far the great bulk of its matter would have been compelled to take the form of primordial black holes* (but do bear in mind that black holes *don't have event horizons*). However those primordial black holes *profoundly modulated* the spectacular events underway by, for example, becoming the active nuclei of galaxies and quasars, with the primordial black holes of lesser mass *being utterly crucial to galaxy formation by supplying the necessary cold, dark gravitational "glue".* With its continued expansion, the universe's density diminished, diminishing the enormous rate of star and galaxy formation of its early post-inflationary era. The James Webb Space Telescope may possibly be registering evidence of unexpectedly rapid galaxy formation in the early universe's post-inflationary era.

The inflationary expansion of the universe was a consequence of gravitational time dilation: the universe's expansion decreased the intensity of its gravitation, which diminished gravitational time dilation, causing the universe's expansion rate to increase, etc. As the universe continued expanding for billions of years after it reached the liberating radius r_0, the intensity of its gravitation would have greatly diminished, but *it still may be enough* that the gravitational time-dilation acceleration of the universe's expansion *overcomes the natural deceleration of the universe's expansion which is caused by its gravitational force.* It has been found observationally that the universe's expansion is still very, very slightly accelerating; *a cosmological term in the Einstein gravitational field equation has been hypothesized as the explanation.* Such a term *upends the crucially important fundamental physical idea* that the Einstein gravitational field equation *flawlessly reproduces Newtonian gravity for static, weak gravitational fields,* so *the far less disruptive hypothesis* that the universe's gravitational intensity *is still sufficient* for gravitational time-dilation acceleration of the universe's expansion *to overcome the natural deceleration of its expansion by its gravitational force* is worthy of very serious consideration indeed.