Any Formal System That Contains Sets Arithmetic and Rational Numbers is Inconsistent

By Jim Rock

Abstract: Gödel proved that any formal system containing arithmetic is incomplete. We show that any such formal system is inconsistent. We establish a collection of nested sets of rational numbers in a descending hierarchy. The sets higher in the descending hierarchy contain element(s) that are not in the sets below them in the hierarchy. Given such a descending set hierarchy, it is easy to develop two arguments that contradict each other. The conclusion of Argument#2 is false. But, Argument#2 is a valid argument.

For rational numbers $a$ in $[0, 1]$ let the collection of $Ra$ sets be $\{y \text{ is a rational number } | 0 \leq y < a \}$

Argument #1: No $Ra$ contains a largest element.
1) Suppose there is a largest element $a'$ in some individual $Ra$.
2) $a' < (a' + a)/2 < a$.
3) Let $b = (a' + a)/2$.
4) Then $b$ is in $Ra$ and $a' < b$.
5) Therefore, no $Ra$ contains a largest element.

When a largest element is assumed in Argument#1, it leads to a contradiction; so there is no largest element. Every $Ra$ set element is in one of the proper subsets below $Ra$ in the set hierarchy. It is a valid proof by contradiction.

Argument #2: Each $Ra$ contains a largest element.
1) Below each $Ra$ for all rationals $x < a$ is a collection of $Rx$ subsets $\{y \text{ is a rational number } | 0 \leq y < x \}$.
2) Each $Ra$ and its collection of $Rx$ subsets comprise a descending set hierarchy.
3) Each $Rx$ is missing its index “$x$”. $Ra$ contains all the “$x$” indices.
4) Since the union of the collection of $Rx$ sets does not contain any element greater than the elements in all the individual $Rx$ sets, the union of the collection of $Rx$ sets does not equal $Ra$.
5) There exists at least one $Ra$ set element $s \geq (\text{all values of } x)$.
6) Let $c$ and $d$ be two elements of a single $Ra$ set with $c > d$.
7) $d$ is an element of $Re$, which is a proper subset of $Ra$.
8) For any two elements in $Ra$ the smaller element is contained in a $Rx$ subset of $Ra$.
9) By steps 6) 7) and 8), there is at most one $Ra$ set element missing from all the $Rx$ subsets.
10) By steps 5) 9), each $Ra$ set contains a largest element $a'$ not in a $Rx$ set below in the hierarchy.
11) There is no $b = (a' + a)/2$. It would be a second element not in a $Rx$ set below $Ra$ in the hierarchy.

We know by step 8) that isn’t possible.

Argument #1 is generally considered correct and its conclusion is true. The first three statements of Argument #2 are generally uncontested. The first part of Statement #4 stating that the union of the collection of $Rx$ sets doesn’t contain any element greater than the elements in the individual $Rx$ sets is not an issue. It is the latter part of Statement #4 that states “the union of the collection of $Rx$ sets does not equal $Ra$” that is a false statement. This causes most people to dismiss Argument #2.

Containing a false statement does not keep the second part Statement #4 from being a valid logical deduction from Statement #3 and the first part of Statement #4, which are true. It simply means that in all formal systems containing sets, arithmetic, and rational numbers this false statement can be deduced and such formal systems are therefore inconsistent. Likewise, false statements #5, #9, #10, and #11 are valid logical deductions from previous statements.

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