The Perfect Cuboid Is Nothing More Than a Myth

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Abstract
An impeccable proof of the impossibility of the existence of a perfect cuboid based on the parametrization of Leonhard Euler.

In mathematics, the Perfect Cuboid is a rectangular cuboid whose edges, face diagonals and space diagonal all have integer lengths: https://en.wikipedia.org/wiki/Euler_brick

A Perfect Cuboid must satisfy the following system of diophantine equations:

\[
\begin{align*}
    a^2 + b^2 &= d^2 \\
    a^2 + c^2 &= e^2 \\
    b^2 + c^2 &= f^2 \\
    a^2 + b^2 + c^2 &= g^2
\end{align*}
\]  

(I)

where: \(a, b, c\) are the edges, \(d, e, f\) are the face diagonals and \(g\) is the space diagonal.

Until now, there was no confirmation of the existence of a Perfect Cuboid, but it had not been proven either that such a cuboid cannot exist, so this has remained a problem for several centuries. However, I will show the reason why the Perfect Cuboid is impossible.

Lemma.

If Perfect cuboid exists, the squares of 3 its face diagonals should construct a Heronian triangle.

Proof.

From simple transformations of (I) we have:

\[
\begin{align*}
    g^2 &= \frac{d^2 + e^2 + f^2}{2} \\
    a^2 &= \frac{d^2 + e^2 + f^2}{2} - f^2 \\
    b^2 &= \frac{d^2 + e^2 + f^2}{2} - e^2 \\
    c^2 &= \frac{d^2 + e^2 + f^2}{2} - d^2
\end{align*}
\]  

(II)

By substitution from (IV) и (VI) it is follows:

\[
\begin{align*}
    a^2b^2c^2g^2 &= \left(\frac{d^2 + e^2 + f^2}{2} - f^2\right)\left(\frac{d^2 + e^2 + f^2}{2} - e^2\right)\left(\frac{d^2 + e^2 + f^2}{2} - d^2\right) \left(\frac{d^2 + e^2 + f^2}{2}\right)
\end{align*}
\]  

(III)

\[
\begin{align*}
    abcg &= \frac{1}{4} \sqrt{(-d^2 + e^2 + f^2)(d^2 - e^2 + f^2)(d^2 + e^2 - f^2)(d^2 + e^2 + f^2)}
\end{align*}
\]  

(IV)

Since \(abcg \in \mathbb{N}\), then the squares of the face diagonals \(d^2, e^2, f^2\) are the edges of a Heronian triangle with area \(abcg\) (https://en.wikipedia.org/wiki/Heronian_triangle). What was required.
Let’s parametrize the indicated Heronian triangle by means of the general parametric solution of Leonhard Euler (https://en.wikipedia.org/wiki/Heronian_triangle#Euler's_parametric_equation):

\[
\begin{align*}
    d^2 &= mn(p^2 + q^2) \\
    e^2 &= pq(m^2 + n^2) \\
    f^2 &= (mq + np)(mp - nq) \\
\end{align*}
\]  
(V)

\[ mn, p, q \in \mathbb{N}, \quad mp > nq \]

Next, we can to express the remaining parameters of the Perfect Cuboid:

\[
\begin{align*}
    a^2 &= nq(mp + np) \\
    b^2 &= np(mp - nq) \\
    c^2 &= mq(mp - nq) \\
    g^2 &= mp(mp + np) \\
\end{align*}
\]  
(VI)

From (V) и (VI) it is follows:

\[
\begin{align*}
    b^2 f^2 g^2 &= mnp^2 (mp - nq)^2 (mq + np)^2 \Rightarrow mn = □ \\
    c^2 f^2 g^2 &= pqm^2 (mp - nq)^2 (mq + np)^2 \Rightarrow pq = □ \Rightarrow m^2 + n^2 = □ \\
\end{align*}
\]  
(VII)

Let’s: \( m = ui^2, \quad n = us^2 \), then:

\[
\begin{align*}
    mn &= u^2 s^2 t^2 = □ \\
    m^2 + n^2 &= u^2 (s^4 + t^4) ≠ □, \text{ which contradicts (VII).} \\
\end{align*}
\]

Conclusion: the assumption of the existence of a Perfect Cuboid is unrealizable.

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