3D-BRANE GRAVITY WITHOUT EQUIDISTANCE POSTULATE.

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Abstract. The 3D-brane universe model is an alternative non-Einsteinian theory of gravity. It was initially built using the so-called equidistance postulate. In this paper we consider a version of the theory without this equidistance postulate.

1. Introduction.

The 3D-brane universe model is based on the idea that the four-dimensional spacetime subdivides into a dense foliation of 3D-branes. Each brane represents some definite instantaneous state of the real 3D-universe in its evolution pathway as shown in Fig. 1.1. Arguments in favor of the 3D-brane universe model opposed to the standard 4D paradigm are given in [1] (see also [2] and [3]).

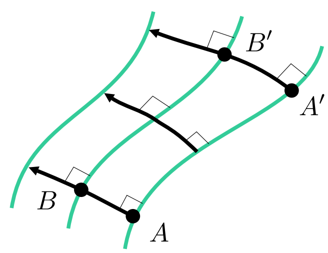

Fig. 1.1

We can choose the normal vector $\mathbf{n}$ of the unit length directed to the future at each point of each 3D-brane. Since 3D-branes are dense in the four-dimensional spacetime, these vectors $\mathbf{n}$ constitute a vector field. Integral curves (see § 2.12 of Chapter 2 in [4]) of the vector field $\mathbf{n}$ are shown as black lines in Fig. 1.1. The arrows at the ends of these curves indicate the direction of the time evolution from the past to the future.

Let's choose some 3D-brane for the initial one, say it is the brane AA' in Fig. 1.1. Let's choose some coordinates x^1, x^2, x^3 in the initial 3D-brane. We can extend them to other 3D-branes along the integral curves of the field $\mathbf{n}$ by setting

$$x^i(B) = x^i(A), \qquad i = 1,\, 2,\, 3, \tag{1.1}$$

for any point A of the initial 3D-brane. The coordinates introduced in such a way are called **comoving coordinates** (see [5]). An observer whose world line (see [6]) coincides with one of the integral curves of the vector field $\mathbf{n}$ in Fig. 1.1 is called a **comoving observer**.

The lengths of the curves AB and $A'B'$ can be used for introducing time intervals:

$$t = \frac{|AB|}{c_{\text{gr}}}, \qquad\qquad t' = \frac{|A'B'|}{c_{\text{gr}}}. \tag{1.2}$$

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The constant c_{gr} from (1.2) in [7] was interpreted as the speed of gravitational waves. The equidistance postulate introduced in [1] sounds as follows.

Postulate 1.1. *Watches of any two comoving observers can be synchronized.*

The postulate 1.1 means that $t = t'$ in (1.2) for any two points A and A' in the initial 3D-brane. In other words it means that the distance between any two 3D-branes is constant throughout all associated points of the two branes. For this reason the postulate 1.1 is called the equidistance postulate. In this paper we exclude this postulate from the theory thus expanding our theory to non-equidistant foliations of 3D-branes in the four-dimensional spacetime.

2. Non-equidistant foliations of 3D-branes.

In the case of a non-equidistant foliation of 3D-branes in the four-dimensional spacetime the time intervals t and t' in (1.2) are not equal to each other. Let's choose one of the integral curves of the vector field $\mathbf{n}$ for the reference curve, say it is the curve AB in Fig. 1.1. Then we can use the time interval t from (1.2) for marking the hypersurface BB' and can declare the product

$$x^0 = c_{\text{gr}}\, t \tag{2.1}$$

to be the fourth coordinate in the foliated spacetime complementary to the comoving coordinates x^1, x^2, x^3 from (1.1). Under this convention the time interval t' becomes the value of some definite function determined by the foliation structure:

$$t' = t'(t, x'^1, x'^2, x'^3) = \frac{|A'B'|}{c_{\text{gr}}}. \tag{2.2}$$

Here x'^1, x'^2, x'^3 are comoving coordinates of both points A' and B'. Let's denote through ν the following partial derivative of the function (2.2):

$$\nu(t, x^1, x^2, x^3) = \frac{\partial t'(t, x^1, x^2, x^3)}{\partial t}. \tag{2.3}$$

Due to (2.1) we can treat ν as a function of the coordinates x^0, x^1, x^2, x^3.

Note that the time variable t in (2.1) is twice relative — it is relative to the choice of some initial 3D-brane and it is relative to the choice of some reference integral curve of the vector field $\mathbf{n}$. The first sort of relativity is removed by using the Big Bang. It is a point in the four-dimension spacetime at which all integral curves of the vector field $\mathbf{n}$ do start (see Fig. 2.1). If we replace the initial 3D-brane AA' by this Big Bang point, then we get the absolute initial point for counting the time t in (2.1). However, this time t remains dependent on the choice of the reference curve AB. The matter is that in the absence of the equidistance postulate 1.1 there is no global age of the 3D-universe. Each point of a 3D-brane has

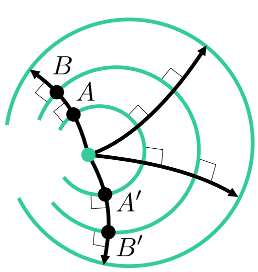

Fig. 2.1

its own age relative to the Big Bang. And conversely, the points with the same age do not constitute a physical 3D-brane. The time t in (2.1) is just a marker time used to mark physical 3D-branes and to distinguish them from each other. This time can be called a **brane time**. Unlike the cosmological time considered in [1] in the case with the equidistance postulate 1.1, the brane time is not uniquely defined. With these words of caution we proceed to build a new version of the theory.

3. Metric in the foliated spacetime.

Assume that the four-dimensional spacetime is foliated into 3D-branes and assume that x^1, x^2, x^3 are comoving coordinates associated with this foliation. Assume that x^0 is a complementary coordinate given by the formula (2.1) where t is a brane time. Under these assumption the four-dimensional metric in the coordinates x^0, x^1, x^2, x^3 is given by the following block-diagonal matrix:

$$G_{ij}=\begin{Vmatrix} g_{00} & 0 & 0 & 0\\ 0 & -g_{11} & -g_{12} & -g_{13}\\ 0 & -g_{21} & -g_{22} & -g_{23}\\ 0 & -g_{31} & -g_{32} & -g_{33}\end{Vmatrix}. \tag{3.1}$$

The diagonal element g_{00} of the matrix (3.1) is given by the formula

$$g_{00}(x^0,x^1,x^2,x^3)=\nu^2, \tag{3.2}$$

where ν is the partial derivative from (2.3). Like the element g_{00} in (3.2), other elements of the matrix (3.1) are functions of the coordinates x^0, x^1, x^2, x^3. They define a time dependent positive 3D metric in a 3D-brane universe:

$$g_{ij}=g_{ij}(x^0,x^1,x^2,x^3),\quad 1\leqslant i,j\leqslant 3. \tag{3.3}$$

The metric (3.3) along with the function (3.2) describes the gravitational field in the present extension of the 3D-brane universe model.

4. Reduction of four dimensional equations.

In this section and further below we follow the scheme of [1] and derive differential equations for the metric (3.3) and for the function (3.2) by substituting the metric (3.1) into the four-dimensional Einstein's equation

$$r_{ij}-\frac{r}{2}\,G_{ij}-\Lambda\,G_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij}. \tag{4.1}$$

Here γ is Newton's gravitational constant (see [8]) and Λ is the cosmological constant (see [9]). The quantities r_{ij} in (4.1) are the components of the four-dimensional Ricci tensor (see § 8 in Chapter IV of [10]) associated with the metric (3.1) and r is the scalar curvature[1] associated with this metric. The quantities

[1] We used lowercase letters for r_{ij} and r in order to reserve capital letters for the 3D Ricci tensor and for the 3D scalar curvature.

T_{ij} in the right hand side of the equation (4.1) are the components of the energy-momentum tensor (see [11]). They represent matter which is the source of the gravitational field in the equation (4.1).

The metric (3.1) produces the metric connection with the components

$$\gamma^k_{ij} = \frac{1}{2} \sum_{s=9}^{3} G^{ks} \left(\frac{\partial G_{sj}}{\partial x^i} + \frac{\partial G_{is}}{\partial x^j} - \frac{\partial G_{ij}}{\partial x^s} \right). \tag{4.2}$$

It is easy to derive that

$$\gamma^k_{ij} = \Gamma^k_{ij} \quad \text{for} \quad 1 \leqslant i, j, k \leqslant 3, \tag{4.3}$$

where Γ^k_{ij} are the components of the metric connection for the metric (3.3):

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{s=1}^{3} g^{ks} \left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} \right). \tag{4.4}$$

The rest of the components (4.2) are distributed as follows:

$$\gamma^0_{ij} = \frac{g^{-1}_{00}}{2} \, \frac{\partial g_{ij}}{\partial x^0} \quad \text{for} \quad 1 \leqslant i, j \leqslant 3, \tag{4.5}$$

$$\gamma^k_{0j} = \gamma^k_{j0} = \frac{1}{2} \sum_{s=1}^{3} g^{ks} \, \frac{\partial g_{sj}}{\partial x^0} = \sum_{s=1}^{3} g_{00} \, g^{ks} \, \gamma^0_{sj} \quad \text{for} \quad 1 \leqslant k, j \leqslant 3, \tag{4.6}$$

$$\gamma^q_{00} = \frac{1}{2} \sum_{s=1}^{3} g^{qs} \, \frac{\partial g_{00}}{\partial x^s} \quad \text{for} \quad 1 \leqslant q \leqslant 3, \tag{4.7}$$

$$\gamma^0_{q0} = \gamma^0_{0q} = \frac{1}{2} \, g^{-1}_{00} \, \frac{\partial g_{00}}{\partial x^q} \quad \text{for} \quad 1 \leqslant q \leqslant 3, \tag{4.8}$$

$$\gamma^0_{00} = \frac{1}{2} \, g^{-1}_{00} \, \frac{\partial g_{00}}{\partial x^0}. \tag{4.9}$$

The formulas (4.5), (4.6), (4.7), (4.8), and (4.9) are easily derived from (4.2) with the use of the formula (3.1).

Now we introduce the quantities which were already used in [1]:

$$b_{ij} = \frac{1}{2} \, \frac{\partial g_{ij}}{\partial x^0}. \tag{4.10}$$

These quantities are components of the symmetric tensor field $\mathbf{b}$. Raising indices in (4.10), we produce the following quantities:

$$b^k_j = \sum_{s=1}^{3} g^{ks} \, b_{sj}, \qquad\qquad b^{ij} = \sum_{s=1}^{3} b^i_s \, g^{sj}. \tag{4.11}$$

Using (4.10) and (4.11), we can rewrite the formulas (4.5) and (4.6) as follows:

$$\gamma^0_{ij} = g^{-1}_{00} \, b_{ij} \quad \text{for} \quad 1 \leqslant i, j \leqslant 3, \tag{4.12}$$

$$\gamma^k_{0j} = \gamma^k_{j\,0} = b^k_j \quad \text{for} \quad 1 \leqslant k, j \leqslant 3. \tag{4.13}$$

The Ricci tensor in Einstein's equation (4.1) is calculated through the curvature tensor by means of the following formula (see § 8 in Chapter IV of [10]):

$$r_{ij}=\sum_{k=0}^{3}r^k_{ikj},\tag{4.14}$$

where the components of the curvature tensor are

$$r^k_{isj}=\frac{\partial\gamma^k_{ji}}{\partial x^s}-\frac{\partial\gamma^k_{si}}{\partial x^j}+\sum_{q=0}^{3}\gamma^k_{sq}\,\gamma^q_{ji}-\sum_{q=0}^{3}\gamma^k_{jq}\,\gamma^q_{si}.\tag{4.15}$$

Due to (4.14) in (4.15) we need only those terms where $s=k$:

$$r^k_{ikj}=\frac{\partial\gamma^k_{ji}}{\partial x^k}-\frac{\partial\gamma^k_{ki}}{\partial x^j}+\sum_{q=0}^{3}\gamma^k_{kq}\,\gamma^q_{ji}-\sum_{q=0}^{3}\gamma^k_{jq}\,\gamma^q_{ki}.\tag{4.16}$$

Applying (4.3), (4.12), and (4.13) to (4.16), we derive

$$r^k_{ikj}=R^k_{ikj}+g_{00}^{-1}\,b^k_k\,b_{ij}-g_{00}^{-1}\,b^k_j\,b_{ki}\quad\text{for}\quad 1\leqslant i,j,k\leqslant 3.\tag{4.17}$$

Here R^k_{ikj} are the components of the 3D curvature tensor. They are given by a formula similar to (4.15) upon setting $s=k$ in it:

$$R^k_{isj}=\frac{\partial\Gamma^k_{ji}}{\partial x^s}-\frac{\partial\Gamma^k_{si}}{\partial x^j}+\sum_{q=1}^{3}\Gamma^k_{sq}\,\Gamma^q_{ji}-\sum_{q=1}^{3}\Gamma^k_{jq}\,\Gamma^q_{si}.\tag{4.18}$$

The 3D connection components in (4.18) are given by the formula (4.4). The 3D Ricci tensor is derived from (4.18) by means of the formula

$$R_{ij}=\sum_{k=1}^{3}R^k_{ikj},\tag{4.19}$$

which is the 3D version of the formula (4.14).

Now let's consider the case $k=0$ and $1\leqslant i,j\leqslant 3$ in (4.15). In this case we have

$$r^0_{i0j}=\frac{\partial\gamma^0_{ji}}{\partial x^0}-\frac{\partial\gamma^0_{0i}}{\partial x^j}+\sum_{q=0}^{3}\gamma^0_{0q}\,\gamma^q_{ji}-\sum_{q=0}^{3}\gamma^0_{jq}\,\gamma^q_{0i}.\tag{4.20}$$

Applying (4.12), (4.8), (4.3), (4.9), and (4.13), to (4.20), we reduce this formula to

$$\begin{aligned}r^0_{i0j}&=g_{00}^{-1}\,\frac{\partial b_{ij}}{\partial x^0}-\frac{1}{2}\,g_{00}^{-1}\,\nabla_{ij}\,g_{00}-\frac{1}{2}\,g_{00}^{-2}\,\frac{\partial g_{00}}{\partial x^0}\,b_{ij}+\\&+\frac{1}{4}\,g_{00}^{-2}\,\nabla_i\,g_{00}\,\nabla_j\,g_{00}-\sum_{q=1}^{3}g_{00}^{-1}\,b_{jq}\,b^q_i\quad\text{for}\quad 1\leqslant i,j\leqslant 3.\end{aligned}\tag{4.21}$$

Applying (4.17) and (4.21) to (4.14), we derive the following formula for the components of the four-dimensional Ricci tensor:

$$\begin{gathered}
r_{ij}=g_{00}^{-1}\,\frac{\partial b_{ij}}{\partial x^0}-\frac{1}{2}\,g_{00}^{-1}\,\nabla_{ij}\,g_{00}-\frac{1}{2}\,g_{00}^{-2}\,\frac{\partial g_{00}}{\partial x^0}\,b_{ij}+\frac{1}{4}\,g_{00}^{-2}\,\nabla_i\,g_{00}\,\nabla_j\,g_{00}\,+\\
+\,R_{ij}+g_{00}^{-1}\sum_{k=1}^{3}b^k_k\,b_{ij}-g_{00}^{-1}\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)\quad\text{for}\quad 1\leqslant i,j\leqslant 3.
\end{gathered}\tag{4.22}$$

The formula (4.22) is an extension of the formula (5.26) from [1].

The next step is $i=0$ and $1\leqslant j,k\leqslant 3$. in (4.16). In this case we have

$$\begin{gathered}
r^k_{0kj}=\frac{\partial b^k_j}{\partial x^k}-\frac{\partial b^k_k}{\partial x^j}+\sum_{q=1}^{3}\Gamma^k_{kq}\,b^q_j-\sum_{q=1}^{3}\Gamma^k_{jq}\,b^q_k\,+\\
+\,\frac{1}{2}\,g_{00}^{-1}\,b^k_k\,\frac{\partial g_{00}}{\partial x^j}-\frac{1}{2}\,g_{00}^{-1}\,b^k_j\,\frac{\partial g_{00}}{\partial x^k}.
\end{gathered}\tag{4.23}$$

We add two terms to (4.23) and rearrange the terms in it:

$$\begin{gathered}
r^k_{0kj}=\frac{\partial b^k_j}{\partial x^k}+\sum_{q=1}^{3}\Gamma^k_{kq}\,b^q_j-\sum_{q=1}^{3}\Gamma^q_{kj}\,b^k_q+\frac{1}{2}\,g_{00}^{-1}\,b^k_k\,\nabla_j\,g_{00}\,-\\
-\,\frac{\partial b^k_k}{\partial x^j}-\sum_{q=1}^{3}\Gamma^k_{jq}\,b^q_k+\sum_{q=1}^{3}\Gamma^q_{jk}\,b^k_q-\frac{1}{2}\,g_{00}^{-1}\,b^k_j\,\nabla_k\,g_{00}.
\end{gathered}$$

Due to the symmetry $\Gamma^q_{kj}=\Gamma^q_{jk}$ two extra terms that was added do cancel each other. But they let us apply the concept of covariant derivatives to the above formula (see § 6 in Chapter IV of [10]). As a result we obtain

$$r^k_{0kj}=\nabla_k\,b^k_j-\nabla_j\,b^k_k+\frac{1}{2}\,g_{00}^{-1}\,b^k_k\,\nabla_j\,g_{00}-\frac{1}{2}\,g_{00}^{-1}\,b^k_j\,\nabla_k\,g_{00}\tag{4.24}$$

for $1\leqslant k,j\leqslant 3$. Then we consider the case $i=k=0$ with $1\leqslant j\leqslant 3$ in (4.16):

$$r^0_{00j}=0\quad\text{for}\quad 1\leqslant j\leqslant 3.\tag{4.25}$$

Applying (4.24) and (4.25) to (4.14), we derive

$$r_{0j}=\sum_{k=1}^{3}\nabla_k\,b^k_j-\sum_{k=1}^{3}\nabla_j\,b^k_k+\frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3}\Bigl(b^k_k\,\nabla_j\,g_{00}-b^k_j\,\nabla_k\,g_{00}\Bigr).\tag{4.26}$$

Due to the symmetry of the Ricci tensor r_{ij} the formula (4.26) yields

$$r_{i0}=\sum_{k=1}^{3}\nabla_k\,b^k_i-\sum_{k=1}^{3}\nabla_i\,b^k_k+\frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3}\Bigl(b^k_k\,\nabla_i\,g_{00}-b^k_i\,\nabla_k\,g_{00}\Bigr).\tag{4.27}$$

The next step is to calculate the component r_{00} of the four-dimensional Ricci tensor. In order to calculate this component we choose $i=0$ and $j=0$ with

$1 \leqslant k \leqslant 3$ in (4.16). As a result we derive

$$\begin{aligned}r^{k}_{0k0}=\frac{1}{2}\sum^{3}_{s=1}g^{ks}\,\nabla_{ks}\,g_{00}-\frac{g^{-1}_{00}}{4}\sum^{3}_{s=1}g^{ks}\,\nabla_{k}\,g_{00}\,\nabla_{s}\,g_{00}\,+\\+\frac{1}{2}\,g^{-1}_{00}\,\frac{\partial g_{00}}{\partial x^{0}}\,b^{k}_{k}-\frac{\partial b^{k}_{k}}{\partial x^{0}}-\sum^{3}_{q=1}b^{k}_{q}\,b^{q}_{k}.\end{aligned}\tag{4.28}$$

The last case is the case where $i=0$, $j=0$, and $k=0$ in (4.16):

$$r^{0}_{000}=0.\tag{4.29}$$

Applying (4.28) and (4.29) to (4.14), we derive

$$\begin{aligned}r_{00}=\frac{1}{2}\sum^{3}_{k=1}\sum^{3}_{s=1}g^{ks}\,\nabla_{ks}\,g_{00}-\frac{g^{-1}_{00}}{4}\sum^{3}_{k=1}\sum^{3}_{s=1}g^{ks}\,\nabla_{k}\,g_{00}\,\nabla_{s}\,g_{00}\,+\\+\frac{1}{2}\,g^{-1}_{00}\,\frac{\partial g_{00}}{\partial x^{0}}\sum^{3}_{k=1}b^{k}_{k}-\sum^{3}_{k=1}\frac{\partial b^{k}_{k}}{\partial x^{0}}-\sum^{3}_{k=1}\sum^{3}_{q=1}b^{k}_{q}\,b^{q}_{k}.\end{aligned}\tag{4.30}$$

The four-dimensional scalar curvature r is calculated through the four-dimensional Ricci tensor (4.14) by means of the formula

$$r=\sum^{3}_{i=0}\sum^{3}_{j=0}r_{ij}\,G^{ij},\tag{4.31}$$

see § 8 in Chapter IV of [10]. Since the matrix (3.1) is block-diagonal, (4.31) yields

$$r=r_{00}\,g^{-1}_{00}-\sum^{3}_{i=1}\sum^{3}_{j=1}r_{ij}\,g^{ij}.\tag{4.32}$$

Applying (4.22) and (4.30) to (4.32) and taking into account (4.10), we derive

$$\begin{aligned}r=g^{-2}_{00}\,\frac{\partial g_{00}}{\partial x^{0}}\sum^{3}_{k=1}b^{k}_{k}+g^{-1}_{00}\sum^{3}_{k=1}\sum^{3}_{q=1}g^{kq}\,\nabla_{kq}\,g_{00}\,-\\-\frac{g^{-2}_{00}}{2}\sum^{3}_{k=1}\sum^{3}_{q=1}g^{kq}\,\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}-2\,g^{-1}_{00}\sum^{3}_{k=1}\frac{\partial b^{k}_{k}}{\partial x^{0}}\,-\\-R-g^{-1}_{00}\sum^{3}_{k=1}\sum^{3}_{q=1}b^{k}_{q}\,b^{q}_{k}-g^{-1}_{00}\sum^{3}_{k=1}\sum^{3}_{q=1}b^{k}_{k}\,b^{q}_{q}.\end{aligned}\tag{4.33}$$

The three-dimensional scalar curvature R in (4.33) is given by the formula

$$R=\sum^{3}_{i=1}\sum^{3}_{j=1}R_{ij}\,g^{ij}.\tag{4.34}$$

The formula (4.34) is an analog of the four-dimensional formula (4.31).

Now we are ready to rewrite Einstein's equations (4.1) in $3+1$ presentation. They are subdivided into three groups. The first group is written as

$$\begin{gathered}
\frac{g_{00}^{-2}}{2}\left(g_{ij}\sum_{k=1}^{3}b^{k}_{k}-b_{ij}\right)\frac{\partial g_{00}}{\partial x^{0}}+\frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\left(g^{kq}\,g_{ij}-\delta^{k}_{i}\,\delta^{q}_{j}\right)\nabla_{kq}\,g_{00}-\\
-\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\left(g^{kq}\,g_{ij}-\delta^{k}_{i}\,\delta^{q}_{j}\right)\nabla_{k}\,g_{00}\,\nabla_{q}\,g_{00}+\\
+g_{00}^{-1}\left(\frac{\partial b_{ij}}{\partial x^{0}}-\sum_{k=1}^{3}\frac{\partial b^{k}_{k}}{\partial x^{0}}\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^{k}_{j}+b_{kj}\,b^{k}_{i})-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^{k}_{q}\,b^{q}_{k}-\right.\\
\left.-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^{k}_{k}\,b^{q}_{q}+\sum_{k=1}^{3}b^{k}_{k}\,b_{ij}\right)+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^{4}}\,T_{ij},
\end{gathered}\tag{4.35}$$

where $1\leqslant i,j\leqslant 3$. The equations (4.35) are derived from (4.1) using (4.22) and (4.33). The quantities δ^{k}_{i} and δ^{q}_{j} are Kronecker deltas.

The second group of equations derived from (4.1) is written as

$$\sum_{k=1}^{3}\nabla_{k}\,b^{k}_{i}-\sum_{k=1}^{3}\nabla_{i}\,b^{k}_{k}+\frac{1}{2}\,g_{00}^{-1}\sum_{k=1}^{3}\left(b^{k}_{k}\,\nabla_{i}\,g_{00}-b^{k}_{i}\,\nabla_{k}\,g_{00}\right)=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^{4}}\,T_{i\,0},\tag{4.36}$$

where $1\leqslant i\leqslant 3$. The equations are immediate from (4.26) or (4.27).

The third group comprises exactly one equation. It is written as

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^{k}_{q}\,b^{q}_{k}+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^{k}_{k}\,b^{q}_{q}+\frac{R}{2}\,g_{00}-\Lambda\,g_{00}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^{4}}\,T_{00}.\tag{4.37}$$

This equation (4.37) is derived from (4.1) using (4.30) and (4.33).

The equations (4.35), (4.36), and (4.37) are extensions of the equations (5.32), (5.33), and (5.34) from [1]. The further progress of the previous version of theory in [12] showed that only one of the three groups of equations (5.32), (5.33), and (5.34) is preserved in the theory. In the present version of theory we preserve two of the three groups of equations (4.35), (4.36), and (4.37). They are the equations (4.35) and (4.37). The reason is because in the present version of theory we have one more dynamic variable g_{00} in (3.2), which is associated with the equation (4.37). The other dynamic variables of the theory are the components of the three-dimensional metric (3.3). They are associated with the equations (4.35).

5. The Schwarzschild black hole metric as an example.

The Schwarzschild black hole metric is a four-dimensional metric (3.1) written in terms of the following four coordinates:

$$x^{0}=c_{\text{gr}}\,t,\qquad x^{1}=\rho,\qquad x^{2}=\theta,\qquad x^{3}=\phi.\tag{5.1}$$

The Schwarzschild black hole metric is diagonal. Its temporal component is

$$g_{00}=1-\frac{r_{\text{gr}}}{\rho}.\tag{5.2}$$

The quantity r_{gr} in (5.2) is a constant which is called the Schwarzschild gravitational radius (see § 100 in [13]). The other diagonal components of the Schwarzschild metric in (3.1) are given by the formulas

$$g_{11}=\frac{1}{1-\dfrac{r_{\text{gr}}}{\rho}}\,,\qquad\qquad g_{22}=\rho^2,\qquad\qquad g_{33}=\rho^2\,\sin^2(\theta).\tag{5.3}$$

The non-diagonal components of the Schwarzschild metric in (3.1) are zero.

The Schwarzschild metric is a stationary metric. This means that its components (5.2) and (5.3) do not depend on the time variable $x^0=c_{\text{gr}}\,t$ in (5.1). Therefore from (4.10) for the Schwarzschild metric we derive

$$b_{ij}=0.\tag{5.4}$$

By means of direct calculations one can find that the four-dimensional Ricci tensor of the Schwarzschild metric is identically zero:

$$r_{ij}=0.\tag{5.5}$$

The same is true for the four-dimensional scalar curvature:

$$r=0.\tag{5.6}$$

Due to (5.5) and (5.6) the Schwarzschild metric with the components (5.2) and (5.3) is a solution of the Einstein equations (4.1) with $\Lambda=0$ and $T_{ij}=0$.

In the 3D-brane universe paradigm the three-dimensional metric (5.3) and the scalar field (5.1) are treated as two separate entities both describing gravity. Non-zero components of the metric connection associated with the 3D metric (5.3) are given by the following formulas:

$$\begin{aligned}&\Gamma^1_{11}=\frac{r_{\text{gr}}}{2\,\rho\,(r_{\text{gr}}-\rho)},&&\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{\rho},&&\Gamma^1_{22}=r_{\text{gr}}-\rho,&&\Gamma^3_{23}=\cot\theta,\\&\Gamma^1_{33}=(r_{\text{gr}}-\rho)\,\sin^2\theta,&&\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{\rho},&&\Gamma^2_{33}=-\frac{\sin(2\,\theta)}{2},&&\Gamma^3_{32}=\cot\theta,\end{aligned}\tag{5.7}$$

Using (5.7) and applying the formulas (4.18) and (4.19), we can calculate the three-dimensional Ricci tensor for the metric (5.3). It is presented by the diagonal 3×3 matrix whose diagonal elements are given by the following formulas:

$$R_{11}=\frac{r_{\text{gr}}}{\rho^2\,(r_{\text{gr}}-\rho)}\,,\qquad\qquad R_{22}=\frac{r_{\text{gr}}}{2\,\rho}\,,\qquad\qquad R_{33}=\frac{r_{\text{gr}}\,\sin^2\theta}{2\,\rho}\,.\tag{5.8}$$

Using (5.8) and applying the formula (4.34), we can calculate the three-dimensional scalar curvature associated with the metric (5.3). It turns out to be zero:

$$R=0.\tag{5.9}$$

In (4.35) we see the gradient of the function g_{00}. Using (5.2), we can calculate the components of this gradient in the coordinates ρ, θ, ϕ from (5.1):

$$\nabla_1 g_{00}=\frac{r_{\text{gr}}}{\rho^2}\,,\qquad\qquad \nabla_2 g_{00}=0,\qquad\qquad \nabla_3 g_{00}=0.\tag{5.10}$$

Apart from the gradient of g_{00}, in (4.35) we see the double gradient of the scalar function g_{00}. Its components are calculated with the use of (5.10) and (5.7). The double gradient $\nabla_{ij}\,g_{00}$ is presented by the diagonal 3×3 matrix

$$\nabla_{ij}\,g_{00}=\begin{Vmatrix}\dfrac{(4\,\rho-3\,r_{\text{gr}})}{2\,(r_{\text{gr}}-\rho)\,\rho^3} & 0 & 0\\ 0 & \dfrac{r_{\text{gr}}\,(r_{\text{gr}}-\rho)}{\rho^2} & 0\\ 0 & 0 & \dfrac{r_{\text{gr}}\,(r_{\text{gr}}-\rho)\,\sin^2\theta}{\rho^2}\end{Vmatrix}.\tag{5.11}$$

The gradient term and the double gradient term in (4.35) are given by the formulas

$$\begin{aligned}A_{ij}&=\frac{g_{00}^{-2}}{4}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j\Bigr)\,\nabla_k\,g_{00}\,\nabla_q\,g_{00},\\ B_{ij}&=\frac{g_{00}^{-1}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}\Bigl(g^{kq}\,g_{ij}-\delta^k_i\,\delta^q_j\Bigr)\,\nabla_{kq}\,g_{00}.\end{aligned}\tag{5.12}$$

The terms (5.12) are given by diagonal matrices. Using (5.10) and (5.11), we get

$$A_{ij}=\begin{Vmatrix}0 & 0 & 0\\ 0 & \dfrac{r_{\text{gr}}\,(r_{\text{gr}}-\rho)}{\rho^2} & 0\\ 0 & 0 & \dfrac{r_{\text{gr}}\,(r_{\text{gr}}-\rho)\,\sin^2\theta}{\rho^2}\end{Vmatrix},\tag{5.13}$$

$$B_{ij}=\begin{Vmatrix}\dfrac{r_{\text{gr}}}{(r_{\text{gr}}-\rho)\,\rho^2} & 0 & 0\\ 0 & \dfrac{r_{\text{gr}}\,(3\,r_{\text{gr}}-2\,\rho)}{4\,(r_{\text{gr}}-\rho)\,\rho} & 0\\ 0 & 0 & \dfrac{r_{\text{gr}}\,(3\,r_{\text{gr}}-2\,\rho)\,\sin^2\theta}{4\,(r_{\text{gr}}-\rho)\,\rho}\end{Vmatrix}.\tag{5.14}$$

Now we are ready to verify the equations (4.35), (4.36), and (4.37) for the Schwarzschild metric in the 3D-brane universe paradigm. Due to (5.4) the components of the tensor field $\mathbf{b}$ in (4.35), (4.36), and (4.37) do vanish. Therefore the equations (4.36) are fulfilled provided $T_{i\,0}=0$.

Applying (5.9) and (5.4) we conclude that the equation (4.37) is fulfilled provided $T_{00}=0$ and provided we assume that $\Lambda=0$.

Due to (5.4), (5.9), and (5.12) the equations (4.35) reduce to the following ones:

$$B_{ij}-A_{ij}+R_{ij}+\Lambda\,G_{ij}=\frac{8\,\pi\,\gamma}{c_{\text{gr}}^4}\,T_{ij}.\tag{5.15}$$

Applying (5.13), (5.14) and (5.8) to (5.15), we conclude that the equations (4.35) are fulfilled provided $T_{ij} = 0$ and provided we assume that $\Lambda = 0$. The ultimate result is formulated in the following theorem.

Theorem 5.1. *The diagonal Schwarzschild's black hole metric with the spacial components* (5.3) *and with the temporal component* (5.2) *provides a solution for the 3D-brane gravity equations* (4.35) *and* (4.37) *for the empty space, i. e. for* $T_{ij} = 0$ *and* $T_{00} = 0$*, in the coordinates* (5.1) *in cosmology with zero cosmological constant.*

The equations (4.36) are excluded from the theorem 5.1 since they are excluded from the theory of gravity in the 3D-brane universe paradigm. This exclusion makes the 3D-brane universe model an alternative non-Einsteinian theory of gravity.

6. Time scaling invariance.

As we noted in Section 2 of the present paper, in the absence of the equidistance postulate 1.1 the choice of the brane time in (2.1) is not unique. However the extent of non-uniqueness is not that large. Provided some comoving coordinates x^1, x^2, x^3 are chosen and fixed, it is given by the following time scaling transformations:

$$\tilde{t} = \tilde{t}(t), \qquad\qquad t = t(\tilde{t}). \tag{6.1}$$

Here $\tilde{t}(t)$ and $t = t(\tilde{t})$ are two strictly monotonic, increasing, differentiable, and mutually inverse functions. Applying (2.1) to (6.1), we get

$$\tilde{x}^0 = \tilde{x}^0(x^0), \qquad\qquad x^0 = x^0(\tilde{x}^0). \tag{6.2}$$

The transformations (6.2) can be extended to mutually inverse coordinate transformations in the four-dimensional spacetime:

$$\begin{cases} \tilde{x}^0 = \tilde{x}^0(x^0), \\ \tilde{x}^1 = x^1, \\ \tilde{x}^2 = x^2, \\ \tilde{x}^3 = x^3, \end{cases} \qquad\qquad \begin{cases} x^0 = x^0(\tilde{x}^0), \\ x^1 = \tilde{x}^1, \\ x^2 = \tilde{x}^2, \\ x^3 = \tilde{x}^3. \end{cases} \tag{6.3}$$

The four-dimensional metric (3.1) obeys the standard tensorial transformation rule with respect to the coordinate transformations (6.3):

$$G_{ij} = \sum_{k=0}^{3} \sum_{q=0}^{3} \frac{\partial \tilde{x}^k}{\partial x^i} \, \frac{\partial \tilde{x}^q}{\partial x^j} \, \tilde{G}_{kq}, \tag{6.4}$$

see (4.7) and (5.2) in Chapter III of [10]. The same tensorial rule applies to the components of the energy-momentum tensor:

$$T_{ij} = \sum_{k=0}^{3} \sum_{q=0}^{3} \frac{\partial \tilde{x}^k}{\partial x^i} \, \frac{\partial \tilde{x}^q}{\partial x^j} \, \tilde{T}_{kq}. \tag{6.5}$$

Due to the special form of the coordinate transformations (6.3) it preserves the block-diagonal structure of the matrix (3.1) in (6.4). The relationships (6.4) subdivide into spatial and temporal parts. In their spacial part we have

$$g_{ij}(x^0,x^1,x^2,x^3)=\tilde{g}_{ij}(\tilde{x}^0(x^0),x^1,x^2,x^3),\qquad 1\leqslant i,j\leqslant 3. \tag{6.6}$$

The temporal part of (6.4) looks slightly different:

$$g_{00}(x^0,x^1,x^2,x^3)=(\tilde{x}^0(x^0)')^2\,\tilde{g}_{00}(\tilde{x}^0(x^0),x^1,x^2,x^3). \tag{6.7}$$

Let's denote through ξ the derivative of the function $\tilde{x}^0(x^0)$ in (6.3). Then the formulas (6.6) and (6.7) are rewritten as follows:

$$g_{00}=\xi^2\,\tilde{g}_{00},\qquad\qquad g_{ij}=\tilde{g}_{ij}. \tag{6.8}$$

Unlike (6.4), the relationships (6.5) subdivide into three parts. Two of them are

$$T_{00}=\xi^2\,\tilde{T}_{00},\qquad\qquad T_{ij}=\tilde{T}_{ij}\ \text{ for }\ 1\leqslant i,j\leqslant 3. \tag{6.9}$$

The third part of the relationships (6.5) is written as

$$T_{i0}=\xi\,\tilde{T}_{i0}\ \text{ and }\ T_{0i}=\xi\,\tilde{T}_{0i}\ \text{ for }\ 1\leqslant i\leqslant 3. \tag{6.10}$$

The transformations (6.8), (6.9), and (6.10) can be extended to all terms in the gravity equations (4.35), (4.36), and (4.37). From (6.8) we derive

$$g^{ij}=\tilde{g}^{ij}. \tag{6.11}$$

Then, applying (6.8) and (6.11) to (4.10), we get

$$b_{ij}=\xi\,\tilde{b}_{ij},\qquad\qquad b^k_q=\xi\,\tilde{b}^k_q. \tag{6.12}$$

Differentiating the first relationship in (6.8) with respect to x^0, we get

$$\frac{\partial g_{00}}{\partial x^0}=\xi^3\,\frac{\partial\tilde{g}_{00}}{\partial\tilde{x}^0}+2\,\xi\,\xi'\,\tilde{g}_{00}. \tag{6.13}$$

Similarly, differentiating (6.12) with respect to x^0, we derive

$$\frac{\partial b_{ij}}{\partial x^0}=\xi^2\,\frac{\partial\tilde{b}_{ij}}{\partial\tilde{x}^0}+\xi'\,\tilde{b}_{ij},\qquad\qquad \frac{\partial b^k_q}{\partial x^0}=\xi^2\,\frac{\partial\tilde{b}^k_q}{\partial\tilde{x}^0}+\xi'\,\tilde{b}^k_q. \tag{6.14}$$

The next step is to apply the second relationship (6.8), (6.3), and (6.11) to (4.4). As a result we derive a transformation rule for the connection components:

$$\Gamma^k_{ij}=\tilde{\Gamma}^k_{ij}. \tag{6.15}$$

The transformation rule (6.15) along with (6.3) yields

$$\nabla_i\,g_{00}=\xi^2\,\nabla_i\,\tilde{g}_{00},\qquad\qquad \nabla_{ij}\,g_{00}=\xi^2\,\nabla_{ij}\,\tilde{g}_{00}. \tag{6.16}$$

Similarly, applying (6.15) and (6.3) to (6.12), we derive

$$\nabla_{i} b_{kq} = \xi\, \nabla_{i} \tilde{b}_{ij}, \qquad\qquad \nabla_{i} b^{k}_{q} = \xi\, \nabla_{i} \tilde{b}^{k}_{q}. \tag{6.17}$$

The transformations (6.3), (6.8), (6.11), and (6.15) applied to the relationships (4.18), (4.19), and (4.34) yield the following formulas:

$$R_{ij} = \tilde{R}_{ij}, \qquad\qquad R = \tilde{R}. \tag{6.18}$$

No we a ready to formulate the following theorem. Its proof is direct calculations.

Theorem 6.1. *The gravitational field equations* (4.35) *and* (4.37) *of the 3D-brane universe model are invariant with respect to the transformations* (6.3)*,* (6.8)*,* (6.9)*,* (6.10)*,* (6.11)*,* (6.12)*,* (6.13)*,* (6.14)*,* (6.15)*,* (6.16)*,* (6.17)*, and* (6.18) *induced by the brane time scaling* (6.1)*.*

Note that the equations (4.36) are also invariant with respect to the above mentioned transformations. However we do not include them to the theorem 6.1 since we are going to exclude them from the theory at all.

7. Conclusions.

The gravitational field equations (4.35) and (4.37) constitute the main result of the present paper. They are written in special coordinates x^0, x^1, x^2, x^3, three of which x^1, x^2, x^3 are comoving coordinates and the fourth one $x^0 = c_{\text{gr}}\, t$ is a brane time coordinate. The existence of such coordinates is based on the idea that the four-dimensional spacetime should be considered as a dense foliation of 3D-branes. This idea was first suggested and motivated in [1]. In [1] and in the succeeding papers [12], [14], [15] [7], and [16] (see also [17], [18], and [19]) the idea was complemented with the equidistance postulate 1.1. As a result an alternative non-Einsteinian theory of gravity was developed. The name of this theory is **3D-brane universe model**.

In the present paper we start a new extended version of the theory excluding the equidistance postulate from it. From the standard relativity this new version of theory inherits Schwarzschild's black hole metric becoming an example of solution for the extended gravity equations (4.35) and (4.37) (see Section 5 above). Along with the solution for gravity equations, the new extended version of theory gains a new feature (or maybe an obstruction) that consists in non-uniqueness of the global brane time. This feature is discussed in Sections 2 and 6 of the present paper.

8. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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