Gravitational Wave Colliding with a Small Mass Having Path Not Approximately a Geodesic

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Abstract

We consider a system of a gravitational plane wave pulse colliding with a point mass of small mass. The path of the mass is shown not to be approximately a geodesic.

1 Gravitational plane wave pulse metric

Define \( u = t - x \) and let the metric \( g_{\mu\nu} \) of the gravitational plane wave pulse be determined by [1]

\[
ds^2 = -dt^2 + dx^2 + [L(u)]^2 \left[ e^{2\beta(u)} dy^2 + e^{-2\beta(u)} dz^2 \right]
\]

and \( g_{\mu\nu}(u) = \eta_{\mu\nu} \) for \( u < 0 \). From the equation \( R_{\mu\nu} = 0 \) the only relation between \( L \) and \( \beta \) is

\[
\frac{d^2 L}{du^2}(u) + \left( \frac{d\beta}{du}(u) \right)^2 L(u) = 0
\]

Let \( L(0) = 1 \) and \( \beta \neq 0 \). We then have by (2) that \( L(u) \) will decrease and become zero at some point \( u_0 > 0 \). Consequently \( g_{22}(u) > 0 \) for \( u < u_0 \).

2 Proper Lorentz transformation

Consider a coordinate transformation from \( t, x, y, z \) to \( t', x', y', z' \) coordinates that is a composition of a rotation by \( \theta \) about the \( z \) axis followed by a boost by \( 2\cos(\theta)/(1 + \cos^2(\theta)) \) in the \( x \) direction followed by a rotation by \( \theta + \pi \) about the \( z \) axis. For \( \theta/\pi \) not an integer this is a proper Lorentz transformation such that

\[
t' = t' \left( 1 + 2 \cot^2(\theta) \right) - 2 x' \cot^2(\theta) + 2 y' \cot \theta
\]

\[
x = 2 t' \cot^2(\theta) + x' \left( 1 - 2 \cot^2(\theta) \right) + 2 y' \cot \theta
\]

\[
y = 2 t' \cot \theta - 2 x' \cot \theta + y'
\]

\[
z = z'
\]

By (3) and (4) we have \( u = t - x = t' - x' = u' \). For the metric (1) and transformation (3)-(6) define the metric \( g'_{\mu\nu}(u') \) by

\[
g'_{\mu\nu}(u') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \eta_{\alpha\beta}(u)
\]

hence we get

\[
ds^2 = \left\{ 1 - 4[1 - g_{22}(u')] \cot^2(\theta) \right\} dt'^2 + \left[ 8[1 - g_{22}(u')] \cot^2(\theta) \right] dx'^2 - 4\left[ 1 - g_{22}(u') \right] \cot \theta dx' dy'
\]

\[
+ \left\{ 1 - 4[1 - g_{22}(u')] \cot^2(\theta) \right\} dx'^2 - 4[1 - g_{22}(u')] \cot \theta dy' dz'
\]

\[
+ 4[1 - g_{22}(u')] \cot \theta dx' dy' + g_{22}(u') dy'^2 + g_{33}(u') dz'^2
\]

Since \( g_{\mu\nu} = \eta_{\mu\nu} \) for \( u < 0 \) we have \( g'_{\mu\nu}(u') = \eta_{\mu\nu} \) for \( u' < 0 \). The metric \( g'_{\mu\nu}(u') \) satisfying \( R'_{\mu\nu} = 0 \) and \( g'_{\mu\nu}(u') = \eta_{\mu\nu} \) for \( u' < 0 \) is then also the metric of a gravitational plane wave pulse.
3 A geodesic of the metric $g'_{\mu\nu}$

The curve

$$t'(\lambda) = (1 + 2 \cot^2 \theta)\lambda - 2 \cot^2 \theta \int_0^\lambda \frac{dw}{g_{22}(w)}$$  \hspace{1cm} (9)

$$x'(\lambda) = 2 \cot^2 \theta \lambda - 2 \cot^2 \theta \int_0^\lambda \frac{dw}{g_{22}(w)}$$  \hspace{1cm} (10)

$$y'(\lambda) = -2 \cot \theta \lambda + 2 \cot \theta \int_0^\lambda \frac{dw}{g_{22}(w)}$$  \hspace{1cm} (11)

$$z'(\lambda) = 0$$  \hspace{1cm} (12)

is a geodesic of the metric $g'_{\mu\nu}(u')$. We have the figure

\[\text{Fig.} \text{Q} \]

4 Path of particle is not approximately a geodesic

Consider a system of a gravitational plane wave pulse that collides with a point mass $A$ initially at rest at the origin. Let $\tilde{g}_{\mu\nu}(t', x', y', z')$ be the metric of the combined system of wave and $A$. The wave comes from infinity so for points having large negative $t'$ and $x' < t'$ the wave is far from $A$ and so is little affected by $A$. Consequently $\tilde{g}_{\mu\nu}(t', x', y', z')$ is approximately $g'_{\mu\nu}(t' - x')$ at points having large negative $t'$ and $x' < t'$. Now $g'_{\mu\nu}(t' - x')$ is finite at all points hence $\tilde{g}_{\mu\nu}(t', x', y', z')$ is finite at points having large negative $t'$ and $x' < t'$.

Assume the path of $A$ is approximately the curve (9)-(12) for an $A$ of small mass. We then have using the figure that $A$ will reach a point $p$ having large negative $t'$ and $x' < t'$. By previous paragraph $\tilde{g}_{\mu\nu}(t', x', y', z')$ is then finite at $p$. Since $A$ is a point mass $\tilde{g}_{\mu\nu}(t', x', y', z')$ at $p$ is not finite. We then have $\tilde{g}_{\mu\nu}(t', x', y', z')$ is both finite and not finite at $p$. This is a contradiction. The path of $A$ is then not approximately a geodesic.

References


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