Novel proof of the four- color theorem

Henri Caillaud

Email : hl.caillaud@free.fr (H. Caillaud)

Affiliation: No affiliation. VILLEURBANNE, FRANCE.

ORCID: 0009-0001-2705-2615

Abstract: HAKEN and APPEL showed in their famous demonstration that the configuration consisting of two neighboring countries each having five neighbors was inevitable and irreducible. The present demonstration proposes a reduction of this configuration.

Keywords: Four color theorem, planar map, planar graph, Hamiltonian path, loop Hamiltonian graph.

Mathematical Classification (2020). 05C15.

INTRODUCTION

The four color theorem was demonstrated by Wolfgang HAKEN and Kenneth APPEL in 1976 (ref [1]). Their demonstration was strongly inspired by that of Alfred KEMPE (1879) which turned out to be false.

Their demonstration is based on two key concepts.

1) The notion of a set of “inevitable” configurations : any minimal pentachromatic map (the smallest map requiring five colors to be colored) will contain at least one of the configurations in this set. For example, the set consisting of the single configuration “a country with five neighbors” is “inevitable”. Any minimal pentachromatic map will contain “a country with five neighbors”.
2) The other notion is “reducibility”: a configuration is reducible if it can be shown that it cannot belong to a minimal pentachromatic map.

The demonstration of HAKEN and APPEL consisted in finding an “inevitable” set of “reducible” configurations, i.e. about 1500 configurations requiring a calculator to verify their reducibility. (This number was subsequently reduced a little but the calculator remained necessary ...) (ref [2])

Just as KEMPE had shown that a country with five neighbors was “inevitable” in a minimal pentachromatic map, HAKEN and APPEL showed that “two neighboring countries each with five neighbors” is an “inevitable” configuration (ref [3]) (see figure 1)

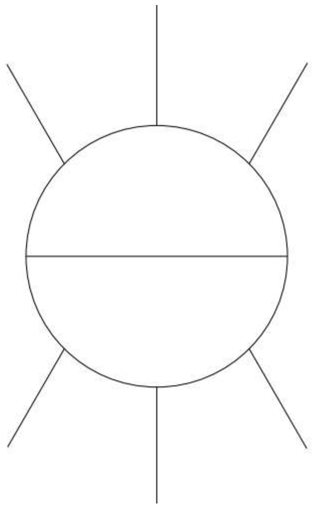


Figure 1 - Two neighboring countries each having five neighbors.

In other words, any minimal pentachromatic map will contain this configuration.

Of course this configuration was considered “irreducible” by the authors, at least with the reduction methods they used.

The whole idea of this demonstration is to propose a reduction of this configuration.

DEMONSTRATION

The planar maps we will work on are the same as those of HAKEN and APPEL, namely the normal maps of KEMPE.

A planar map is normal if at no point do more than three countries meet. Furthermore, these maps do not have any cuts of degree 0 or degree 1 (a cut being a closed line passing through borders and separating the graph formed by the map into exactly two subgraphs. The degree of the cut being the number of borders crossed).

A cut of degree 0 would indicate that the map is not connected and is actually formed of two separate maps.

A degree 1 cut would indicate that the border crossed by the cut is not a border between two countries.

KEMPE showed that if the four color theorem is true for normal maps then it is true for all planar maps.

HAKEN and APPEL worked on the “dual” maps of these normal maps (the dual of a map is obtained by replacing the countries by points and the borders between two countries by connections between these points)

These dual maps are called “triangulation”. The original map is obviously equivalent to its dual map (see figure 2).

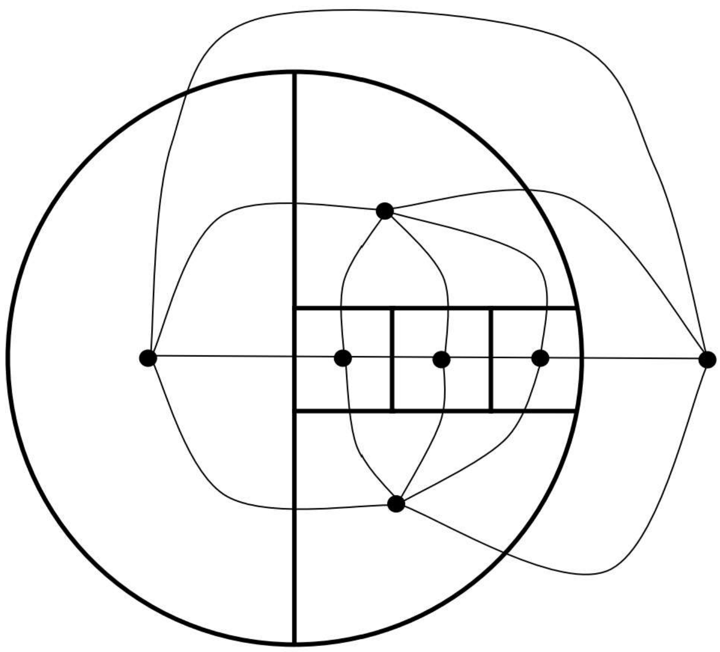


Figure 2 – A normal planar map and its dual map (triangulation)

For our part, we will work directly on the normal map rather than on its triangulation.

The first thing we're going to do is replace the property “being colorable in four colors” with “being Hamiltonian in even loops”.

A planar P3 graph (all its points have three neighbors) is Hamiltonian in even loop if it contains one or more loops consisting of an even number of points and passing through all the points of the graph (see Figure 3).

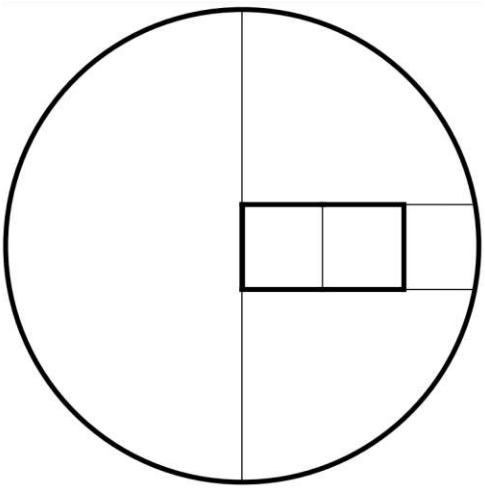


Figure 3 - Hamiltonian planar *P3 graph* in even loops.

In the planar case, we notice that this graph is easy to color in four colors. It is enough to choose two colors for the inside of the loops and the other two for the outside.

Conversely, if a planar P3 graph is colored in four colors and if we surround the groups of countries having two arbitrarily chosen colors, we then obtain even loops passing through all the points of the graph (see figure 4).

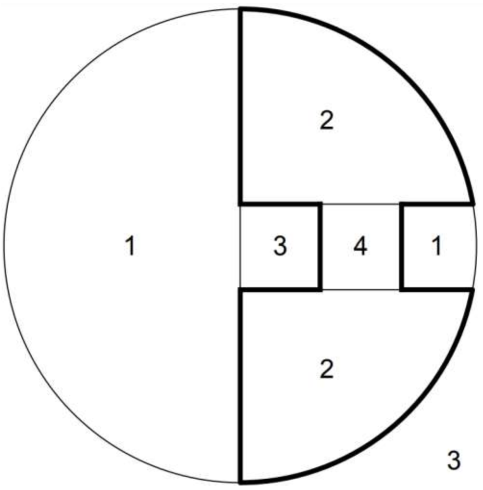


Figure 4 - Countries of color 1 and 3 are surrounded and the graph is indeed Hamiltonian in even loops.

We will no longer say "colorable in four colors" but "Hamiltonian in even loop (HEL)" assuming the equivalence of these two properties in the case of planar graphs.

There is a simple algorithm (denoted $algo_e$) to move from one set of even loops to another (see the detailed description of $algo_e$ in Appendix 1). In the case of planar graphs this algorithm corresponds to the permutation of colors (see Figure 5).

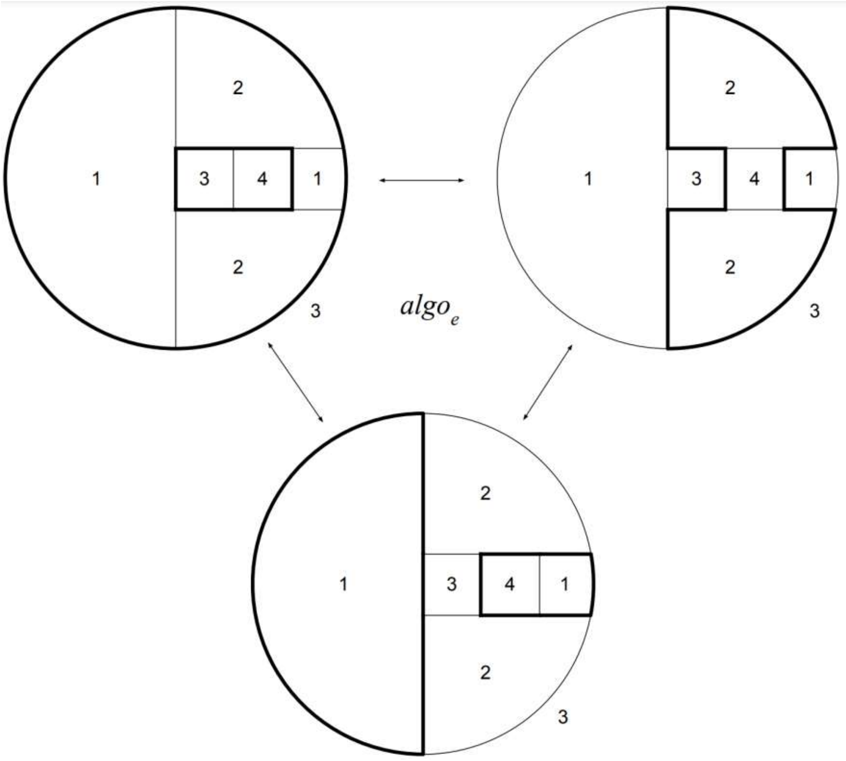


Figure 5 – Illustration of $algo_e$

Note : This algorithm can be used on any *P3 graph* having a Hamiltonian in even loops solution (HEL) even if this graph is not planar.

From now on, we will only be concerned with two colours and they concern "connections" and no longer "countries".

A connection is said to be "black" (denoted C=B) if a loop passes through it and it is said to be "white" (denoted C=W) otherwise (see figure 6).

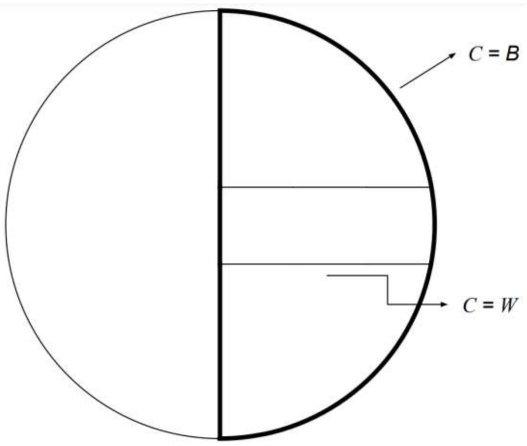
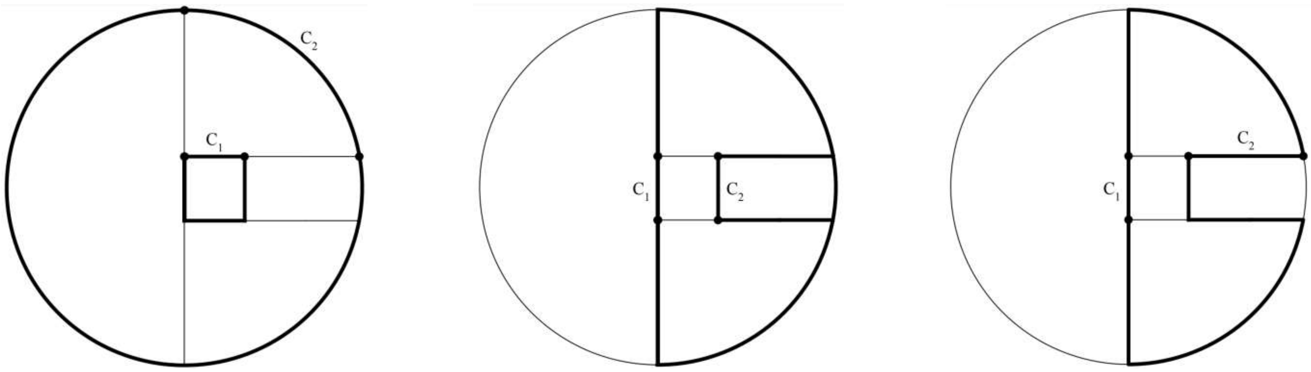


Figure 6 – Black connection and white connection.

An important property of $algo_e$ is that whatever two connections C_1 and C_2 belonging to a Hamiltonian in even loops graph *P3* there exists a particular HEL solution such that $C_1 = B$ and $C_2 = B$.

There are therefore three possible statuses for a pair of connections C_1 and C_2 in a *P3* HEL graph :

1. C_1 and C_2 each belong to a different even loop. We will say that they have a “separate e-e” status (see figure 7a).
2. C_1 and C_2 belong to the same even loop and are connected by an even path. We will say that they have an even status (see figure 7b).
3. C_1 and C_2 belong to the same even loop and are connected by an odd path. We will say that they have an odd status (see figure 7c).



(7a) “separate e-e” status (7b) “even” status (7c) “odd” status

Figure 7 – the three statuses linked to the ownership of $algo_e$

Note : A couple C_1 and C_2 can have several statuses.

We will now demonstrate a theorem that will be necessary for the demonstration of the main theorem. It is the double status theorem.

Double Status Theorem:

In a planar *P3* graph of rank less than n_0 (n_0 is the rank of the minimal map) if two non-consecutive connections C_1 and C_2 belonging to the same country have an "odd" status and if no degree 3 cut passes through these two connections then they have at least one other status among the following three statuses: "separated e-e" or "separated o-o" or "even".

The "separated o-o" status means that C_1 and C_2 each belong to a different odd loop (a graph *P3* can be HEL and have Hamiltonian in odd loops solutions (HOL)).

Demonstration:

If two connections C_1 and C_2, non-consecutive and belonging to the same country, have an odd status then consider the following transformation (see figure 8):

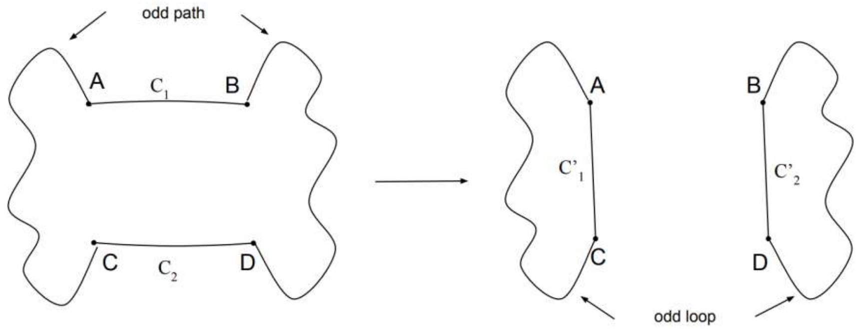
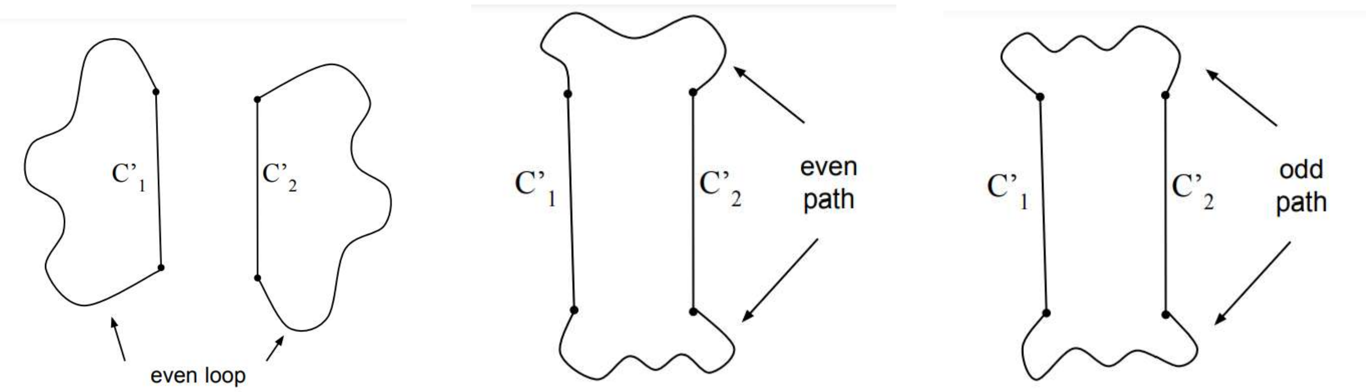


Figure 8 - Transformation

This transformation preserves planarity because C_1 and C_2 belong to the same country.

We see that this transformed graph is Hamiltonian in odd loops but since its rank has not changed and it does not have a degree 1 cut by virtue of the hypotheses, it is also HEL.

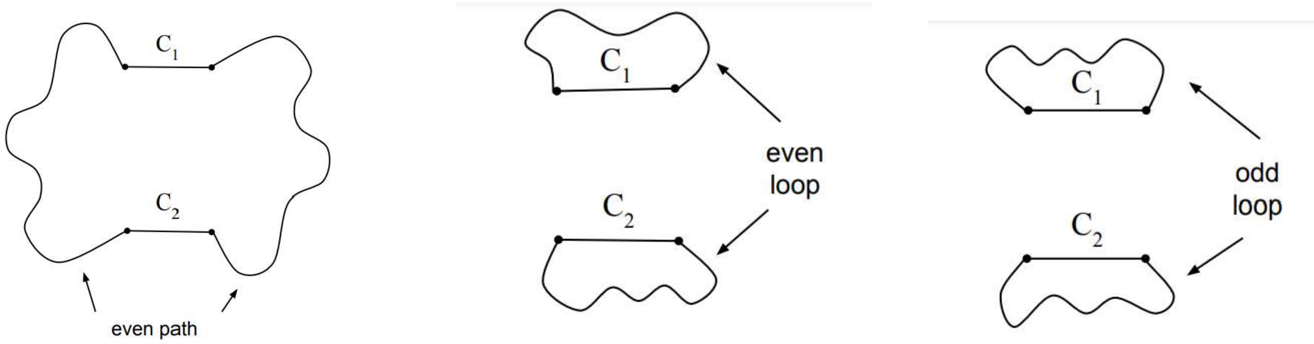
Given an HEL solution passing through C'_1 and C'_2, we therefore have the following three cases (see figure 9):



(9a) separate e-e status (9b) even status (9c) odd status

Figure 9 – The three possible statuses for C'_1 and C'_2

Which gives after the restoration of C_1 and C_2



(10a) even status (10b) separate e-e status (10c) separate o-o status

Figure 10 – the three statuses of C_1 and C_2 after recovery

We have clearly demonstrated the double status theorem (see Appendix 2 for an example of the application of this theorem).

Note : If we could have demonstrated as easily that two connections with even status could have another status among the statuses “odd”, “separated e-e” and “separated o-o” then the demonstration of the four color theorem would have been even simpler.

Let us now return to the proof of the four color theorem and present our reduction (see Figure 11):

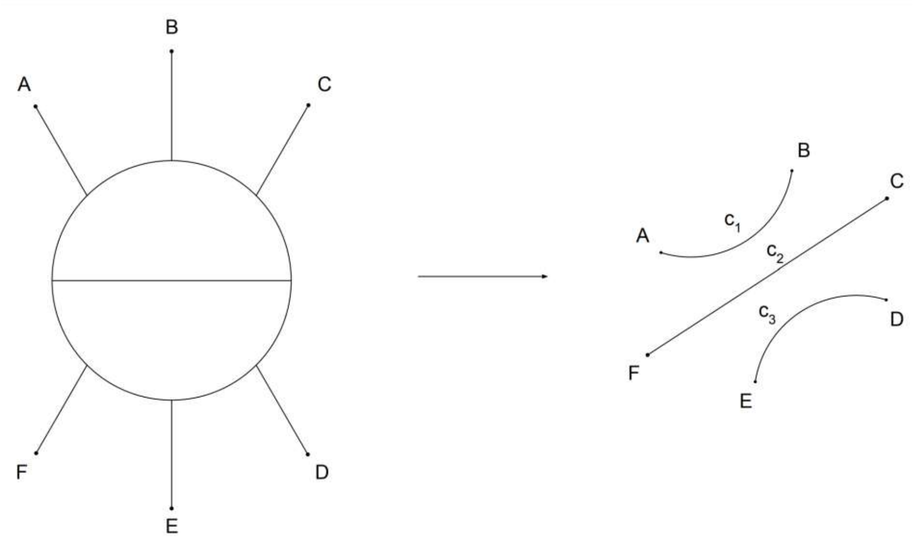


Figure 11 - we replace the configuration “two neighboring countries each having five neighbors” by the three connections AB, FC and ED

The reduced graph is obtained by removing four connections (see Figure 12).

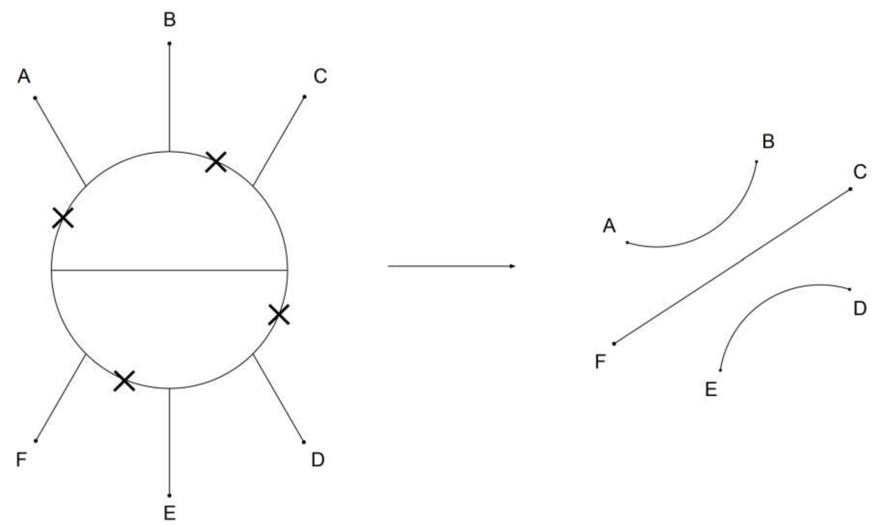


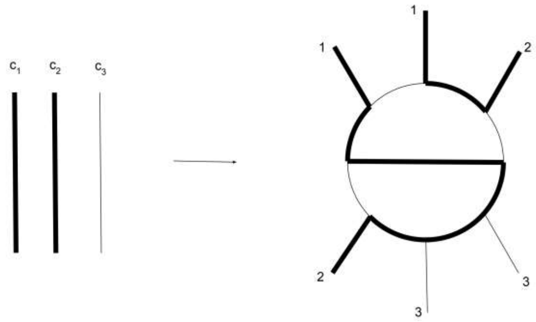
Figure 12 - The four connections deleted.

It is therefore of rank $n_0 - 4$ and is consequently HEL

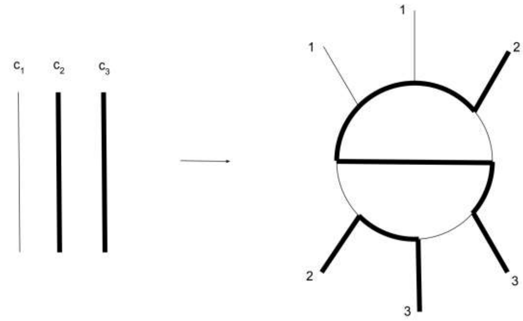
We will assume that planarity is preserved and that there are no cuts of degree 0 or degree 1.

We notice that whatever the colors (B and W) taken by C_1 , C_2 and C_3 in the reduced graph, it is always possible to “restore” the initial configuration.

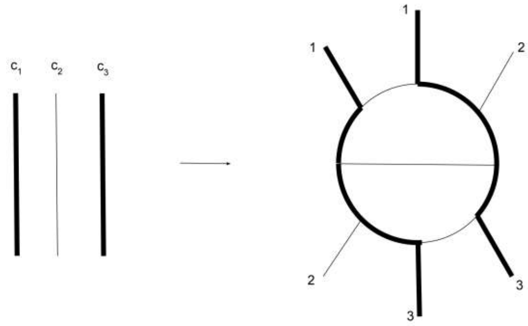
Figure 13 illustrates seven of the eight possible cases and their restoration. In the case where all three connections C_1, C_2 and C_3 are black then there are two possible restorations (see Figure 14).



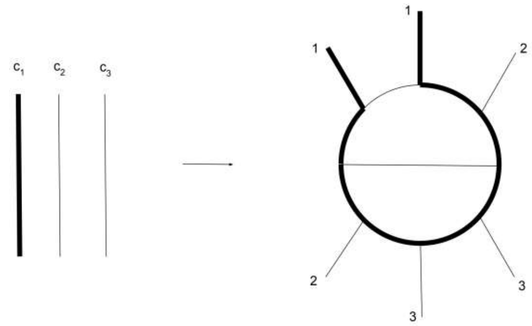
(a)



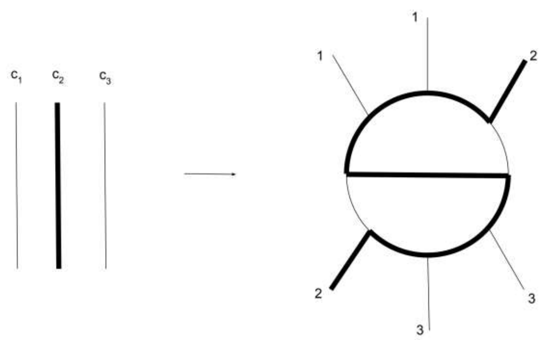
(b)



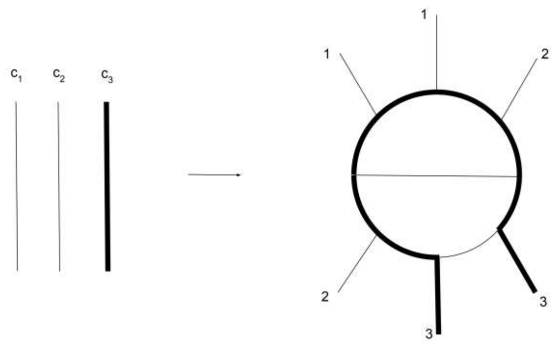
(c)



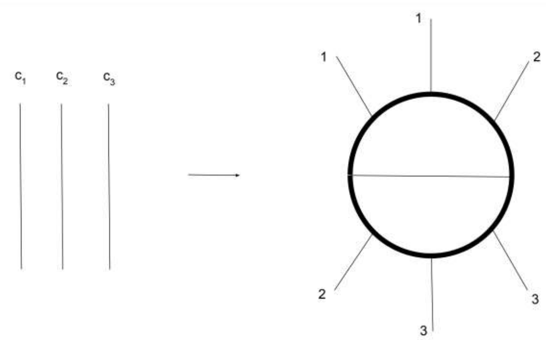
(d)



(e)

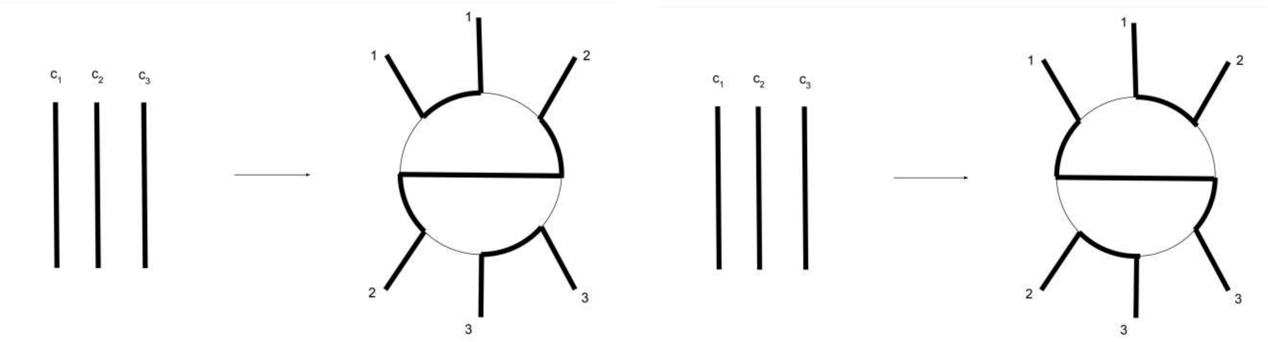


(f)



(g)

Figure 13 - recovery results.



(a) (b)

Figure 14 – The two restorations in the case of three black connections.

We see that if all these restorations were in even loops then the four color theorem would be demonstrated. But is this really the case?

There are only three cases where this recovery is not HEL. This is when C_1 = B, C_2 = B and C_3 = W, C_1 and C_2 having odd status (Figure 15), respectively C_1 = B, C_2 = W and C_3 = B, C_1 and C_3 having odd status (Figure 16) and C_1 = W, C_2 = B and C_3 = B and C_2 and C_3 having odd status (Figure 17).

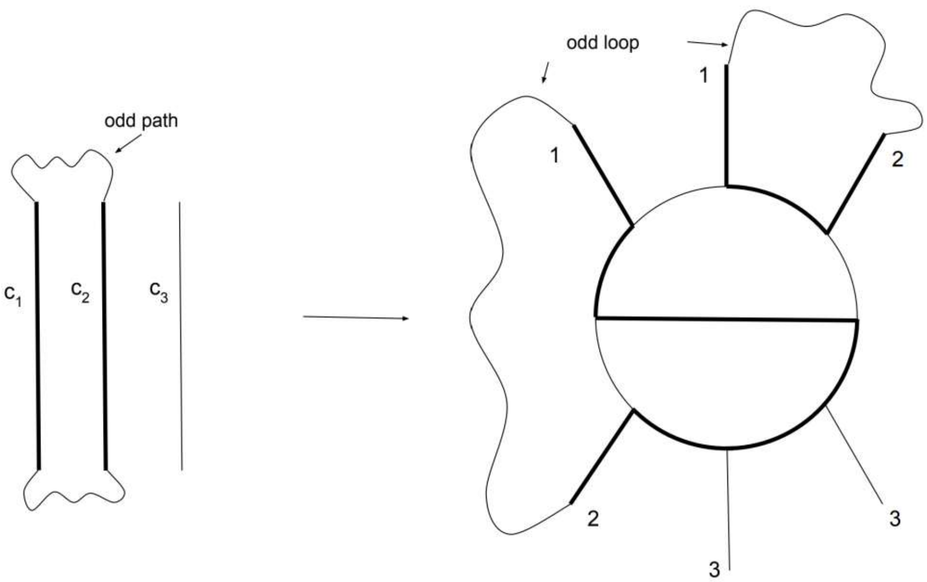


Figure 15 - Case 1: C_1 =B, C_2 =B and C_3 =W, C_1 and C_2 having an odd status and its restoration. We see that two odd loops are formed.

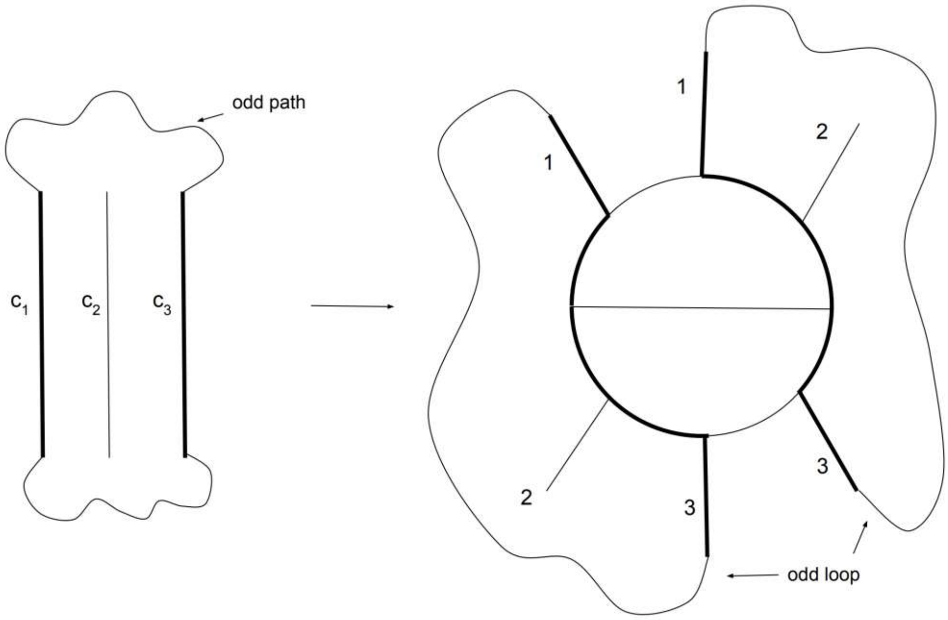


Figure 16 - Case 2: C_1 =B, C_2 =W and C_3 =B, C_1 and C_3 having an odd status and its restoration. We see that two odd loops are formed.

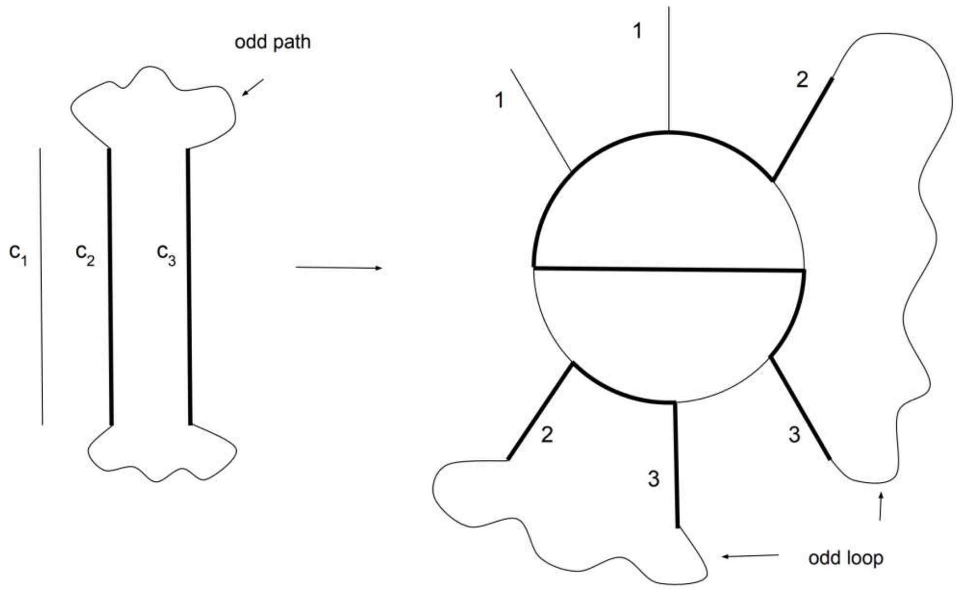


Figure 17 - Case 3: C_1 =W, C_2 =B and C_3 =B and C_2 and C_3 having an odd status and its restoration. We see that there is formation of two odd loops.

These three cases are actually the same because it is possible to move from one to the other by $algo_e$ (applied to the reduced graph) (see figure 18).

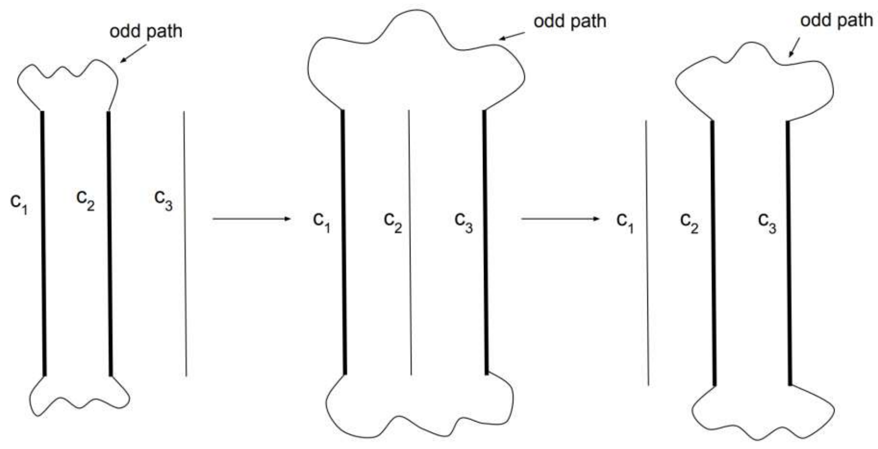


Figure 18 - According to $algo_e$ if, in case 1, we choose to keep C_1 black for the application of $algo_e$ then $algo_e$ will make C_2 white and C_3 black with C_1 and C_3 of odd status (case 2). Similarly if we now apply $algo_e$ keeping the connection C_3 black then C_1 will become white and C_2 will become black with C_2 and C_3 of odd status (case 3) (see appendix for description of $algo_e$).

Let us take for example case 1 (figure 15).

For the pair C_1 and C_2 we can invoke the double status theorem demonstrated previously and claim that C_1 and C_2 can also have one of the following three statuses:

C_1 and C_2 "even"; C_1 and C_2 "separated e-e"; C_1 and C_2 "separated o-o"

The two statuses "even" and "separated e-e" bring us back to the previous cases and the recovery will be HEL whatever the color of C_3

What about the case where C_1 and C_2 have separate o-o status ?

In this new Hamiltonian in odd loops solution C_3 can be either W or B.

If C_3 = W the recovery will be (figure 19):

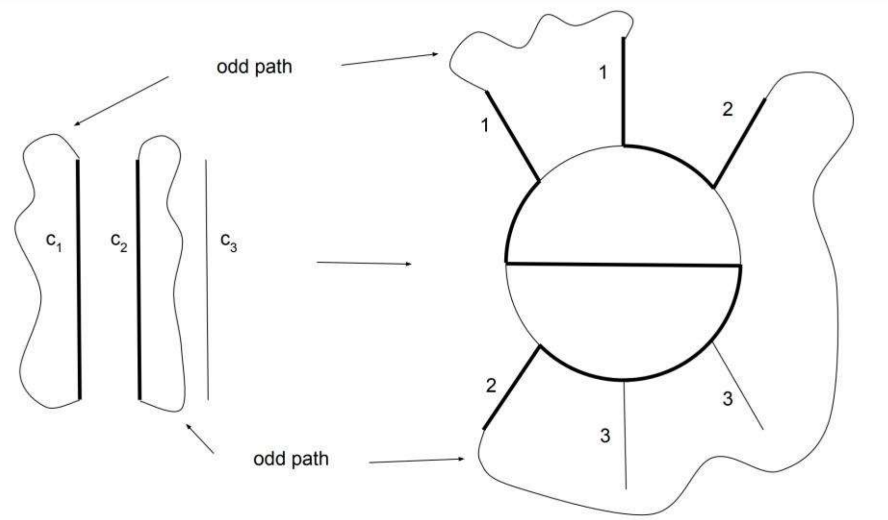


Figure 19 – Recovery if C_3 is white.

We see that the two odd paths are now connected and the recovery is therefore HEL.

If C_3 =B the recovery will be (figure 20):

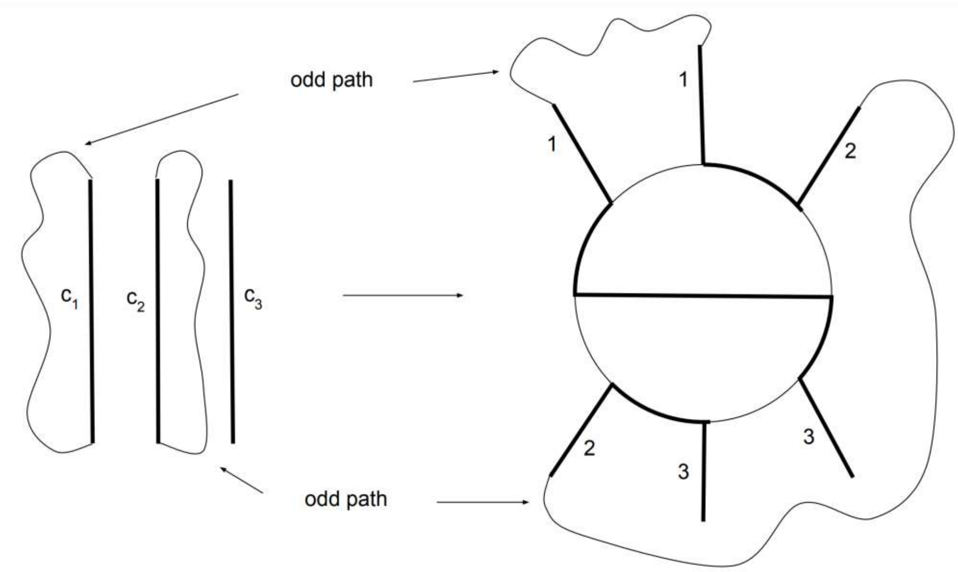


Figure 20 – Recovery if C_3 is black.

We see that this recovery is HEL.

It is possible to show that we are indeed in the conditions of application of the double status theorem for the connections C_1 and C_2 .

CONCLUSION

We have clearly demonstrated using the double status theorem that the inevitable configuration of W. HAKEN and K. APPEL consisting of two neighboring countries each having five neighboring countries is in fact reducible, which completes the demonstration of the four colors theorem.

ANNEXES

Appendix 1: Description of $algo_e$

Let there be a planar map and an HEL solution for this map.

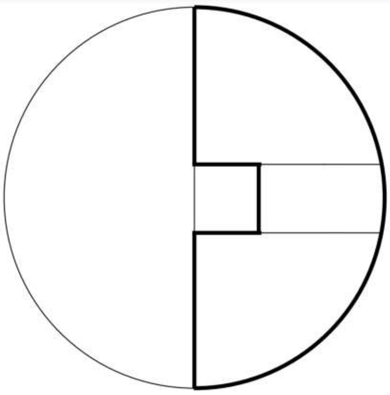


Figure A1-1 – A planar map and a HEL solution for this map.

Let's apply $algo_e$ to this HEL solution

The first step is to choose a black connection, here called C.

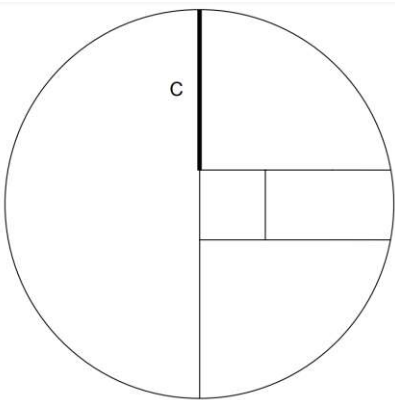


Figure A1-2 – Choice of a connection C = B.

The second step is to keep black all connections that have even status with C.

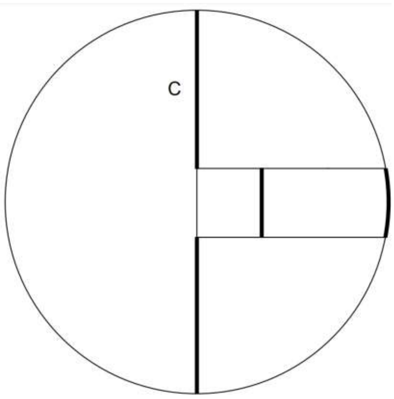


Figure A1-3 – Illustration of the second step of $algo_e$.

The third step is to make black all the connections that were white in the original HEL solution.

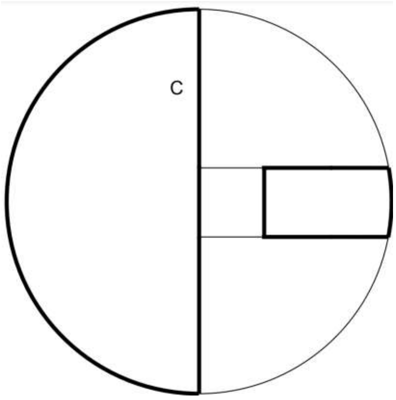


Figure A1-4 – Result of $algo_e$.

It is easy to show that this algorithm always converges to a Hamiltonian in even loops solution.

Case of multiple even loops: The first step is to choose one of the loops and choose a black connection on this loop. Connection called here C.

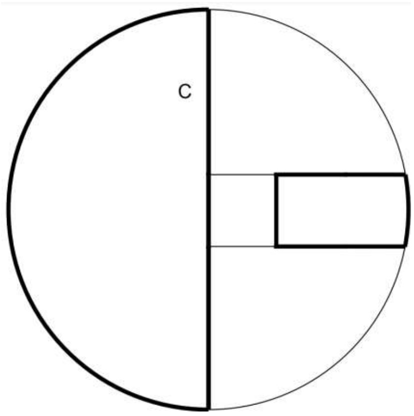


Figure A1-5 – A planar map and a HEL solution in two even loops.

The second step is to apply the procedure previously described but only to the chosen loop.

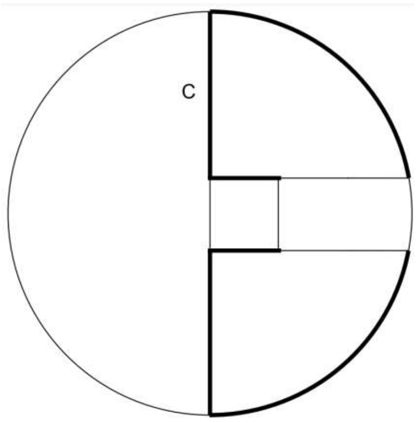


Figure A1-6 – Selection of a connection C = B in the first loop.

We see that the algorithm stops near the second loop.

The third step is to choose a black connection on the second loop of the original HEL solution (here C') and apply the algorithm to this second loop.

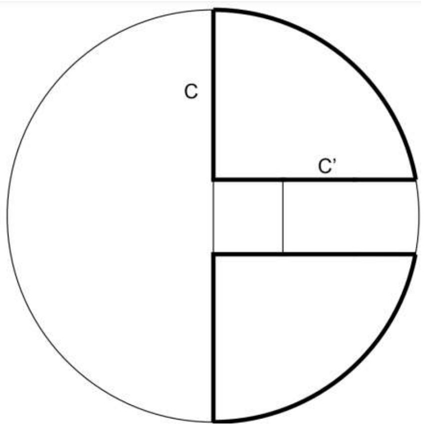


Figure A1-7 – Result of $algo_e$.

We can always choose the same connection as long as the Hamiltonian solution is in several even loops.

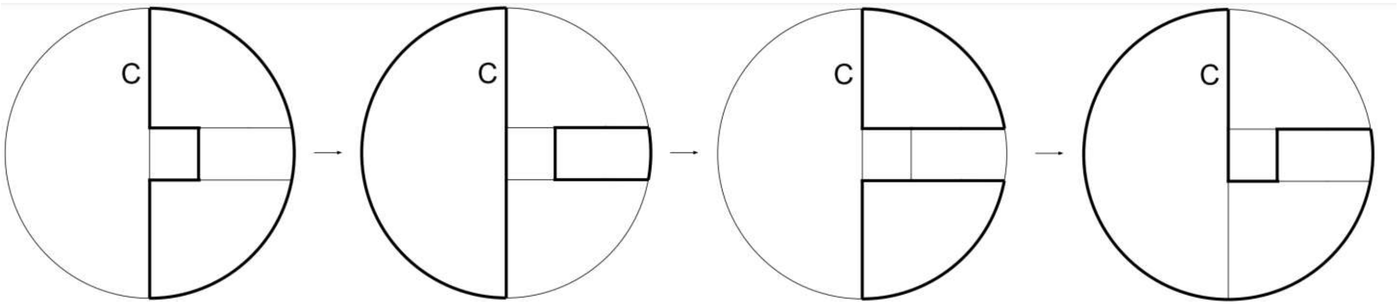


Figure A1-8 – Sequence of steps of $algo_e$ applied to the connection C.

Note : one can similarly define an $algo_o$ that can eventually converge to odd loops. Many very interesting conjectures can be made from these two algorithms in relation to the original Hamiltonian path problem.

Appendix 2: An example of applying the double status theorem to a particular planar P3 graph

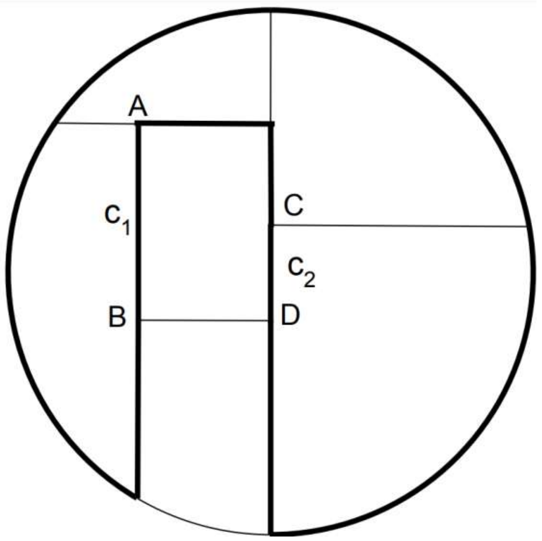


Figure A2-1 – A planar map and a HEL solution passing through C_1 and C_2.

In this planar (normal) map with a HEL solution we see that C_1 and C_2 have an odd status.

Let us apply the transformation described in the double status theorem (see Figure 8).

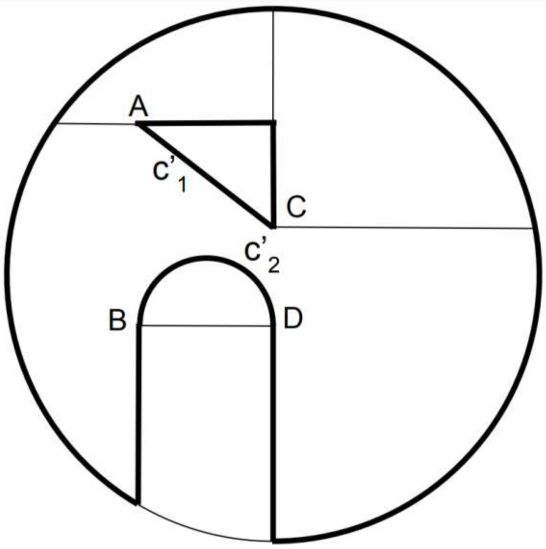


Figure A2-2 – Application of the transformation to C_1 and C_2

We see that C'_1 and C'_2 have a "separate o-o" status as expected.

This transformed graph is always planar and always of rank lower than n_0 it is therefore HEL.

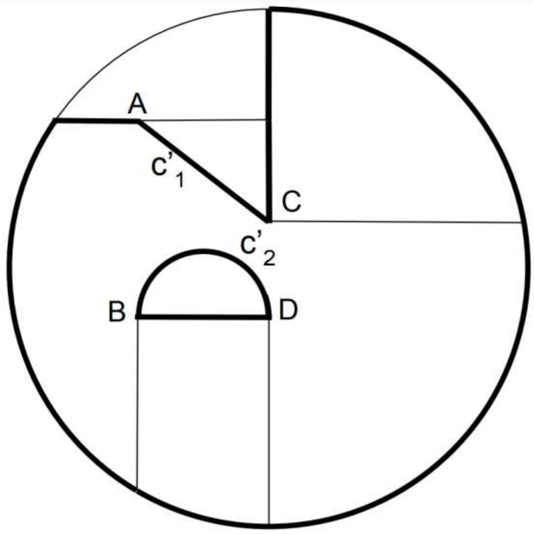


Figure A2-3 – A HEL solution of the transformed graph.

Given an HEL solution for this graph, we see that C'_1 and C'_2 now have a “separate e-e” status.

Now let's restore C_1 and C_2.

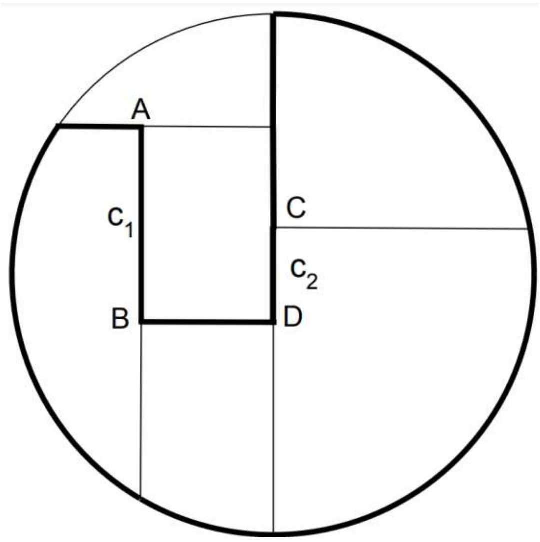


Figure A2-4 – Recovery of C_1 and C_2.

We see that C_1 and C_2 now have an even status.

Appendix 3: example of application of the reduction

Let the planar graph be assumed to be minimal pentachromatic (non-HEL of rank n_0).

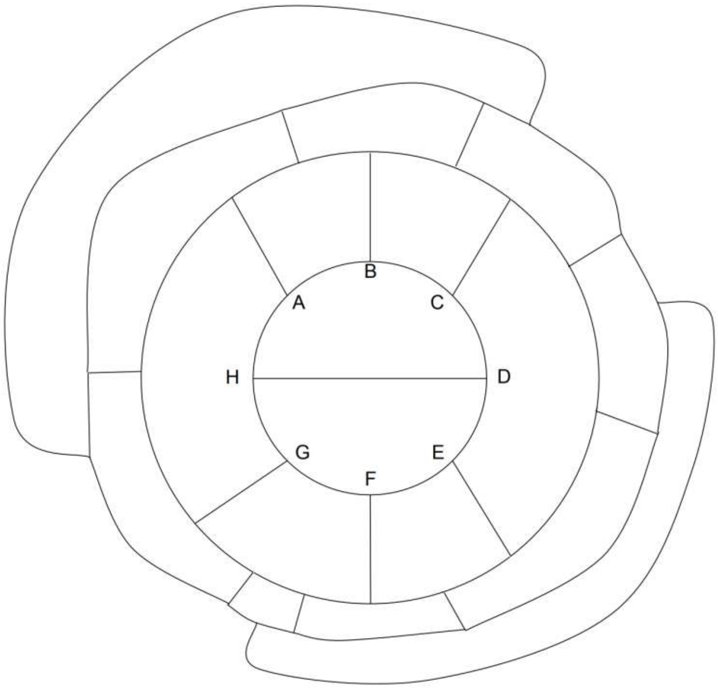


Figure A3-1 – A planar map that we assume minimal pentachromatic.

Let us apply our reduction to the loop ABCDEFGH. We obtain the following figure.

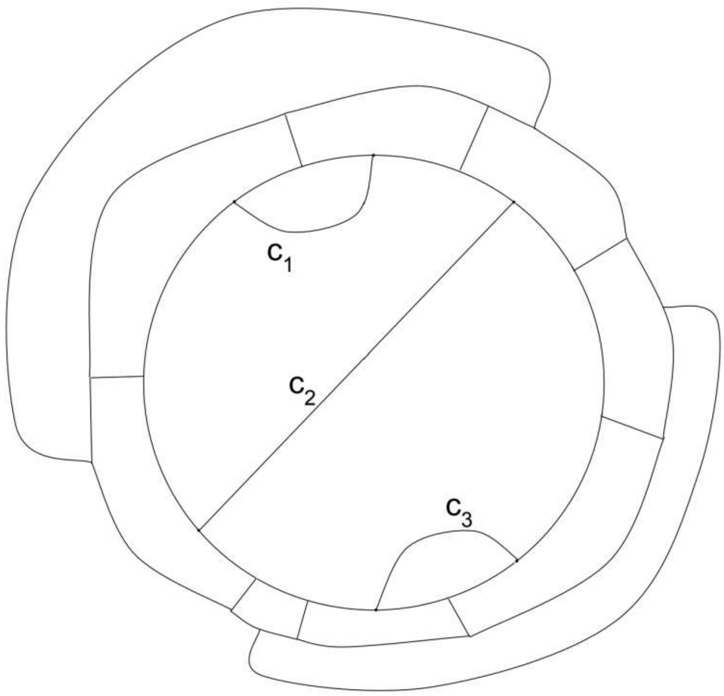


Figure A3-2 – Apply the reduction to the ABCDEFGH configuration.

We find our three connections C_1, C_2, C_3 and a planar graph of rank $n_0 - 4$ which is therefore HEL.

Let the following HEL solution be given.

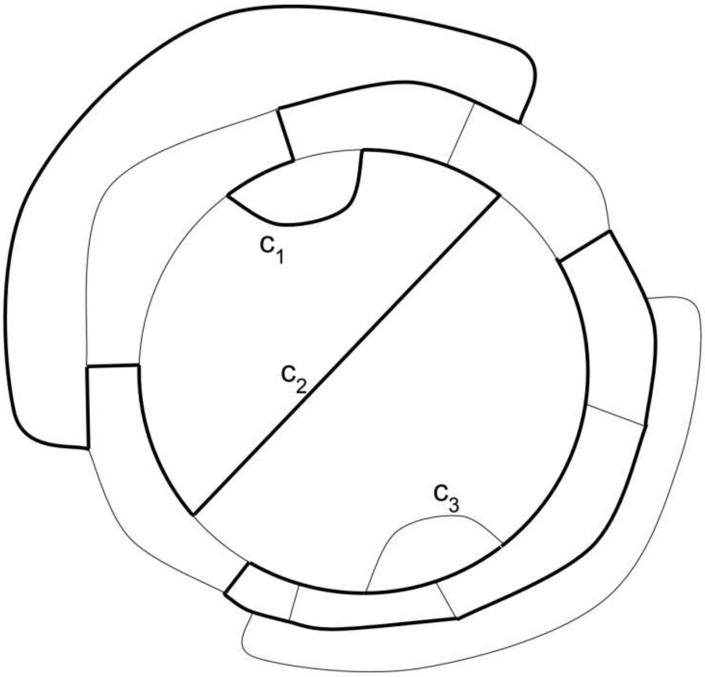


Figure A3-3 – A HEL solution of the reduced graph.

We are in the unfavorable case where C_1 = B, C_2 = B and C_3 = W and C_1 and C_2 have an “odd” status.

If we reestablish the original graph for this solution we would obtain:

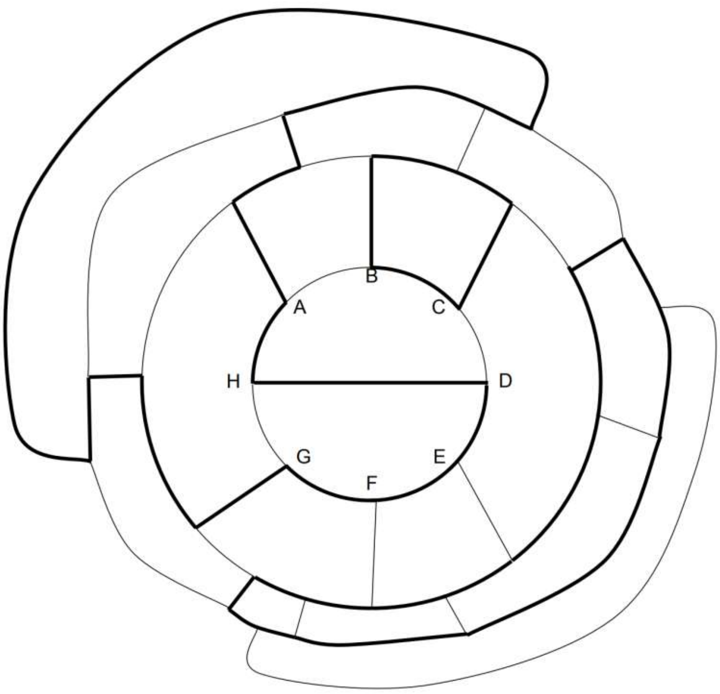


Figure A3-4 – Recovery of the original graph.

And we find that indeed this recovery is not HEL.

By virtue of the double status theorem we invoke another status for C_1 and C_2 in the reduced loop.

For example the following HEL solution.

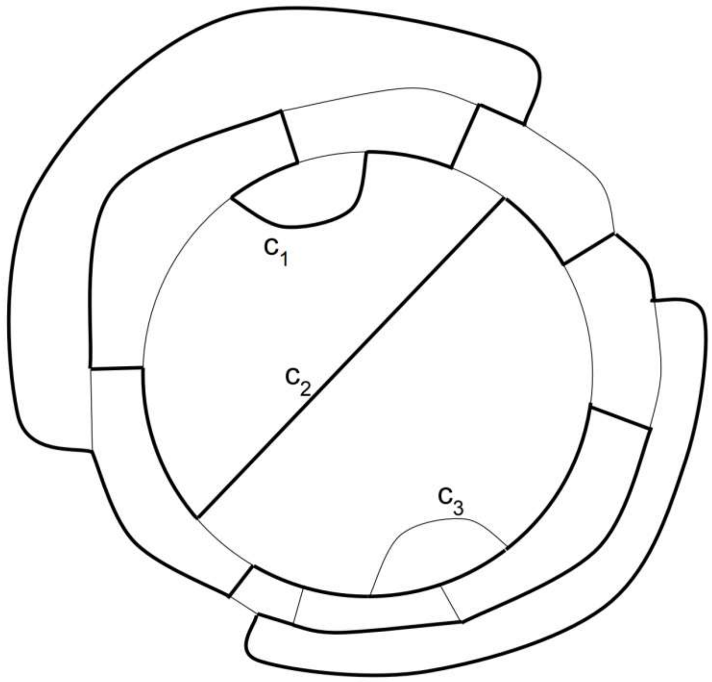


Figure A3-5 – A new HEL solution for the reduced graph.

We see that in this solution the pair C_1 , C_2 now has an "even" status.
We can restore the original graph:

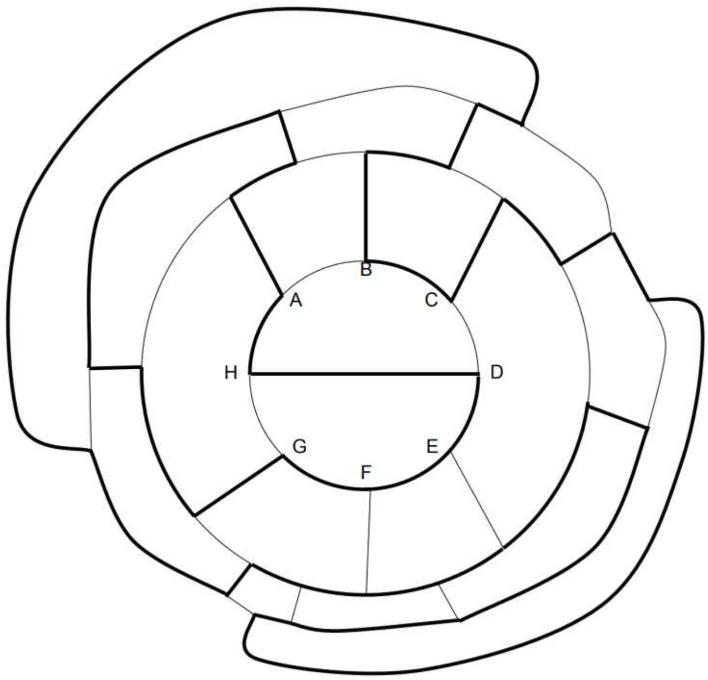


Figure A3-6 – Getting a new HEL solution for the original graph.

This recovery is indeed HEL which contradicts the initial hypothesis.

Thanks

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Data availability *There is no data availability.*