The Reality

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Zhengchen Liang†
Department of Physics, Tsinghua University, Beijing 100084, P. R. China
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Abstract
This document introduces the genesis and cardinal structure of existences, philosophically derived by the two concepts of certainty and diversity. All existences constructed by the $x$ diversity maps is the base structure of The Reality, on which sequential excitations of $z$ diversity maps found realities. Natural and other classes of existences in The Reality are partially characterized.

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†This is not a paper for publication nor a research for you, whatever chaebol you are. Also don’t pretend as if you have somewhere in history the philosophy, definition, and structure of RR maps in your poorly applied but highly dominated set theories.
1 NR vertex

One tries to recognize the genesis of everything.

Suppose one has the concept of certainty, that finds an existence. One names the existence first found. This existence can have arbitrary aliases such as the null-reality (NR) vertex, or \( n_0 \). Aliases or expressions attached to it are only for the convenience to speak but irrelevant to itself. As illustrated in Fig. 1, one is only certain about the existence of the NR vertex.

![Figure 1: The NR vertex.](image1.png)

2 QR vertices

One wants to be certain about other existences. Based on the concept of certainty, one can find existence. However, any existence found is not a second existence unless one tells that the found existence is different from the first one i.e. the NR vertex \( n_0 \). Suppose one has also the concept of diversity, to name an existence diverse from existence(s) one already named. A map of diversity

\[
 x_0 : \begin{cases} 
 R_0 \rightarrow C_0 \\
 n_0 \mapsto n_1 
 \end{cases}
\]

(1)

\[
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 R_0 \rightarrow C_0 \\
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 \end{cases}
\]

(1)

demonstrates how the existence of \( n_1 \) is named given that the existence of \( n_0 \). For \( x_0 \), the known existence is \( n_0 \). The term \( R_0 \) denotes all named existence(s), where for \( x_0 \) it is \( n_0 \). The term \( C_0 \) denotes all possible existence(s) to-be-named that is different from named existence(s) in \( R_0 \).

The existence found by the concept of certainty and named by \( x_0 \) is labeled by \( n_1 \), or any other aliases. As illustrated in Fig. 2, the integrity of co-existing \( n_0 \) and \( n_1 \), as well as other named existences called quantized-reality (QR) vertices, is only originated from the map of diversity which defines a certain existence only from named existences.

![Figure 2: Two QR vertices and the map of diversity.](image2.png)

3 CR set

One clarifies the map of diversity, generically

\[
 x_s : \begin{cases} 
 R_S \rightarrow C_S \\
 R_S \supseteq \{ n_s \} \mapsto n_r \notin R_S 
 \end{cases}
\]

(2)

in which \( S \) is the label for a set \( R_S \) of QR vertices, and \( s \) is the label for elements \( n_s \) in a subset of \( R_S \). At \( x_s \), a new existence \( n_r \) is named a.k.a. defined by the concept of diversity from the QR vertices \( \{ n_s \} \) already named in \( R_S \).

The incompleteness of the concept of diversity lies in that the diversity map itself is a certain thing but not a certain existence. A similar incompleteness in Axiomatic Set Theory is that the set operation \( s : p \mapsto \{ p \} \) collecting element(s) \( p \) itself is an object but not in constructed sets.

Here, any named existence is above the “surface” of conceptual incompleteness. One choose not to have iterative set operation (fictitious abstraction) like \( \{ \{ n_0 \} \} \) or else. The only fact is the co-existence of QR vertices linked by diversity maps, which restricts the term set in use here to only the collection of existences named by diversity maps, and the diversity maps from named existences. This allows one to focus on the exact structure revealing The Reality.

One asks how large a realized set \( R_S \) can exist. One questions whether at last for some \( S \), one cannot name any existence in \( C_S \). One is indeed asking what is the structure of all existences.

One first focus on the case of special \( x \) maps of diversity satisfying \( x_s : | \{ n_s \} | = 1 \). Under this kind of maps only, one finally reach a sequence of named existences: \( n_0, n_1, n_2, \ldots \).

To see it clearly, one first define a natural relation \( < \) from the maps of diversity between two named existences \( n_k \) and \( n_l \) \((k, l = 0, 1, 2, \ldots)\) in a realized set \( R \): the relation \( n_l < n_k \) holds iff \( n_l \) maps to \( n_k \) under some \( x \) map(s), otherwise \( n_l \not< n_k \). By definition of diversity maps one knows that i) \( n_k < n_k \); ii) \( n_k < n_l \Rightarrow n_l \not< n_k \); iii) \( n_k < n_l \land n_l < n_m \Rightarrow n_k < n_m \). Hence \( < \) (or \( \leq \)) from the \( x \) maps forms an order on arbitrary set \( R_S \). Samely \( R_S \) is endowed with the \( > \) order (or \( \geq \)), i.e. \( n_k > n_l \) iff \( n_l \) maps to \( n_k \) under some \( x \) map(s), otherwise \( n_k \not> n_l \).

One considers for any \( R_k = \{ n_0, n_1, \ldots, n_k \} \) and any \( n_l < n_k, \bar{x}_1 : n_l \mapsto n_r \), one knows that both \( n_l \) and the existing \( n_{l+1} \in R_k \) are defined only by the diversity from \( n_l \) hence \( n_r \) is already named as \( n_{l+1} \). Since \( \bar{x}_1 : | \{ n_l \} | = 1 \), \( n_r \) has no diversity map from \( n_{l+1} \). The only new map of such
\( \bar{x} \) from \( R_k \) is \( \bar{x}_k : n_k \mapsto n_r \) hence one derives the chain of existences under only \( \bar{x} \) maps, as illustrated in Fig. 3. One calls it a QR chain \( Q_0 \). One may not know what are natural numbers yet but indeed this QR chain can be aliased by \((\mathbb{N},<)\) with the < usual order. The number of QR vertices a.k.a. the cardinality of the QR chain is denoted by \( \aleph_0 \) and called countably infinite. The term “countably infinite” at present is only originated from the \( \bar{x} \) maps s.t. \( \forall n_k \in \mathbb{Q}_0, \exists \aleph_0 : k < \aleph_0 \), since the above property of \( \bar{x} \) maps holds \( \exists Q_0 : n_0 \in Q_0 \land (\forall n_k \in Q_0, \exists n_r \equiv n_{r+1} : n_r \in Q_0) \).

If one has to axiomize the concept of diversity, then this trivial property stands for the Axiom of Infinity.

\[
\begin{align*}
\text{The Reality 7-1-0} & \quad 3 \\
\text{On this realized QR chain, one the}\footnote{The general form of operators \(-\) and \(+\) are defined by the crystal order \( n_t > n_k \) or their \( z \) maps, see later the Section 5.} \text{define an operation \(-\) on } n_k > n_t > n_0 \text{ as } = \begin{cases} 
(n_k, n_t) \mapsto n_s \text{ where } n_t \text{ maps to } n_k \text{ under the same number of } \bar{x} \text{ maps (s) that } n_0 \text{ maps to } n_s. \text{The named existence } n_s \in R \text{ can be denoted as } n_{k-1}. \text{For any three QR vertices satisfying } n_k - n_t = n_s, \text{rewriting } n_k + n_s = n_k \text{ define } + : (n_t, n_s) \mapsto n_k. \text{Since } i) n_k - n_t = n_s \Rightarrow n_k > n_s > n_0 \Rightarrow n_k - n_s = n_k, \text{ one knows if } + : (n_t, n_s) \mapsto n_k \text{ then } + : (n_s, n_t) \mapsto n_k \text{ which is the commutative law for } +; \text{ ii) } (n_m - n_k) - n_t = n_s \Rightarrow n_m - n_k = n_u = n_k + n_t, (n_m - n_t) - n_k = n_t \Rightarrow n_m - n_t = n_v = n_t + n_k \text{ so } n_m = n_u + n_k = n_v + n_t = (n_u + n_k) + n_t = (n_t + n_k) + n_k, \text{while } n_s = n_u - n_t, n_k = n_v - n_k \text{ so } n_u > n_t > n_0, n_u > n_t > n_0 \Rightarrow n_u = (n_u - n_k) + n_t \Rightarrow n_u - n_t = (n_v - n_k) \Rightarrow n_s = n_t, \text{ one knows } (n_s + n_t) + n_k = (n_s + n_k) + n_t \text{ which is the associative law for } +. \text{From the derivations } i) \text{ and } ii) \text{ one knows that the } + \text{ commutativity requires only } \forall n_k > n_t \Rightarrow n_k - n_t < n_k, \text{and the } + \text{ associativity does not rely on the commutativity but requires just } \forall n_t > n_k, n_m > n_t, n_m - n_k > n_t \Rightarrow n_{m} - n_k \text{, so the requirements are naturally met.}
\end{cases}
\]

One then considers the general \( \bar{x} \) maps of diversity
\[
\begin{cases}
R_S \to CS \quad & \bar{x}_S : R_S \supseteq \{n_s\} \mapsto n_r \notin R_S \\
\{|\{n_s\}\} > 1
\end{cases}
\]

where the cardinality of the predecessor set \( \{n_s\} \) can be finite from several QR vertices, or countably infinite from the entire QR chain, or else.

\[
\begin{align*}
\text{In order to derive the possibly largest realized set } R_S, \text{ one calls it the constructed-reality (CR) set } CR, \text{ and starting from the NR vertex, one first has derived the QR chain } Q_0 \subset CR \text{ from only the } \bar{x} \text{ maps. At } R_S = Q_0, \text{ due to the uniqueness of } Q_0 \text{ one cannot name another QR vertex using the } \bar{x} \text{ map hence one has to use the } \bar{x} \text{ maps starting from } Q_0. \text{ Under a single } \bar{x} \text{ map from } Q_0, \text{ one names the existences diversified from QR vertices in } Q_0 \text{ subsets. As illustrated in Fig. 4 one denotes all the QR vertices named by the } \bar{x} \text{ maps from } Q_0 \text{ as the set } P_0.
\end{align*}
\]

\[
\begin{align*}
\text{Figure 3: The countable chain } Q_0 \text{ of QR vertices is derived under only } \bar{x} \text{ maps.}
\end{align*}
\]

\[
\begin{align*}
\text{Figure 4: The uncountable set } P_0 \text{ of QR vertices is derived under a } \bar{x} \text{ map from the countable QR chain } Q_0.
\end{align*}
\]

\[
\begin{align*}
\text{Figure 5: The set } Q_1 \text{ including uncountable numbers of QR chains is derived under } \bar{x} \text{ maps from } Q_0 \cup P_0.
\end{align*}
\]
For any QR vertices \(n_1\) and \(n_2\) in \(P_0\), their predecessor sets \(\{n_1\}\) and \(\{n_2\}\) of the \(\vec{x}_1\) and \(\vec{x}_2\) maps, are two different subsets of \(Q_0\). By definition of the \(x\) diversity maps, \(n_1\) and \(n_2\) are indeed two different QR vertices. This ensures \(P_0\) being unique. If axiomized, the obvious property of \(\vec{x}\) maps \(\forall (\forall\{n_1\} \subseteq Q_1, \forall n_2, \forall n_3, (\vec{x} : \{n_k\} \rightarrow n_1 \wedge \vec{x} : \{n_k\} \rightarrow n_2) \Rightarrow n_1 = n_2), \forall n_1, \exists\{n_p\} \subseteq P_1; \forall n_{pl} \in \{n_p\}, \exists\{n_1, n_{pl}\} \subseteq Q_1; \vec{x} : \{n_1\} \rightarrow n_{pl}\) explains the Axiom of Replacement, and here \(i = 0\).

One can see \(|P_0| > |Q_0|\) very straightforward by considering that if there is a bijective map \(\vec{b} : Q_0 \rightarrow P_0\) as some directed edges selected from all the \(\vec{x}\) maps, then QR vertices \(\{n_1\} \subseteq Q_0\) s.t. \(\forall n_1 \in \{n_1\}\) \((\vec{x} : \{\ldots, n_k, \ldots\} \rightarrow b(n_1)), \forall\{n_1\} = n_{pl} \in P_0, \exists n_0 \in Q_0: b(n_0) = n_{pl}\) however contradicts with \(n_0 \in \{n_1\} \Rightarrow b(\vec{x}) : \{\ldots, n_k, \ldots\} \rightarrow b(n_0)\). One can thus denote the cardinality of \(P_0\) as \(\aleph_1 > \aleph_0\) and call every cardinality greater than \(\aleph_0\) uncountable.

Starting from \(Q_0\) and \(P_0\), under only the \(\vec{x}\) maps one derives uncountably infinite numbers of QR chains, each begins with a QR vertex in \(P_0\). As illustrated in Fig. 5 one denotes \(P_0\) and the QR vertices in the QR chains derived in this step as set \(Q_1\), so \(P_0 \subset Q_1\), and \(|Q_1| = \aleph_1\).

At \(R_S = Q_0 \cup Q_1\), one cannot name another QR vertex using the \(\vec{x}\) map hence one has to use the \(\vec{x}\) map starting from \(Q_0 \cup Q_1\). Under a single \(\vec{x}\) map from \(Q_0 \cup Q_1\), one names the existences diversified from elements in the power set of \(Q_0 \cup Q_1\). As illustrated in Fig. 5 one denotes all the QR vertices named by the \(\vec{x}\) maps from \(Q_0 \cup Q_1\) as set \(P_1\).

Figure 6: The uncountable set \(P_1\) of QR vertices is derived under a \(\vec{x}\) map from \(Q_0 \cup Q_1\).

Following the above prove of \(\aleph_1 > \aleph_0\), one can see from the property of \(\vec{x}\) maps that the cardinality of \(P_1\) is \(\aleph_2 > \aleph_1\). The sequence of diversity maps

\[
\vec{x}_\ldots, \vec{x}_n \vec{x}_\ldots, \vec{x}_n \vec{x}_\ldots, \ldots \ldots
\]

starting from the NR vertex, derives the CR set. The total number of each \(\vec{x} \ldots \vec{x}\) period must be countable, since each period only relies on the realized set from its previous period that these periods, in the same way as existences named by \(\vec{x}\) diversity maps, form a QR chain which is countable. The cardinality of \(Q_i\) is \(\aleph_i\) for all \(i \in \mathbb{N}\). The CR set can be expressed as

\[
CR = \bigcup_{i \in \mathbb{N}} Q_i
\]

To prove Eq. 5 one considers a QR vertex \(n_r\) not named at \(CR\). The NR vertex \(n_0\) has a directed path to \(n_r\), since \(n_r\) is a named existence and only diversity maps can name it, starting from the NR vertex. If the directed path is countable then the proof is trivial. If the directed path is uncountable, the cardinality of such definition through \(x\) maps to reach this QR vertex contradicts with the countably infinite \(\vec{x} \ldots \vec{x}\) period i.e. Peano predecessor in Eq. 4.

The CR set is what one can get with only the two concepts of certainty and diversity, starting from the NR vertex. The Axiomatic Set Theory has derived similar structure by seeing the NR vertex as the empty set \(\emptyset\), the \(x\) maps as the set operation \(s\), and the directed edges here as the \(\in\) relation between sets. However the set operation brings nonessential vagueness that may blind human from seeing The Reality, since one cannot buy that the QR vertices, i.e., those ordinal numbers in Axiomatic Set Theory, already represented everything one observes in reality.

For any QR vertex in the CR set, i.e., countably reachable existence by maps of diversity, one can mark it with a natural number \(m\), indicating that it is the \(m\)-th QR vertex in its QR chain. Then the starting point of the QR chain \(n_r\) is marked by a repeatable set of natural numbers \(\{m_{r_1}, r_1\}, \ldots\), indicating that each QR vertex in the predecessor set of \(n_r\) is the \(m_{r_1}\)-th QR vertex in its QR chain. By marking all predecessor sequences down to the NR vertex,

\[
\{m, m_{r_1}, m_{r_1 r_2}, m_{r_1 r_2 r_3}, \ldots\} \rightarrow m_{r_1 r_2 r_3}, \ldots
\]

one locates the QR vertex in the CR set.

4 RR maps

From definition of \(x\) maps there is no directed loop in the directed graph of the CR set. This is because the directed edges are only the \(x\) maps. Referring to the concept of diversity, known diverse existences can be confirmed by diversity maps too. One calls these maps post-CR diversity maps, or represented-reality (RR) maps, or \(z\) maps.

Note that for any two QR vertices \(n_k\) and \(n_l\), one and only one must hold among i) \(n_k < n_l\) i.e. there exists a directed path from \(n_k\) to \(n_l\); ii) \(n_l < n_k\) i.e. there exists a directed path from \(n_l\) to \(n_k\); iii) \(n_k \not< n_l\) and \(n_l \not< n_k\) under the order of directed path. A \(z\) map from \(n_k\) to \(n_l\) in the case ii) forms a finite directive loop in the QR path where \(n_k\) and \(n_l\) exists. A \(z\) map from \(n_k\) to \(n_l\) in the case i) or iii) indeed is an existing \(x\) map with \(\{n_k\} \cup P(n_l)\) its predecessor set, where \(P(n_l)\) denotes the predecessor set of \(n_l\). Therefore, an RR map must map between two QR vertices connected by a directed path and form a directed loop in the CR set.

The special case is for all the QR vertices in any path from \(n_l\) to \(n_k\), not including \(n_l\), its predecessor set only includes the QR vertices \(n_k\) that \(n_l \leq n_k\). This special case is denoted \(n_l < n_k\). One can mark the special RR map from \(n_k\) to \(n_l\), i.e., the relative location between \(n_k\) and \(n_l\) by the repeatable set

\[
\{m, m_{r_1}, m_{r_1 r_2}, m_{r_1 r_2 r_3}, \ldots\} \rightarrow m_{r_1 r_2 r_3}, \ldots
\]

tracing only the directed paths of \(x\) maps from \(n_l\) to \(n_k\).
This mark is the same as marking a QR vertex $n$, starting from the NR vertex $n_0$.

The general case is $n_l \not{\propto} n_k$, i.e., $\forall n_l < n_k, \exists n_c < n_l, n_c < n_k$. One can select the QR vertex $n_{c0}$ closest to $n_l$ and $n_k$ such that $\exists n_{c0}: n_{c0} < n_l, n_{c0} < n_k, \forall n_c \neq n_{c0}: n_c < n_{c0}$. In this case, the general RR map can be marked by two repeatable sets in Eq. 7. One is for $n_{c0} < n_k$ which is called the minuend set, and the other one is for $n_{c0} < n_l$ which is called the subtrahend set. The two sets keep the complete information of the general RR map from $n_k$ to $n_l$ where $n_l < n_k$ but $n_l \neq n_k$.

The relation $\prec$ (or $\preceq$), as well as its dual $\succeq$ (or $\geq$), forms a partial order on the CR set. For convenience one says $n_l \not{\preceq} n_k \Rightarrow n_l \neq n_k$. One calls $\prec$ the crystal order.

Figure 7: An example of special RR map from $n_f$ to $n_i$ where $n_i \prec n_f$.

For instance, one considers the example of special RR map from $n_f$ to $n_i$ shown in Fig. 7. The repeatable set for $n_i \prec n_f$ or equivalently this RR map can be written as:

$m = 2$;

$m_1 = 2, m_2 = 2$;

$m_{11} = 1, m_{12} = 0, m_{13} = 2, m_{21} = 1, m_{22} = 3$;

$m_{111} = 0, m_{112} = 1, m_{121} = 0, m_{122} = 1, m_{123} = 2$.

The mark is equivalent to the tree shown in Fig. 8 which is called the tree representation of $n_i \prec n_f$ or the RR map. In this tree, the root is $m = 2$. All nodes are numbers corresponding to the repeatable set. All leaves are QR vertices on the QR chain where $n_i$ exists. Starting from this QR chain and recursively, two nodes in the tree can be recognized as on a same QR chain if and only if their branching nodes are the same numbers that correspond to the same recognized QR chains.

The tree representation is free from the lower indices of $m$'s in the set representation. The permutation of lower indices is redundant information, so that the tree representation is faithful to the crystal order and special RR maps.

On the CR set, one root, any natural numbers and more-than-one branches for any branching nodes, and also any layers are allowed for the representation tree of a possible RR map, as long as any nodes of the same number must not exist on the same QR chain, due to the uniqueness of QR vertices in the CR set. Starting from the leaves, this condition can be implemented layer by layer.

For general RR maps on $n_l < n_k \land n_l \not{\propto} n_k$ from $n_k$ to $n_l$, the minuend set and the subtrahend set correspond to the minuend tree and the subtrahend tree, respectively.

For fast comprehension, the CR set is the base space, the vacuum, or the ground state upon all the named existences, while all the possible RR maps are its excitations.

5 Realistics

A reality or a physical world is a sequence of some RR maps, with the sequence called time, endowed with a total order that is derived from certain rules in a sense of minimal diversity. A moment in the sense of reality is some RR maps excited on the base of The Reality i.e. the CR set. In that sense of non-minimal diversity, the total order of a reality can also be embedded into some partial ordered lattices of realities containing the sequence of that reality. RR maps are not unique thus allow replicas of the CR set, and thus direct products on QR vertices or $x/z$ maps.

The role of a reality in The Reality is as small as the rate between $\aleph_0$ (its RR excitations) and $\aleph_{\aleph_0}$ (the all possible RR excitations). Hence normally it is not an efficient way to exhaust all possibilities of RR maps in The Reality to solve realistic problems.

However one wants to take whatever imaginable tools in The Reality into consideration when facing its reality. These tools, recognized as mathematical structures, are sequences of $x/z$ maps under certain constraints. Before exploring the uncountable possibilities of RR maps and hidden rules of their sequences, one starts with the $x$ maps of diversity which in natural construct the CR base.
5.1 Natural basis

Naturalness refers to the exact and unique structure of CR set, the static base of The Reality. The x maps of diversity in the CR set first define the natural conditions for if-then branches. These conditions include:

- $\forall n_1, n_2: n_1 < n_2$ or $n_2 < n_1$.
- $\forall n_1, n_2, n_3: n_1 < n_2$ and $n_2 < n_3$ imply $n_1 < n_3$.
- $\forall n_1, n_2: n_1 > n_2$ implies $n_2 < n_1$.
- $\forall n: n < n$.

For the simplest case $\forall n_1, n_2$ in a QR chain, one and only one of these three cases holds: $n_1 < n_2$ or $n_2 < n_1$ or $n_1 = n_2$.

The structure of diversity maps in the CR set defines the natural operations to forall-exists statements. These operations include:

- $\forall n_1 < n_2, \exists! n_{\text{min}}: n_{\text{min}} = n_2 - n_1$ i.e. the tree representation of $n_1 < n_2$ is the same tree representation of $n_{\text{min}} < n_2$.

For the simplest case $\forall n_1, n_2, n_3$ in the QR chain $Q_0$, $\forall n_1 < n_2, \exists! n_{\text{min}}: n_{\text{min}} = n_3 - n_2$ if $n_2 < n_3$ or $n_1 < n_2$ or $n_1 = n_2$.

In this QR chain $Q_0$ starts with the NR vertex $n_0$, natural operations of for-loop iterations are:

- $++: Q_0 \times Q_0 \to Q_0, \forall n_1, n_2: n_1++ = n_1 + 1$.
- $+: Q_0 \times \mathbb{N} \to Q_0, \forall n_1, n_2, n_3: n_1+n_2 = n_1 + n_2$.
- $\times: Q_0 \times \mathbb{N} \times \mathbb{N} \to Q_0, \forall n_1, n_2, n_3: n_1 \times n_2 = n_1 \times n_2$.
- $\div: Q_0 \times \mathbb{N} \times \mathbb{N} \to Q_0, \forall n_1, n_2, n_3: n_1 \div n_2 = n_1 \div n_2$.

Note that here $x^{(0)}$ is $++$, $x^{(1)}$ is $+$, $x^{(2)}$ is $\times$. The procedure to calculate formulas up to $x^{(n)}$ are: i) $\forall n < n$ expands prior to low n; ii) $x^{(n)}$ expand in left associativity, expanded bulks in square brackets hold until all $x^{(n)}$ is expanded; iii) Square brackets are released by the $\#$ indices after $x^{(n)}$ is expanded and before the $x^{(n-1)}$'s expansion.

The natural numbers $n_0$ as operation loop indices can be elements in $Q_0 = \mathbb{N}$, hence $x^{(n)}: \bigoplus_{0}^{n} Q_0 \to Q_0$ holds. By simply expanding and counting the x maps in chain $Q_0$, lower levels of $x^{(n)}$ can extract the known properties:

1. $+: Q_0 \times Q_0 \to Q_0$, commutativity $n_1 + n_2 = n_2 + n_1$, associativity $(n_1 + n_2) + n_3 = n_1 + (n_2 + n_3)$.
2. $\times: Q_0 \times Q_0 \to Q_0$, commutativity $n_1 \times n_2 = n_2 \times n_1$, associativity $(n_1 \times n_2) \times n_3 = n_1 \times (n_2 \times n_3)$, distributivity $(n_1 + n_2) \times n_3 = n_1 \times n_3 + n_2 \times n_3$.

Any higher levels of operations can’t be directly extracted to a binary algebraic operator of $Q_0 \times Q_0 \to Q_0$ due to the expansion procedure, e.g., $\times$’s expansion relies on $h_1$. However, one can simplify $x^{(n)}$ by letting $n_k = n_0$, $k = 1$ to get the exponential operation $\text{Pow}(1): Q_0 \times Q_0 \to Q_0$, $(h_2, h_3) \mapsto h_2^{h_3}$. Higher level of $n$ can always be simplified in the way: i) Let $n_k = n_0$, $k = 1$; ii) Let $h_k = h_0 = \cdots = h_{n-1}$.

Obviously commutativity fails and only left associativity is obeyed. Distributivity is only trivial from counting the numbers following the above procedure i)-iii).

For $++$, it is not necessary to start from the NR vertex. Standing at the middle of $Q_0$, in a finite range, the natural operations $-\$, $\times$, $\div$, $\ldots$, $x^{(n)}$ holds for the infinite $\mathbb{Z}$. One can induce two QR chains to construct the uncountably infinite $\mathbb{Z}$ by artificial rules but then its $<$, $\geq$, and $>$ conditions are not natural (from the $x$ maps of diversity).

The $(n+1)$-ary operations $x^{(n)}$ (or $x^{(n)}$) are the only natural operations inheriting the structure of $x$ maps in a QR chain. The algebra they form can be denoted by $N_1 = (\mathbb{N}, \ldots, x^{(n)}, \ldots, x^{(n)} + , ++, 0)$. The above simplifications form $N_2 = (\mathbb{N}, \ldots, \text{Pow}(n), \ldots, \text{Pow}(1), +, +, +, 0)$ which is not natural (only rely on counting $x$ maps), since the simplifications in fact bring restrictions on $x^{(n)}$. Obviously $\forall n_k \in Q_0 (n_k \neq n_0), \forall n \in \mathbb{N}, \exists! n_{\text{min}} < n, \forall h_1, h_2, \ldots, h_n \in \mathbb{N}: n_k = n_1 + h_2 + h_3 + \cdots + x^{(n)} h_n$, however not all $n_k \in Q_0$ are reachable by operation(s) of some restricted $x^{(n)}$. For instance, $\forall n_k \in Q_0 (n_k > n_1), \forall h_1, h_2 \in \mathbb{N} (h_1, h_2 > 1): n_k = n_1 + h_2$ is not true, which leads to the classes of prime and composite numbers. Another example is the radix $\forall r \in \mathbb{N} (r > 1), \forall n_k \in Q_0, \exists h \in \mathbb{N}, \exists \{h_k\}_{k \geq \in \mathbb{N}} (r > h \in \mathbb{N}): n_k = n_{h_1} + h \times \text{Pow}(1)(r, i)$ leads to the base-$r$ representation of natural numbers.

Vitaly, any restrictions on the operations on the direct products $\bigoplus Q_0$ is not only about naturally counting the $x$ maps in $Q_0$, but imposing rules on replicas of $Q_0$ which induce classes i.e. properties of QR vertices in $Q_0$ in its sense. One has to distinguish other properties induced by restricted $x^{(n)}$ operations, from the natural $=$ equivalence defined only by the count of $x$ maps in $Q_0$.

For the general case where $n_1 < n_k$ are QR vertices in the CR set but not necessarily in the NR chain above, one has $n_m - n_0 = n_k - n_1$ defining the associative + $= CR \times CR \to CR$, $(n_1, n_m) \mapsto n_k$, but not commutative.
since the CR set does not always hold $n_m < n_k$. The general + operation under $\preceq$, as the inverse operation of $-$ in the CR set, is guaranteed to be associative.

However, subsets of the CR set may hold commutativity, associativity, and other identities for operation $+$ defined under the crystal order $\prec$. From now on, one refers tree to that crystal tree faithfully representing the crystal order between two QR vertices $n_l < n_k$ as in Fig. 8 and paths to some directed paths that links $n_l < n_k$ in their corresponding crystal tree. As shown in Fig. 8, the crystal diagram starts with the NR vertex $n_0$ and goes to $n_a \succ n_0$

More generally, a set of generator trees $\{a_i\}_{i \in I}$ s.t. $\forall i \neq j \in I$: $n_{a_i} \neq n_{a_j} \land n_{a_i} \neq n_{a_j}$, extracts the CR subset $L = \{n_{\mu a_i}\}_{i \in I}$ (Einstein auto sum) where $\forall j \in I$, $n_{\mu a_i} \succ n_{\mu a_j}$ by tree $a_j$, that forms a lattice in mathematics by defining $\gamma : L \times L \to L$, $n_{\mu a_i} \gamma n_{\mu a_j} = n_{\min(\mu a_i, \mu a_j)}$, as well as $\lambda : L \times L \to L$, $n_{\mu a_i} \lambda n_{\mu a_j} = n_{\min(\mu a_i, \mu a_j)}$. As direct to verify the commutative laws, the associative laws, the idempotent laws, and the absorption laws of lattice $(L, \gamma, \lambda)$.

It also preserves the commutativity, associativity, and closure of the binary operator $+$ when preceeding a tree $c_{\lambda_1 \lambda_2 \cdots \lambda_l} : n_{c_{\lambda_1 \lambda_2 \cdots \lambda_l}} \succ n_0$ where $n_{c_{\lambda_1 \lambda_2 \cdots \lambda_l}} \prec n_{a_i} \land n_{a_i} \prec n_{a_j}$ to the simply generated lattice $L = \{n_{\mu a_i}\}_{i \in I}$ raising $n_{(\mu a_i + \lambda)} a_i \succ n_{\mu a_i}$ by a tree including the paths of $c_{\lambda_1 \lambda_2 \cdots \lambda_l}$ with any constants $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ \( \subseteq \prod_{i \in I} \mathbb{N} \setminus \{(0, \cdots, 0), (0, \cdots, 0, 1, 0, \cdots, 0)\} \). By preceeding countable trees of this kind, one gets a different subset of the CR set under the crystal order, that is still a lattice, which can be denoted as $\prod_i a_i + \sum_i c_{\lambda_i}$ with its dimension $\dim L = \{|a_i|\}$.

For instance, the simply generated tree in Fig. 9 can be accompanied by trees $c_{\rho \sigma}$ that raise $n_{(\mu a + \rho) a + (\nu + \sigma) b}$ to the tree including the paths of $c_{\rho \sigma}$, for any constants $(\rho, \sigma)$ in $\mathbb{N} \times \mathbb{N} \setminus \{(0,0), (0,1), (1,0)\}$. As shown in Fig. 10, one preceedes trees $c_{11}$ and $c_{21}$ to the simply generated lattice. The resulting lattice contains the QR vertices in the CR set, except for $\{n_{a_i}, n_{b_i}\}$, indeed different from those in Fig. 5. The CR vertex $n_{2a+b} \succ n_0$ by tree $r = (a+a+b) \cup (a+c) \cup (a+b+a) \cup (c+1+a) \cup (b+a+a)$. Different paths of sums of embedded trees in the crystal tree shows the identities of operation $+$ in the lattice. Here for example $n_{2a+b}$ yields $a + c_1 \approx c_1 + a$ and $a + a + b \approx a + b + a \approx b + a + a$. It is direct to see that the relation $\approx$, called paths equivalence, is an equivalence relation on paths, with equivalence classes separated by only the different QR vertices. The lattice ensures that $\forall a, b, c; a + b \approx b + a \land b + c \approx c + b \land a + c \approx c + a$ as well as the associativity under $\approx$ inherited from the crystal order $\prec$ for every path in $L$, as $L$ is a subset of the CR set.

![Figure 9: The lattice simply generated by trees $a$ and $b$ under the crystal order.](image)

![Figure 10: The lattice generated by trees $a$ and $b$ accompanied by trees $c_{11}$ and $c_{21}$ under the crystal order.](image)

Given some specific generator trees $\{a_i\}_{i \in I}$, a badly specified lattice $(L, \gamma, \lambda)$ without the constraint of $\forall i \neq j \in I$: $n_{a_i} \neq n_{a_j} \land n_{a_i} \neq n_{a_j}$ can be reduced to $(L', \gamma, \lambda)$ if some elements of $L$ indeed refer to the same QR vertex. Such a reduction due to bad definition of a lattice $L$ from abstract notations $a, b, \cdots$ of trees, has no complexity other than the identical expressions with embedded trees $\{a_{i_{a_i}}\}_{i_{a_i} \in I_{a_i}}$ of a tree $r$ s.t. $\forall i_k \in I_k$: $\lambda^k < n_0$ and $r = \sum_{i_{a_i} \in I_{a_i}} \lambda_{i_{a_i}} a_{i_{a_i}} = \sum_{i_{a_i} \in I_{a_i}} \lambda_{i_{a_i}} a_{i_{a_i}} = \cdots = \sum_{i_{a_i} \in I_{a_i}} \lambda_{i_{a_i}} a_{i_{a_i}} = \cdots (9)$ requiring that $\forall k \neq l$: $\{a_{i_{a_i}}\} \cap \{a_{i_{b_i}}\} = \emptyset$ since the lattice has already the commutativity and the associativity for + so that any tree $a_{i_{a_i}} \in \{a_{i_{a_i}}\} \cap \{a_{i_{b_i}}\}$ can be eliminated in
The expression Eq. [3] i.e. identification of lattice elements is originated from the structure of a tree $r$ that includes the trees $\{ai\}_r$. Under the tree equivalence $\equiv$ with respect to the crystal order (strictly than the equivalence $\equiv$ which is applied to paths), meaning that the same QR vertex is described by the associative sums of trees $\lambda^i a_i$ and $\lambda^j a_j, \exists i, j \in I_k \cup I_l$: All QR vertices in tree $a_i \succ n_{a_i} \succ n_0$ is a subset of QR vertices in tree $a_j : n_{a_j} \succ n_0$. More importantly, $\forall L (\lambda^i a_i = \lambda^j a_j), \forall i, j \in I_k, \exists \text{ tree } e_i : \forall i \in I_k, \exists l \in \lambda^i a_i \in N^+, \exists e_{ik} \in \{e_{ik} : \lambda^i a_i \prec n_0 \wedge \lambda^i a_i = \lambda^i e_{ik} \wedge \lambda^i e_{ik} = \text{ The proof}$. The proof does one takes the tree $e_i$ as the greatest common divisor (gcd) of $a_i$ and all $a_i$ that paired with $a_i$, i.e. $a_i \prec a_j \vee a_i \succ a_j$. Here the $<$ between trees in short denotes the crystal order $<$ between the QR vertices that the two trees goes from $n_0$. The lattice s.t. $\forall i \in I_k \cup I_l, \lambda^i a_i = \lambda^i a_i (\text{no sum}) + \sum_{i \in I_k} \lambda^i a_i = \lambda^i a_i (\text{no sum}) + \sum_{i \in I_l, i \notin I_k} \lambda^i a_i,$ and vice versa, thus ensures that $\forall i \in I_k \cup I_l$, $a_i$ is paired with an $a_i$ and $a_i$ (due to $a_i \neq \lambda^i a_i \wedge a_i \neq \lambda^i a_i \Rightarrow r = a_i + \cdots = a_i + \cdots$ LHS and RHS is only paths of $r$ but not the tree $r$). The tree $e_i$ s.t. $a_i = \lambda^i e_i \wedge a_i = \lambda^i e_i$ exists otherwise $a_i (a_k, \lambda^i a_i \in N^+, r = \lambda^i a_i = \lambda^i a_i$ (the difference between ordered trees in $r$ is still a tree in the minutest tree, and by doing the difference iteratively one at least reaches a gcd tree that is the divisor of original minutest and subrahend trees). The order $<$ is a total order in $\{a_i, a_j\}_i \cup \{a_i, a_j\}_j$ since for any trees $a_i$ and $a_j$ in the lattice, $\exists l_i, l_j$ such that $l_i = \{a_i, a_j\}_i \cup \{a_i, a_j\}_j (\text{the } \equiv \text{ and commutative, by doing the difference between the their gcd trees one must get a finite gcd tree since the QR vertices in the CR set are discrete), hence $\lambda^i a_i \prec a_j \vee a_j \succ a_i$. The finiteness of least common multiple $\text{lcm}(\lambda^i, \lambda^j)$ ensures the uniqueness of tree $e_i$ under the total order $<$ among $a_i$ and all $a_i$, such that $\forall l_i, l_j \prec a_i \wedge a_i = \lambda^i e_{ij}, a_i = \lambda^i e_{ij}, \lambda^i e_{ij}$, and (proof omitted) $\lambda^i a_i, \lambda^i a_j \leq \text{lcm}(\lambda^i, \lambda^j)$. In this case of $\lambda^i a_i = \lambda^j a_j$, one calls $L$ reducibly generated. Suppose $|I_k| \leq |I_l|$, one can induce the reduction $p : L \rightarrow L', \mu^i a_i : \text{ such that}$

$$
\mu^i a_i = \begin{cases} 
\mu^i a_i, & a_i \notin \{a_i \cup \{a_i\}_i \cup \{a_i\}_j \text{ i.e. } i \notin I_k \cup I_l \\
\text{gcd}(\lambda^i a_i) e_i, & i \in I_k \\
\text{gcd}(\lambda^j a_j) e_j, & i \in I_l
\end{cases}
$$

(10)

In which $\text{gcd}(\lambda^i, \lambda^j)$ or $\text{lcm}(\lambda^i, \lambda^j)$ includes all $\lambda^i$ for $a_i$ paired with $a_i$ inside the parenthesis. The dimension lowers down by $\dim L - \dim L = \max \{|I_k|, |I_l|\} = |I_l|$. Starting from a reducibly generated $L$, by doing all reduction(s) one will finally get an irreducibly generated $L$ by generator trees $\{a_i\}_i \in I_k \cup I_l$. The tree $r : J \cap K = 0 \wedge r = \sum_{j \in J} \lambda^j a_j = \text{ all } \lambda^j a_j$. Clearly, by using generator trees $\{a_i\}_i \in I_k \cup I_l, a_i \neq a_j \wedge a_i \neq a_i$, one can avoid the specific lattice $L$ to be a reducibly generated one.

The generator trees of an irreducibly generated lattice $L$ is called principal trees in $L$, with their corresponding QR vertices from $n_0$ called principal vertices in $L$. For QR vertices $n_k$ and $n_l$ generating a lattice $L$, $n_k \neq n_l \wedge n_l \neq n_k$ infers that $n_k$ and $n_l$ are coprincipal, but coprincipal vertices $n_k$ and $n_l$ can hold $n_k \prec n_l \wedge n_l \prec n_k$ as long as they don’t have a common divisor vertex $n_e$. In the case of $n_k \prec n_l, L = a_i \times a_k$ is instead generated by $a_i : n_i > n_0$ and $a_i^{-1} : n_k > n_l$.

Principal QR vertices are those QR vertices that are principal in all possible lattices $L$. There is no principal QR vertices other than $n_1$ in $Q_0$. All QR vertices in $Q_1$ is principal, but not all QR vertices in $Q_n (n > 1)$ is principal. For instance, the QR vertex $\{m = 0; m_1 = 0, m_2 = 4; m_1 = 0, m_2 = 4, m_2 = 0, m_2 = 4\}$ in $P_1 \subset Q_2$ is not principal, but $(m = 0; m_1 = 1, m_2 = 3, m_1 = 0, m_2 = 4, m_2 = 0, m_2 = 4)$ is principal, and $\{m = 1; m_1 = 0, m_2 = 2; m_1 = 0, m_2 = 3\}$ is principal.

In summary, the $x$ maps hold the solid structure of all QR vertices in the CR base. This structure is understood to be of two main aspects:

1. Intra QR chains: Algebra $\lambda_i$ counting the specific numbers of $x$ maps. The count leads to natural relations $<, >, =$ and natural operations $\chi^{(m)}$ that hold their corresponding basic properties on QR vertices.

2. Inter QR chains: CR subsets forming lattices $L$ under the crystal order of $x$ maps. The structure of specified CR subsets $L$ leads to natural relations $<, >, =$ (tree equivalence), $\equiv$ (paths equivalence) and natural operation + (and its iterated operations) that hold their corresponding basic properties including the algebra of trees under $\approx$ with the commutative, associative + and the principality of QR vertices.

5.2 Sequential space
On the CR base, all possible RR maps allow for sequential excitations which are characterized by direct products on replicas of CR set and the RR maps. Restrictions on maps involving direct products of CR subsets and $x/z$ maps lead to properties or classes of QR vertices and $x/z$ maps other than the natural one. As previously shown in $N_2$, possible restrictions and their derived properties are severely uncontrollable. It is extremely impossible to examine all of them as they count for all realities of The Reality yet one is in only its own reality. However, rules of one’s own reality is not uncontrollable thus one has to comply to pragmatism on mathematics, if one wants only to cover the characteristics of its own physical world.

The methodology and its implementations for one to fetch mathematical tools and examine its reality is diverse and mostly out of the scope of current documentation. Below shows only some examples of constructions on the CR set replicas with the RR maps.

In mathematics, an irreducibly generated lattice $(L, <, \wedge)$ of a CR subset under $<$ is distributive (distributive i.e. $\forall a, b, c \in L : a \wedge (b \vee c) = (a \wedge b) \vee (c \wedge a)$ i.e. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$) hence modular (modular i.e. $\forall a, b, c \in L : a \leq b \Rightarrow a \wedge (b \vee c) = b \wedge (a \vee c)$), but neither a complete lattice (complete i.e. $\forall A \subseteq L, \exists \sup A, \inf A \subseteq L \wedge \inf A \subseteq L$ nor a compactly generated lattice (compactly generated i.e. $\forall n_1 \in L, \exists G \subseteq L : n_1 \in G \wedge \forall n_j \in G, \forall A \subseteq L (A \subseteq L \wedge n_j \subseteq \sup A) \sup B \subseteq A : |B| < n_0 \wedge n_j \subseteq \sup B$) hence not algebraic. However, it is obvious that any finite sublattice of such $(L, <, \wedge)$ is algebraic.

To fetch some algebraic lattices under the crystal order, one induces a projection, which is a collection of RR maps.
\[\theta_{\lambda^1\lambda^2...\lambda^i} : L \to \bar{L}, n_{\mu^i a_i} \mapsto n_{\overline{\mu^i} a_i}, \]
\[\bar{\mu}^i = \begin{cases} \mu^i, & a_i \notin \{a_i k \} \cup \{a_i \} \text{ i.e. } i \notin I_k \cup I_l \\ \mu^i \text{ mod } \lambda^i, & i = i_k \in I_k \lor i = i_l \in I_l \end{cases} \]

(11)

appeared the binary operator \(\text{mod} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}\) of the usual definition on natural numbers. The \(\text{mod}(\mu, \lambda)\) operator is defined by fixing \(\lambda\) to be a constant in \(\times : \nu_\mu = \nu_\mu + h_1 \times \lambda \in \mathbb{N}_1\) and inverting the order of variables in the tuple, from \((\nu_\mu, \lambda) \mapsto \nu_\mu\) to \((\nu_\mu, \lambda) \mapsto \nu_\mu\), which is the same tuple inversion as in the definition of + from - on QR vertices in the previous Sections. The image \(\bar{L}\) is a finite thus algebraic lattice. One can define the unary operation \(-1 : \bar{L} \rightarrow \bar{L}\) by \(n_{\mu^i a_i} \mapsto n_{(\lambda - \mu^i)a_i}\) and define the nullary operation \(e\) as \(n_0\) so that \((\bar{L}, \bar{+}, -1, e)\) forms an Abelian group, with \(\bar{+}\) compatibly induced in \(\bar{L}\) by \(\theta\) from the natural +. It is direct to see the structure of finite Abelian groups \(\prod_i Z_i\) corresponds to this lattice construction \(n_{\mu^i a_i}\) with any principal trees \(a_i (i \in \mathbb{N})\). As shown in Fig. 11 the quotient lattice \(\bar{L} \triangleright n_{\mu a}\) with \(\lambda = 4\) presents the group \(Z_4\), and the quotient lattice \(\bar{L} \triangleright n_{\mu^1 a_1} + \mu^2 a_2\) \((a_1 = a, a_2 = b)\) with \(\lambda^3 = 3\) and \(\lambda^2 = 2\) presents the group \(Z_3 \times Z_2\). Morphisms, direct products, actions, classes, and other properties in Group Theory can also be defined on replicas of \(\bar{L}\).

Figure 11: The quotient lattice \(\bar{L}\) presenting the Abelian groups \(Z_4\) and \(Z_3 \times Z_2\).

For presenting non-Abelian groups, the lattice \(\bar{L}\) is no longer commutative with respect to +, while it inherits the associativity of + from the crystal order \(\prec\). Given \(n\) generators \(\{a_i\}_{1 \leq i \leq n} \in \mathbb{N}\), the number of permutations \(P(a_1^i \cdot a_2^i \cdot \cdots \cdot a_k^i) (k \in \mathbb{N}^*)\) is \(n^k\). The above congruent projection \(\theta_{\lambda^1\lambda^2...\lambda^i} : L \rightarrow \bar{L}\) no longer holds for \(\bar{L}\) because of the non-commutativity of tree sums. If one finds another congruence \(\theta\) that close the entire \(n^k\) space without deriving the commutativity then it appears a non-Abelian finite group. This means that the subspace \(\kappa^i a_i (k^i \in \mathbb{N}^*)\) must be closed by \(\theta\), hence in any non-Abelian finite group \((\bar{L}, \bar{+}, -1, e)\) any generator \(a_i\) obeys a relation \(\forall i \leq n, \exists \lambda^i \in \mathbb{N} : e \simeq n_0 \simeq n_{\lambda^i a_i}\) or denoted as \(\lambda^i a_i \simeq e\). One has to estimate the reduction of permutation space \(n^k\) by this relation \(\simeq\) when \(k\) is sufficiently large, and that the additional relations which \((\bar{L}, \bar{+}, -1, e)\) holds together with the \(\lambda^i\)’s should eventually lead to a finite permutation space when \(k\) is very large. For instance, one lets \(a_1 \equiv r, a_2 \equiv s, \lambda^1 = 3, \lambda^2 = 2, \theta_1 : rr \simeq srs, \theta_2 : rs \simeq srr, \theta_3 : sr \simeq rrs, \theta_4 : s \simeq rsr\), then the quotient \(\bar{L} = \bar{L}/\{\lambda^1, \lambda^2, \theta_1, \theta_2, \theta_3, \theta_4\}\) is finitely closed, presenting the dihedral group of order 6 a.k.a. \(D_3\) as shown in Fig. 12 in which the dashed red lines mark the congruence from \(\lambda\), dashed blue lines from \(\theta\), and dashed pink/cyan lines derived by the lattice associativity.

Figure 12: The quotient lattice \(\bar{L}\) presenting the non-Abelian group \(D_3\).

Overall, the discovery of such \(\theta\) congruences is out of the scope of the current documentation. One should always be aware of that understanding the structure of CR base does not necessarily mean that one exhausts all the properties or classes in mathematical structures derived from restricted natural operations on replicas of the CR subsets and the \(x/z\) maps. Even the finite rules founding one’s own reality is far from one’s limited border of observations and knowledge. For a quick guess, one may name the RR maps of its reality as “quintessences” with \(\text{mass at moment } h/c^2\) \([\mathbf{M} \cdot \mathbf{T}]\) forming the space and \(\text{mass assemblies}\) of constant numbers of RR maps per moment \([\mathbf{T}^{-1}]\) ruled by physical laws with their spins and charges. The RR maps provide only the framework but not the direct results for physics.