Via Geometric Algebra: A Solution to the Snellius-Pothenot Resection (Surveying) Problem

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Abstract

Using geometric algebra (GA), we derive a solution to the classic Snellius-Pothenot problem. We note two types of cases where that solution does not apply, and present a GA-based solution for one of those cases.

The points $A$, $B$, $C$ and the angles $\alpha$ and $\beta$ are known. Determine the location of the observer point $P$. 
1 Statement of the Problem

As viewed from point $P$, the angles between the known points $A$, $B$, and $C$ are as shown in Fig. 1. What is the position of point $P$ in terms of the positions of $A$, $B$, and $C$?

2 Some of the Ideas that We Will Find Useful

1. An angle that is inscribed in a circle is half as large as the central angle that subtends the same chord (Fig. 2).

2. As a consequence of the first idea: If a chord of length $d$ is subtended by an inscribed angle whose measure is $\theta$, then the half-chord is subtended by a central angle with that same measure (Fig. 3).

3. The “rejection” of one vector from another (Fig. 4). See also [1].
Figure 2: An angle that is inscribed in a circle is half as large as the central angle that subtends the same chord.

Figure 3: If a chord of length $d$ is subtended by an inscribed angle whose measure is $\theta$, then the half-chord is subtended by a central angle that has that same measure.
3 Preliminary Analysis, and Formulation in GA Terms

3.1 Preliminary Analysis

$P$ and $B$ are the points of intersection of the two circles shown in Fig. 5. The points $B$ and $P$ are equidistant from the line that connects the centers of the two circles that are shown, because (1) the perpendicular bisector of any chord of a circle passes through that circle’s center, and (2) the chord $BP$ is common to the two circles (Fig. 6).

3.2 Formulation in GA Terms

The problem is formulated via the vectors (with point $B$ as origin) shown in Figs. 7 and 8.

4 The Solution, and Its Limitations

4.1 The Solution

As shown in Fig. 9, the vector from point $B$ to $P$ is twice the rejection of the vector from $B$ to the center of either circle, with respect to the vector between the circles’ centers.
Figure 5: Point \( P \) is one of the two points of intersection of the two circles that are shown here. Point \( B \) is the other.

Figure 6: The points \( B \) and \( P \) are equidistant from the line that connects the centers of the two circles that are shown, because (1) the perpendicular bisector of any chord of a circle passes through that circle’s center, and (2) the chord \( BP \) is common to the two circles.
Figure 7: Formulation of the problem in terms that will allow us to use GA.

Figure 8: Identifying the vectors from $B$ to the two circles’ centers.

$m_A = \frac{a}{2} + \frac{ai}{2 \tan \alpha}$

$m_B = \frac{b}{2} - \frac{bi}{2 \tan \beta}$
Figure 9: The vector $p$ is twice the rejection of $m_A$ (or also $m_B$) with respect to the vector between the circles’ centers.

5 Limitations of this Solution

Professor Francisco G. Montoya, of the Universidad de Almería, Spain, has pointed out that the solution presented above does not work when $P$ is aligned with $AB$ or $BC$, because in those cases the radius of one of the circles becomes infinite. Fig. 10 shows one such case, and its solution.

References

Figure 10: When point $P$ is aligned with $A$ and $B$ as shown here, we can use the relation $\tan \beta = \|c \wedge \hat{a}\|/\|p - (c \cdot \hat{a}) \hat{a}\|$. Consequently, $p = \begin{bmatrix} \|c \wedge \hat{a}\| & -c \cdot \hat{a} & \tan \beta \end{bmatrix} \hat{a}$.