Linear dynamical systems and transient terms
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Abstract
Under very general hypotheses, the behavior of dynamical systems described by a linear first
order differential equation is independent of the initial condition.

Theorem 1 Hp.
\[ \dot{y} + \alpha(t) y = \beta(t), \]  \hspace{1cm} (1)
where the coefficients \( \alpha(t) \) and \( \beta(t) \) are functions of class \( C^1(X) \) being \( X = [t_0, +\infty) \). Moreover \( \alpha(t) \) and each of its primitives diverges positively for \( t \to +\infty \).

Th. The general integral of the (1) is
\[ y(t, K) = y_0(t, K) + y_1(t), \]  \hspace{1cm} (2)
where \( K \) where \( K \) is a constant of integration, and
\[ y_1(t) = \frac{\beta(t)}{\alpha(t)}, \quad y_0(t, K) \to 0 \]
\( t \to +\infty \)
for which the asymptotic behavior of the general integral is
\[ y(t, K) \to \frac{\beta(t)}{\alpha(t)} \]
\( t \to +\infty \)

Dimostrazione. We apply the standard procedure for integrating (1). Precisely, an integral factor is
\[ I(t) = e^{\int \alpha(t) dt} \]

Multiplying the first and second sides of (1) by \( I(t) \):
\[ \dot{y} e^{\int \alpha(t) dt} + \alpha(t) y(t) e^{\int \alpha(t) dt} = \beta(t) e^{\int \alpha(t) dt} \]
i.e.
\[ \frac{d}{dt} \left[ y(t) e^{\int \alpha(t) dt} \right] = \beta(t) e^{\int \alpha(t) dt} \]
from which
\[ y(t) e^{\int \alpha(t) dt} = K + \beta(t) e^{\int \alpha(t) dt} \]
where \( K \) is a constant of integration. It follows that by integrating, the constant of integration will not appear as incorporated in \( K \). Therefore the general integral is
\[ y(t, K) = Ke^{-\gamma(t)} + e^{-\gamma(t)} \int \beta(t) e^{\gamma(t)} dt \]  \hspace{1cm} (4)
having defined $\gamma (t) \equiv \int \alpha (t) \, dt$. Performing an integration by parts in the integral a second member of the (4):

$$\int \beta (t) e^{\gamma (t)} dt \bigg|_{\gamma(t)=\alpha(t)} = \beta (t) e^{\gamma (t)} - \int \frac{\dot{\beta} (t)}{\alpha (t)} e^{\gamma (t)} dt$$

It follows

$$y (t, K) = Ke^{-\gamma (t)} + \frac{\beta (t)}{\alpha (t)} - e^{-\gamma (t)} \int \frac{\dot{\beta} (t)}{\alpha (t)} e^{\gamma (t)} dt$$

hence the assertion:

$$y (t, K) \longrightarrow \frac{\beta (t)}{\alpha (t)}$$

since by hypothesis $\gamma (t) \longrightarrow +\infty$. ■

From the theorem just proved it follows that $y_0 (t)$ is the so-called transitional term, while $y_1 (t)$ expresses the steady state behaviour. The latter does not depend on $K$, and therefore on the initial condition $y (t_0) = y_0$. 

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