COMPLEX CIRCLES OF PARTITION AND THE EXPANSION PRINCIPLES

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Abstract. In this paper, we further develop the theory of circles of partition by introducing the notion of complex circles of partition. This work generalizes the classical framework, extending from subsets of the natural numbers as base sets to partitions defined within the complex plane, which now serves as both the base and bearing set. We employ the expansion principles as central tools for rigorously investigating the possibility to partition numbers with base set as a certain subset of the complex plane.

1. **Introduction**

In our previous work [1], inspired in part by the binary Goldbach conjecture and its related problems (see [2] ... [5]), we introduced a method for studying the partition of numbers into subsets of the natural numbers. This method, though elementary in nature, establishes a geometric framework that we found particularly effective for analysing partitions in a combinatorial setting.

Let us consider a natural number $n \in \mathbb{N}$ that can be expressed as $n = u + v$, where $u, v \in \mathbb{M} \subset \mathbb{N}$. The method we propose associates each of these summands with distinct points on a circle, which is generated by the number n for all $n > 2$. A line joining these points on the circle provides a geometric correspondence that encapsulates the partition structure. This approach leverages geometric intuition to offer new insights into the combinatorial properties of partitions. As such, we refer to this structure as the **circle of partition** (CoP).

The **CoP** framework not only offers a new perspective on partition theory but also provides a versatile tool for exploring arithmetic and additive properties of numbers under certain constraints. The geometric interpretation has proven beneficial for illuminating various partition-related problems that arise in number theory.

In this paper, we extend this concept by introducing the notion of *complex circles of partition*. This generalization extends the domain from subsets of $\mathbb{N}$ to the complex plane, where both the base and bearing sets are now considered as a subset of the complex field. The resulting structure offers an enriched geometric and combinatorial framework, suitable for studying partition and additive problems in a broader and more intricate setting. We demonstrate the utility of the squeeze principle as a fundamental technique for rigorously analysing the feasibility of partitioning numbers in this extended framework.

The development of complex circles of partition is a natural continuation of our previous work and provides a deeper understanding of the underlying combinatorial structures. This extension opens new avenues for research in both partition theory

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and its applications to problems in analytic and additive number theory, offering fresh perspectives on well-established conjectures and inviting further investigation into geometric representations of partitions.

2. **Background**

Now, we restate the foundation of our method.

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote by

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, n - x \in \mathbb{M}\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this as **CoP**. We refer to the elements of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. In the special case where $\mathbb{M} = \mathbb{N}$, we denote the CoP simply as $\mathcal{C}(n)$.

We define $\|[x]\| := x$ as the **weight** of the point $[x]$, and similarly, we define the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $\|\mathcal{C}(n, \mathbb{M})\|$. Clearly, we have

$$\|\mathcal{C}(n)\| = \{1, 2, \dots, n - 1\}. \tag{2.1}$$

Definition 2.2. We denote $\mathbb{L}_{[x],[y]}$ as an **axis** of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say that $[y]$ is the axis partner of $[x]$, and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, as they represent the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is called the **center** of the CoP. If it exists, we refer to it as a **degenerate axis** $\mathbb{L}_{[x]}$, in contrast to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}),$$

which implies $[x], [y] \in \mathcal{C}(n, \mathbb{M})$ with $x + y = n$.

From now on, we will focus solely on real axes, and thus we will omit the term *real* in this section.

Proposition 2.3. *Each axis is uniquely determined by the points* $[x] \in \mathcal{C}(n, \mathbb{M})$.

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose $\mathbb{L}_{[x],[z]}$ is also an axis, with $z \neq y$. By Definition 2.2, it follows that $n = x + y = x + z$, implying $y = z$. This contradiction establishes the uniqueness of the axis, proving the claim. □

Proposition 2.4. *Each point of a CoP* $\mathcal{C}(n, \mathbb{M})$*, except for its center, has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner, assuming $[x]$ is not the center of the CoP. Then, for every point $[y] \neq [x]$ with $y \in \mathbb{M}$, we have

$$x + y \neq n.$$

This violates Definition 2.1, since $[x] \in \mathcal{C}(n, \mathbb{M})$. By Proposition 2.3, the possibility of more than one axis partner is excluded. This completes the proof. □

Notation. We denote by

$$\nu(n,\mathbb{M}) := |\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})\}| \tag{2.2}$$

the number of real axes of the CoP $\mathcal{C}(n,\mathbb{M})$. It is evident that

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor$$

if $\mathcal{C}(n,\mathbb{M})$ has k members.

It is not clear whether the axes $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ are lines joining points $[x]$ and $[y]$. In the present study, we will see that a transition from the base set $\mathbb{M} \subseteq \mathbb{N}$ into a certain subset $\mathbb{C}_{\mathbb{M}}$ of the complex plane reveals this natural geometric feature of circles of partition.

3. **Complex Circles of Partition**

First, we define a special subset of the complex numbers to use as the base set of CoPs.

Definition 3.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and

$$\mathbb{C}_{\mathbb{M}} := \{z = x + iy \mid x \in \mathbb{M},\ y \in \mathbb{R}\} \subset \mathbb{C}$$

be a subset of the complex numbers where the real part is from $\mathbb{M} \subseteq \mathbb{N}$. Then, a CoP with the special requirement

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_{\mathbb{M}},\ \Im(z)^2 = \Re(z)\,(n - \Re(z))\}$$

will be referred to as a **complex Circle of Partition**, abbreviated as **cCoP**. The special requirement is called the *circle condition*. The components x and y are referred to as the *real weight* and *imaginary weight*, respectively. The CoP $\mathcal{C}(n,\mathbb{M})$ is termed the *source CoP*. Since in the case $\mathbb{M} = \mathbb{N}$ the source CoP is abbreviated as $\mathcal{C}(n)$, we set

$$\mathcal{C}^o(n) := \mathcal{C}^o(n,\mathbb{C}_{\mathbb{N}}). \tag{3.1}$$

To distinguish between points $[z]$ of cCoPs and points z in the complex plane $\mathbb{C}$, we refer to the latter as *complex points*.

Definition 3.2. Let $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ be a cCoP and $[z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ a point of it with $z = x + iy$. Then $[n - z]$ with the weight $(n - x) - iy$ denotes the axis partner of $[z]$.

With this, the first requirement of a CoP is fulfilled:

$$\|[z]\| + \|[n-z]\| = x + iy + n - x - iy = n.$$

Important: For axis partners $[z_1]$ and $[z_2] = [n - z_1]$, it always holds that

$$\Im(z_1) = -\Im(z_2). \tag{3.2}$$

Definition 3.3. Let $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ be a cCoP and $[z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ a point of it with $z = x + iy$. Then $[\overline{z}]$ with the weight $x - iy$ denotes the *conjugate point* of $[z]$.

Definition 3.4. Let $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ be a cCoP and $\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ an axis of it. Then

$$\mathbb{L}_{[\overline{z}],[\overline{n-z}]}$$

denotes the *conjugate axis* of $\mathbb{L}_{[z],[n-z]}$. We do not distinguish between axes $\mathbb{L}_{[z],[n-z]}$ and $\mathbb{L}_{[n-z],[z]}$, since we do not consider axes as different up to the rearrangement of resident points.

Definition 3.5. Corresponding to Definition 2.2, we define

$$\nu^o(n, \mathbb{C}_\mathbb{M}) := |\{\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})\}|$$

as the number of axes of the cCoP $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$. Evidently,

$$\nu^o(n, \mathbb{C}_\mathbb{M}) = \begin{cases} 2\nu(n, \mathbb{M}) & \text{if the CoP } \mathcal{C}(n, \mathbb{M}) \text{ does not contain a degenerated axis,} \\ 2\nu(n, \mathbb{M}) + 1 & \text{if the CoP } \mathcal{C}(n, \mathbb{M}) \text{ contains a degenerated axis.} \end{cases} \tag{3.3}$$

We will see that the circle condition

$$\Im(z)^2 = \Re(z)\,(n - \Re(z)) \tag{3.4}$$

guarantees that all points of a cCoP lie on a circle in the complex plane $\mathbb{C}$.

Theorem 3.6. *Let $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ be a non-empty cCoP. The weights of all its points are located on a circle in the complex plane $\mathbb{C}$ with its center on the real axis at $\frac{n}{2}$ and a diameter n.*

Proof. Consider an arbitrary point $[z] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and its axis partner $[n-z]$. Set $x := \Re(z)$ and $y := \Im(z)$ [1]. Using the circle condition (3.4), we have

$$y^2 = x(n - x). \tag{3.5}$$

By Definition 3.1, $x \in \mathbb{M} \subseteq \mathbb{N}$. Hence, $x > 0$. The second requirement for $[z] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ is $n - x \in \mathbb{M}$. Therefore, $0 < x < n$. We now find the greatest imaginary part of the complex point z_o such that $[z_o] \in \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$. This means finding the root of the derivative of (3.5):

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\sqrt{x(n-x)} \\ &= \frac{1}{2}\frac{n - 2x}{\sqrt{x(n-x)}} = 0. \end{aligned}$$

Thus, we obtain $x_o = \frac{n}{2}$, provided that the denominator does not become zero. Substituting $x = \frac{n}{2}$ into (3.5), we get

$$y_o^2 = \frac{n}{2}\left(n - \frac{n}{2}\right) = \left(\frac{n}{2}\right)^2$$

and hence $|y_o| = |\Im(z_o)| = \frac{n}{2}$. Clearly, $\Im(n - z) = \Im(\overline{z}) = -\Im(z)$. Therefore, the points $[z]$, $\overline{[z]}$, and $[n-z]$ form a right-angled triangle with the hypotenuse $\mathbb{L}_{[z],[n-z]}$

[1] This setting will be used also in the sequel.

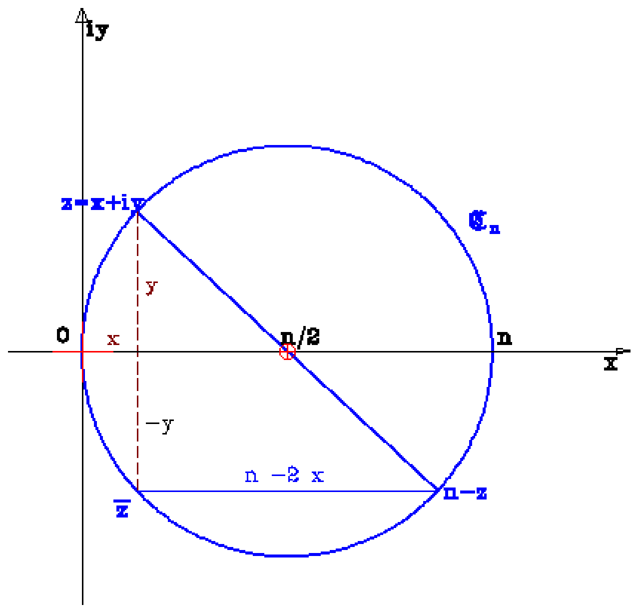

FIGURE 1. Diameter as axis of a cCoP

and the legs $2y$ and $n-2x$. By the Pythagorean Theorem (see Figure 1), we have

$$\begin{aligned}|\mathbb{L}_{[z],[n-z]}|^2 &= (2y)^2 + (n-2x)^2\\ &\text{and using (3.5)}\\ &= 4nx - 4x^2 + n^2 - 4nx + 4x^2\\ &= n^2 \text{ and thus}\\ |\mathbb{L}_{[z],[n-z]}| &= n.\end{aligned}$$

Since the sum of z and $n-z$ equals n, both points $[z]$ and $[n-z]$ are endpoints of an axis $\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and simultaneously form the diameter of a circle containing the complex points z, $\overline{z}$, and $n-z$, as their imaginary parts satisfy the circle condition. This is a circle with center on the real axis at $\frac{n}{2}$ and a diameter n. □

Definition 3.7. The circle in the complex plane $\mathbb{C}$ with center on the real axis at $\frac{n}{2}$ and diameter n will be denoted as the **embedding circle** $\mathfrak{C}_n$ of the cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. It holds that

$$\mathfrak{C}_n = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n, \Im(z)^2 = \Re(z)(n - \Re(z))\}.$$

Additionally, define

$$\begin{aligned}\mathfrak{I}_n &:= \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq n, \Im(z)^2 < \Re(z)(n - \Re(z))\},\\ \mathfrak{X}_n &:= \mathbb{C} \setminus (\mathfrak{I}_n \cup \mathfrak{C}_n)\end{aligned}$$

as the sets of all complex points $z \in \mathbb{C}$ inside and outside of the embedding circle $\mathfrak{C}_n$, respectively.

Definition 3.8. A diameter of the embedding circle $\mathfrak{C}_n$ through the complex points z and $n-z$ is denoted by

$$\mathfrak{D}_n(z, n-z).$$

It is evident that for a non-empty cCoP $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ with $m < n$, the following holds:

$$||\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})|| \subset \mathfrak{C}_m \subset \mathfrak{I}_n, \tag{3.6}$$
$$\mathfrak{I}_m \subset \mathfrak{I}_n \quad \text{and} \quad \mathfrak{X}_n \subset \mathfrak{X}_m.$$

Corollary 3.9. *For all subsets $\mathbb{M} \subseteq \mathbb{N}$, the cCoPs $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ for a fixed generator n have the same embedding circle $\mathfrak{C}_n$.*

Proposition 3.10. *Let $\mathfrak{C}_m$ and $\mathfrak{C}_n$ be two embedding circles with $m \neq n$. Then both circles have the origin as their only common point:*

$$\mathfrak{C}_m \cap \mathfrak{C}_n = \{(0,0)\}.$$

Proof. Let $z_m \in \mathfrak{C}_m$ and $z_n \in \mathfrak{C}_n$. Assume that $z_m = z_n$ as a common complex point of both circles. Then, $\Re(z_m) = \Re(z_n)$. For the imaginary weights, by the circle condition (3.4), we get

$$\Im(z_m)^2 = \Re(z_m)\left(m - \Re(z_m)\right),$$
$$\Im(z_n)^2 = \Re(z_n)\left(n - \Re(z_n)\right) = \Re(z_m)\left(n - \Re(z_m)\right),$$

and as a difference:

$$\Im(z_m)^2 - \Im(z_n)^2 = \Re(z_m)(m-n) = \Re(z_n)(m-n).$$

Since $m \neq n$, this is only zero if $\Re(z_m) = \Re(z_n) = 0$. Thus, for the imaginary weight, by the circle condition, we get

$$\Im(z_m)^2 = 0(m-0) = 0(n-0) = \Im(z_n)^2.$$

Hence, the origin is the only common point of $\mathfrak{C}_m$ and $\mathfrak{C}_n$. □

Corollary 3.11 (Big Bang)**.** *If $m < n$, then the circle $\mathfrak{C}_m$ resides fully inside the circle $\mathfrak{C}_n$, except at the common origin. Thus, the origin is the only common complex point of* ***all*** *embedding circles with increasing diameters, serving as the "Big Bang" of all embedding circles (refer to Figure 2).*

Theorem 3.12. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs with $m \neq n$. Then both cCoPs have no common point*

$$\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}}) \cap \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}}) = \emptyset.$$

Proof. In virtue of (3.6) and Proposition 3.10 the origin could be the only common point of both cCoPs. Since $\mathbb{M} \subseteq \mathbb{N}$, the real weight of a point of any cCoP cannot be 0. Hence both cCoPs have no common point. □

Proposition 3.13. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs with $n \neq m$. They have points $[z_m] \in \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $[z_n] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with a common real weight $\Re(z_m) = \Re(z_n) = x \in \mathbb{M}$ if and only if their source CoPs $\mathcal{C}(m, \mathbb{M})$ and $\mathcal{C}(n, \mathbb{M})$ share a common point $[x]$.*

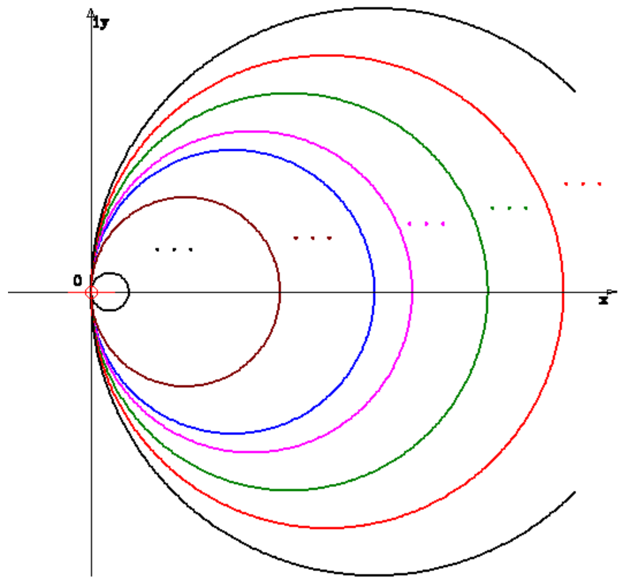

FIGURE 2. The "Big Bang"

Proof. Let $[x]$ be a common point of $\mathcal{C}(m,\mathbb{M})$ and $\mathcal{C}(n,\mathbb{M})$. Then $m-x$ and $n-x$ are members of $\mathbb{M}$, and $m-x-iy_m$ and $n-x-iy_n$ are members of $\mathbb{C}_{\mathbb{M}}$. Consequently, their axis partners $x+iy_m$ and $x+iy_n$ are also members of $\mathbb{C}_{\mathbb{M}}$. Therefore, with $z_m = x+iy_m$ and $z_n = x+iy_n$, we have

$$[z_m] \in \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) \text{ and } [z_n] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$$

with $x = \Re(z_m) = \Re(z_n)$. This reasoning can be reversed, and thus, from $x = \Re(z_m) = \Re(z_n)$, it follows that $[x] \in \mathcal{C}(m,\mathbb{M}) \cap \mathcal{C}(n,\mathbb{M})$. □

Corollary 3.14. *From Proposition 3.13, it follows that a cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ is non-empty if and only if its source CoP $\mathcal{C}(n,\mathbb{M})$ is non-empty.*

Proposition 3.15. *In the special case $\mathbb{M} = \mathbb{N}$, all cCoPs $\mathcal{C}^o(n)$ for integers $n \geq 2$ are non-empty.*

Proof. The source CoPs of such cCoPs are $\mathcal{C}(n)$ by virtue of Definition 2.1, and these are non-empty for all integers $n \geq 2$ by virtue of (2.1). Due to Corollary 3.14, their corresponding cCoPs are also non-empty. □

Corollary 3.16. *Since the considered points $[z]$ in Theorem 3.6 were arbitrary, it follows that all axes of a cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ have equal lengths:*

$$|\mathbb{L}_{[z],[n-z]}| = n \textit{ for all points } [z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}).$$

We now specify the calculation of the length of a chord joining arbitrary points of a cCoP under the circle condition.

Theorem 3.17. *Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a non-empty cCoP, and let $[z_1], [z_2] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two arbitrary points of it. Then the length $\Gamma([z_1], [z_2])$* [2] *of the chord $\mathcal{L}_{[z_1],[z_2]}$ is given by:*

$$|\mathcal{L}_{[z_1],[z_2]}| = \Gamma([z_1], [z_2]) = |\sqrt{x_1(n - x_2)} \pm \sqrt{x_2(n - x_1)}|,$$

where the "$-$" sign is used if $\text{sign}(y_1) = \text{sign}(y_2)$*, and the "$+$" sign otherwise.*

Proof.

$$\begin{aligned}
|\mathcal{L}_{[z_1],[z_2]}|^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
&= x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 \pm 2|y_1y_2| \\
&\text{and using (3.5)} \\
&= x_1^2 + x_2^2 - 2x_1x_2 \pm 2|y_1y_2| + nx_1 - x_1^2 + nx_2 - x_2^2 \\
&= nx_1 - x_1x_2 + nx_2 - x_1x_2 \pm 2|y_1y_2| \\
&= x_1(n - x_2) + x_2(n - x_1) \pm 2\sqrt{x_1(n - x_1)} \cdot \sqrt{x_2(n - x_2)} \\
&= x_1(n - x_2) + x_2(n - x_1) \pm 2\sqrt{x_2(n - x_1)} \cdot \sqrt{x_1(n - x_2)} \\
&= \left(\sqrt{x_1(n - x_2)} \pm \sqrt{x_2(n - x_1)}\right)^2.
\end{aligned}$$

Thus, the function $\Gamma([z_1], [z_2])$ for the chord length simplifies to:

$$\Gamma([z_1], [z_2]) = |\sqrt{x_1(n - x_2)} \pm \sqrt{x_2(n - x_1)}|. \tag{3.7}$$

□

If $[z_2]$ becomes $[n - z_1]$, then the chord $\mathcal{L}_{[z_1],[z_2]}$ becomes a diameter. In this case, $y_2 = -y_1$ and $x_2 = n - x_1$, and therefore:

$$\begin{aligned}
\Gamma([z_1], [n - z_1]) &= |\sqrt{x_1(n - x_2)} + \sqrt{x_2(n - x_1)}| \\
&= |\sqrt{x_1x_1} + \sqrt{(n - x_1)(n - x_1)}| \\
&= |x_1 + n - x_1| = n.
\end{aligned}$$

If $[z_2]$ is the axis partner of the conjugate point of $[z_1]$, then $x_2 = n - x_1$ and $y_2 = y_1$. Since the signs of both y values are equal, we obtain:

$$\begin{aligned}
\Gamma([z_1], [n - z_1]) &= |\sqrt{x_1x_1} - \sqrt{x_2x_2}| \\
&= |x_1 - x_2|.
\end{aligned}$$

This result coincides with the chord length in a CoP by virtue of its definition in [1].

A degenerated axis of a cCoP coincides with the diameter that is parallel to the imaginary axis. It is a real diameter but has the property that it equals its conjugate axis. In this case, using (3.7) with $x_2 = x_1 = \frac{n}{2}$ and $y_2 = -y_1$, we have:

$$\Gamma([z_1], [n - z_1]) = \left|\sqrt{\left(\frac{n}{2}\right)^2} + \sqrt{\left(\frac{n}{2}\right)^2}\right| = n.$$

[2] See [1, p. 2] Definition 2.2

4. Interior and Exterior Points of complex Circles of Partition

Theorem 4.1. *Let $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ be a non-empty cCoP. Then the distance from every complex point of $||\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})||$ to every complex point in $\mathfrak{I}_n$ is less than n, and from some complex point in $||\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})||$ to every complex point in $\mathfrak{X}_n$ is greater than n.*

Proof. The diameter of $\mathfrak{C}_n$ is the longest line from any complex point on this circle to any complex point inside or on the circle. Hence, all complex points of $\mathfrak{I}_n$ have a smaller distance to any complex point on $\mathfrak{C}_n$ than the diameter. By (3.6), this relation is also valid between any complex points of $||\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})||$ and $\mathfrak{I}_n$. Therefore, their distances are less than the diameter of $\mathfrak{C}_n$, which is n. Conversely, the distances between some complex point of $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ and every complex point in $\mathfrak{X}_n$ are greater than n, since $\mathfrak{X}_n$ consists of complex points outside of the embedding circle $\mathfrak{C}_n$. This completes the proof. □

Corollary 4.2. *For two non-empty cCoPs $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ with $m<n$, all distances between points of these sets are less than n, and some are greater than m.*

Definition 4.3. Since $\mathfrak{I}_n, \mathfrak{X}_n$ are defined in Definition 3.7 as all complex points **inside** and **outside** of the embedding circle $\mathfrak{C}_n$, we call the points $z \in \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M}$ **interior** points with respect to $\mathfrak{C}_n$ and denote the set of all such points as $\mathrm{Int}[\mathfrak{C}_n]$. Correspondingly, we call the complex points $z \in \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}$ **exterior** points with respect to $\mathfrak{C}_n$ and denote the set of all these points as $\mathrm{Ext}[\mathfrak{C}_n]$.

We observe that

$$\mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} \quad \text{and} \quad \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}.$$

Definition 4.4. Let $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ be a non-empty cCoP, and let $\mathfrak{C}_n$ be its embedding circle. Then we call the complex point $z \in \mathrm{Int}[\mathfrak{C}_n]$ an **interior** point with respect to the cCoP $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ if and only if for **all** points $[w] \in \mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$, $|z-w|<n$. We denote the set of all such points as $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$. Correspondingly, we call the complex point $z \in \mathrm{Ext}[\mathfrak{C}_n]$ an **exterior** point with respect to $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ if and only if for **some** points $[w] \in \mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$, $|z-w|>n$, and denote the set of all such points as $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$.

Let $n_o \in \mathbb{N}$ be the least generator for all cCoPs. If $n > n_o$ and $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ is an empty cCoP, then $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$ and $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})]$ are empty by definition.

Corollary 4.5. *If $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ is a non-empty cCoP, then by virtue of Theorem 4.1, we have*

$$\begin{gathered}\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] = \mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} \\ \textit{and} \\ \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] = \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_\mathbb{M}.\end{gathered} \tag{4.1}$$

Proposition 4.6. *Let $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ be two non-empty cCoPs. Then $m<n$ if and only if*

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \textit{ and } \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].$$

Proof. Let $m<n$, then by (4.1) we have

$$\begin{aligned}\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] &= \mathfrak{I}_m \cap \mathbb{C}_\mathbb{M} \text{ and since (3.6)} \\ &\subset \mathfrak{I}_n \cap \mathbb{C}_\mathbb{M} = \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})].\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] &= \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}} \text{ and since (3.6)}\\ &\subset \mathfrak{X}_m \cap \mathbb{C}_{\mathbb{M}} = \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].\end{aligned}$$

Conversely, with the embedding $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$, it follows that $\mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} \subset \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$, which is only valid with $m < n$. Analogously, by using the embedding $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$, it follows that $m < n$. □

Proposition 4.7. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs. Then $m < n$ if and only if*

$$||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \text{ and } ||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

Proof. Suppose $m < n$, then by (3.6) and the embedding $||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})|| \subset \mathbb{C}_{\mathbb{M}}$, we deduce

$$\begin{aligned}||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})|| &\subset \mathfrak{C}_m \cap \mathbb{C}_{\mathbb{M}}\\ &\subset (\mathfrak{C}_m \cap \mathbb{C}_{\mathbb{M}}) \cup \mathfrak{I}_m\\ &\subset (\mathfrak{C}_m \cup \mathfrak{I}_n) \cap \mathbb{C}_{\mathbb{M}} \text{ and since } \mathfrak{C}_m \subset \mathfrak{I}_n\\ &= \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}} \text{ and because of (4.1)}\\ &= \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})].\end{aligned}$$

In a similar manner, $||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$ can be easily verified. Conversely, the embedding $||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$ implies $\mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} \subset \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$, which is only valid for $m < n$. Analogously, by using the embedding $\mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$, we deduce that $m < n$. □

Proposition 4.8. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. If $[z_1],[z_2]$ are axis partners of the cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $|\mathbb{L}_{[z_1],[z_2]}| = n > m$, then $z_1, z_2 \in \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$.*

Proof. From the requirement $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ with $n > m$ and Proposition 4.6, it follows that

$$\begin{aligned}||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| &\subset \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \text{ and therefore}\\ z_1, z_2 &\in \mathrm{Ext}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].\end{aligned}$$

□

Proposition 4.9. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. If $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$, then $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$.*

Proof. The conditions above with Definition 4.3 implies that $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \neq \emptyset$ and $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \supset \emptyset$, and hence $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. □

We state a sort of converse of the above result in the following theorem.

Theorem 4.10. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}), \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) \neq \emptyset$. If $m < n$ and $[z] \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, then $z \notin \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$.*

Proof. By virtue of Definition 3.7, it follows that $\mathfrak{C}_n \cap \mathfrak{I}_n = \emptyset$ and $||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| \subset \mathfrak{C}_n$. It follows easily that $\mathfrak{I}_n \cap ||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})|| = \emptyset$. We know that for each point $[z] \in \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ then $z \notin \mathfrak{I}_n$. Because $m < n$, we deduce additionally that $\mathfrak{I}_m \subset \mathfrak{I}_n$ and hence

$$z \notin \mathfrak{I}_n \supset \mathfrak{I}_m \supset \mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} = \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

□

5. The Expansion Principles

In this section, we do not distinguish between the axes $\mathbb{L}_{[z],[n-z]}$ and $\mathbb{L}_{[n-z],[z]}$ and we consider only axes with $\Re(z) < \Re(n-z)$.

Lemma 5.1 (Axial Points Ordering Principle)**.** *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$ be non–empty cCoPs with integers $n, t \in \mathbb{Q} \subseteq \mathbb{N}$. Consider axes $\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathbb{L}_{[w],[n+t-w]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$. We have*

$$\Re(z) < \Re(w) \text{ and } \Re(n-z) < \Re(n+t-w) \tag{5.1}$$

if and only if

$$\Re(z) < \Re(w) < \Re(z) + t. \tag{5.2}$$

Proof. We note that the left inequalities are equivalent. Hence, we need to demonstrate that the right inequalities are also equivalent. Initially, we assume (5.1). From the right inequality and the existence of $\mathbb{L}_{[w],[n+t-w]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$, we obtain

$$\begin{aligned} \Re(n-z) &< \Re(n+t-w) = n + t - \Re(w) \\ \Longrightarrow \Re(w) &< n + t - \Re(n-z) = \Re(z) + t. \end{aligned}$$

This corresponds to the right side of (5.2).

Conversely, if the right side of (5.2) holds, we combine it with $\Re(w) = n + t - \Re(n+t-w)$:

$$\begin{aligned} \Re(w) = n + t - \Re(n+t-w) &< \Re(z) + t = n - \Re(n-z) + t \\ \Longrightarrow \Re(n-z) &< \Re(n+t-w). \end{aligned}$$

This establishes the right inequality of (5.1). □

Next we state the fundamental theorem about the existence of a "new" non–empty cCoP **depending** of two "known" cCops.

Theorem 5.2 (Expansion Principles)**.** *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $n, t, s \in \mathbb{Q} \subseteq \mathbb{N}$. Let $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ and $\mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$ two non–empty cCoPs with the axes*

$$\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M}) \text{ and } \mathbb{L}_{[w],[n+t-w]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M}).$$

If also

a. *$w, n-z \in \mathbb{C}_\mathbb{B}$ and $\Re(w) = \Re(z) + s$, then the axis*

$$\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_\mathbb{B})$$

exists with $\Re(u) = \Re(w)$ and $\Re(n+s-u) = \Re(n-z)$.

b. *$z, n+t-w \in \mathbb{C}_\mathbb{B}$ and $\Re(n+t-w) = \Re(n-z) + s$, then the axis*

$$\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_\mathbb{B})$$

exists with $\Re(u) = \Re(z)$ and $\Re(n+s-u) = \Re(n+t-w)$.

The corresponding imaginary parts are deduced from the circle condition. Thus, $\mathcal{C}^o(n+s, \mathbb{C}_\mathbb{B})$ is a non–empty cCoP.

Proof. We consider the requirements of both cases:

a. From the requirement $\Re(w) = \Re(z) + s$ we get

$$\begin{aligned} n + t - \Re(n + t - w) &= n - \Re(n - z) + s \\ \Re(n + t - w) &= \Re(n - z) + t - s. \end{aligned} \tag{5.3}$$

Since $w \in \mathbb{C}_{\mathbb{B}}$ as well as $n - z \in \mathbb{C}_{\mathbb{B}}$ and because of (5.3) we have

$$\Re(w) + \Re(n - z) = \Re(w) + \Re(n + t - w) - t + s = n + s.$$

Hence there exists an axis $\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}})$ with

$$\begin{aligned} \Re(u) &= \Re(w) \text{ and} \\ \Re(n + s - u) &= n + s - \Re(u) \\ &= n + s - \Re(w) \\ &= n + s - (\Re(z) + s) \\ &= n - \Re(z) = \Re(n - z). \end{aligned}$$

b. From the requirement $\Re(n + t - w) = \Re(n - z) + s$ we get

$$\begin{aligned} n + t - \Re(w) &= n - \Re(z) + s \\ \Re(z) &= \Re(w) - t + s. \end{aligned} \tag{5.4}$$

Since $z \in \mathbb{C}_{\mathbb{B}}$ as well as $n + t - w \in \mathbb{C}_{\mathbb{B}}$ and because of (5.4) we have

$$\Re(z) + \Re(n + t - w) = \Re(w) - t + s + \Re(n + t - w) = n + s.$$

Hence there exists an axis $\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}})$ with

$$\begin{aligned} \Re(u) &= \Re(z) \text{ and} \\ \Re(n + s - u) &= n + s - \Re(u) \\ &= n + s - \Re(z) \\ &= n + s - (\Re(w) - t + s) \\ &= n + t - \Re(w) = \Re(n + t - w). \end{aligned}$$

The corresponding imaginary parts are deduced from the *circle condition*. Thus, the cCoP $\mathcal{C}^o(n + s, \mathbb{C}_{\mathbb{B}})$ is non–empty. □

Depending of the value of s, we obtain the following statements:

Equality Principle. If $s = t$, then the axis

$$\mathbb{L}_{[w],[n+t-w]} \equiv \mathbb{L}_{[u],[n+t-u]} \hat{\in} \mathcal{C}^o(n + t, \mathbb{C}_{\mathbb{M}})$$

is an axis of $\mathcal{C}^o(n + t, \mathbb{C}_{\mathbb{B}})$ too and with (5.3) resp. (5.4) we get

a. $\Re(z) < \Re(w)$ and $\Re(n - z) = \Re(n + t - w)$.
b. $\Re(z) = \Re(w)$ and $\Re(n - z) < \Re(n + t - w)$.

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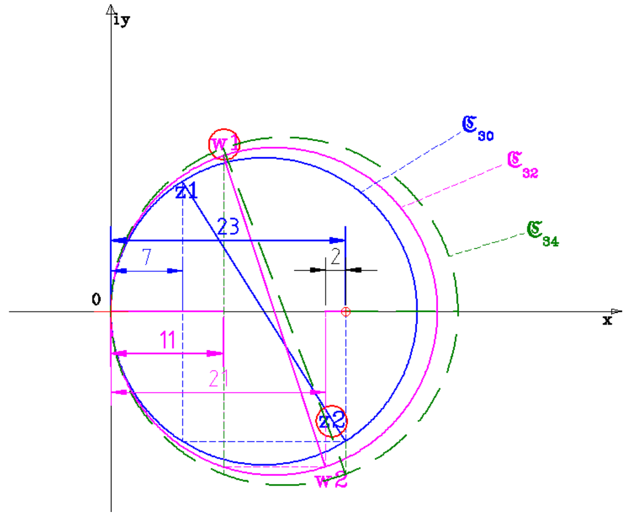

FIGURE 3. Forecasting of $\mathcal{C}^o(34, \mathbb{C}_{\mathbb{P}})$ by $\mathcal{C}^o(30)$ and $\mathcal{C}^o(32)$

Forecast Principle. If $s > t$, then we have determined from two "known" non–empty cCoPs $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ **after them** a "new" non–empty cCoP $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$. By virtue of (5.3) resp. (5.4) holds

a. $\Re(z) < \Re(w)$ and $\Re(n-z) > \Re(n+t-w)$.
b. $\Re(z) > \Re(w)$ and $\Re(n-z) < \Re(n+t-w)$.

Figure 3 shows how $\mathcal{C}^o(30)$ and $\mathcal{C}^o(32)$ "forecast" $\mathcal{C}^o(34, \mathbb{C}_{\mathbb{P}})$.

Squeeze Principle. If $0 < s < t$, then we have determined from two "known" non–empty cCoPs $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ **between them** a "new" non–empty cCoP $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$. By virtue of (5.3) resp. (5.4) holds in both cases

$$\Re(z) < \Re(w) \text{ and } \Re(n-z) < \Re(n+t-w)$$

and by virtue of Lemma 5.1

$$\Re(z) < \Re(w) < \Re(z) + t.$$

Figure 4 shows how $\mathcal{C}^o(30)$ and $\mathcal{C}^o(38)$ "squeeze" $\mathcal{C}^o(34, \mathbb{C}_{\mathbb{P}})$.

If we set the positive integers $\mathbb{N}$ instead of $\mathbb{M}$ and the set of all odd primes $\mathbb{P}$ instead of $\mathbb{B}$, we get the following corollary from the theorem above.

Corollary 5.3 (Special Expansion Principles)**.** *Let $n, t, s \in 2\mathbb{N}$. Let $\mathcal{C}^o(n)$ and $\mathcal{C}^o(n+t)$ two non–empty cCoPs with the axes*

$$\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n) \textit{ and } \mathbb{L}_{[w],[n+t-w]} \hat{\in} \mathcal{C}^o(n+t)$$

and the base set $\mathbb{C}_{\mathbb{N}}$[3]. If also

a. *$w, n-z \in \mathbb{C}_{\mathbb{P}}$ and $\Re(w) = \Re(z) + s$, then the axis*

$$\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{P}})$$

[3]see (3.1)

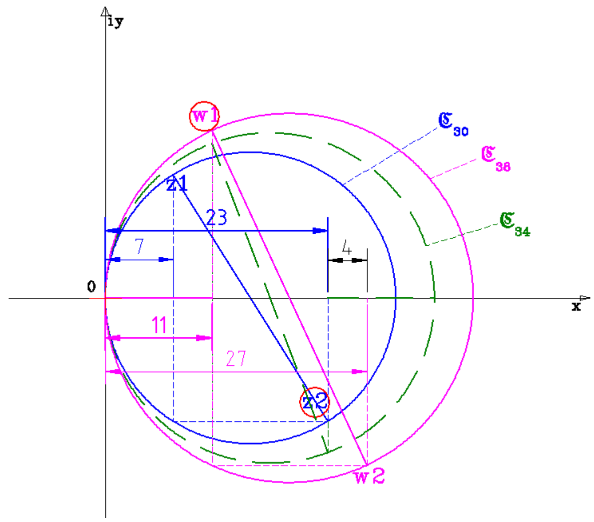

FIGURE 4. Squeezing of $\mathcal{C}^o(34, \mathbb{C}_\mathbb{P})$ by $\mathcal{C}^o(30)$ and $\mathcal{C}^o(38)$

exists with $\Re(u) = \Re(w)$ *and* $\Re(n+s-u) = \Re(n-z)$.

b. $z, n+t-w \in \mathbb{C}_\mathbb{P}$ *and* $\Re(n+t-w) = \Re(n-z)+s$, *then the axis*

$$\mathbb{L}_{[u],[n+s-u]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_\mathbb{P})$$

exists with $\Re(u) = \Re(z)$ *and* $\Re(n+s-u) = \Re(n+t-w)$.

The imaginary parts yield corresponding to the circle condition . Thus, $\mathcal{C}^o(n+s, \mathbb{C}_\mathbb{P})$ *is a non–empty cCoP.*

If $n+t-w \in \mathbb{C}_\mathbb{B}$ as well as of $n-z$, then we get a wide forecasting.

Proposition 5.4. *Let* $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ *and* $n, t, s \in \mathbb{Q} \subseteq \mathbb{N}$. *Let* $\mathcal{C}^o(n, \mathbb{C}_\mathbb{M})$ *and* $\mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M})$ *be two non–empty cCoPs with the axes*

$$\mathbb{L}_{[z],[n-z]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_\mathbb{M}) \text{ and } \mathbb{L}_{[w],[n+t-w]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_\mathbb{M}).$$

If also

$$n-z \text{ as well as } n+t-w \in \mathbb{C}_\mathbb{B} \text{ and } \Re(w)+\Re(z) = t+s,$$

then the axis $\mathbb{L}_{[u],[2n-s-u]} \hat{\in} \mathcal{C}^o(2n-s, \mathbb{C}_\mathbb{B})$ *exists with*

(1) *if* $\Re(n-z) \leq \Re(n+t-w)$, *then*

$$\Re(u) = \Re(n-z) \text{ and } \Re(2n-s-u) = \Re(n+t-w).$$

(2) *if* $\Re(n-z) > \Re(n+t-w)$, *then*

$$\Re(u) = \Re(n+t-w) \text{ and } \Re(2n-s-u) = \Re(n-z).$$

The corresponding imaginary parts can be deduced from the circle condition. Thus, $\mathcal{C}^o(2n-s, \mathbb{C}_\mathbb{B})$ *is a non–empty cCoP.*

Proof. Since $n+t-w \in \mathbb{C}_\mathbb{B}$ as well as $n-z \in \mathbb{C}_\mathbb{B}$ and because of $\Re(w)+\Re(z) = t+s$ we have

$$\Re(n+t-w) + \Re(n-z) = n+t-\Re(w)+n-\Re(z) = 2n-s.$$

Hence there exists an axis $\mathbb{L}_{[u],[2n-s-u]} \hat{\in} \mathcal{C}^o(2n-s, \mathbb{C}_\mathbb{B})$ with

(1) if $\Re(n-z) \leq \Re(n+t-w)$, then

$$\begin{aligned}\Re(u) &= \Re(n-z) \text{ and}\\ \Re(2n-s-u) &= 2n-s-\Re(u)\\ &= 2n-s-\Re(n-z)\\ &= 2n-s-(n-\Re(z)), \text{ since } t+s=\Re(z)+\Re(w)\\ &= n+t-(\Re(w)+\Re(z))+\Re(z)=\Re(n+t-w)\end{aligned}$$

(2) if $\Re(n-z) > \Re(n+t-w)$, then

$$\begin{aligned}\Re(u) &= \Re(n+t-w) \text{ and}\\ \Re(2n-s-u) &= 2n-s-\Re(u)\\ &= 2n-s-\Re(n+t-w)\\ &= 2n-s-(n+t-\Re(w)), \text{ since } t+s=\Re(z)+\Re(w)\\ &= n-(\Re(z)+\Re(w))+\Re(w)=\Re(n-z)\end{aligned}$$

and the corresponding imaginary parts are deduced from the *circle condition*. Thus, the cCoP $\mathcal{C}^o(2n+t-s,\mathbb{C}_{\mathbb{B}})$ is non–empty. □

For the sake of completeness we note that the case $z \in \mathbb{C}_{\mathbb{B}}$ and $w \in \mathbb{C}_{\mathbb{B}}$ results in a "new" cCoP before $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$.

These principles and their derivations serve as a fundamental tool for examining the partitioning of numbers of a specific parity, where components belong to a designated subset of integers. The method involves selecting two non-empty cCoPs with a shared base set and identifying additional non-empty cCoPs with generators that are situated between these two resp. after these two. These can be applied innovatively to explore the broader issue of partitioning numbers such that each summand originates from the same subset of positive integers.

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