Rutherford Cross Section in the laboratory frame-PartII

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In this pedagogical article, we extend the direct derivation of the classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame of two equal mass particles, ala relativistic quantum mechanics as presented in the book of Bjorken and Drell up to \( (\nu/c)^2 \) order.

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I. INTRODUCTION

The classical non-relativistic Rutherford scattering cross section, differential, in the laboratory frame is computed in the classical mechanics in two steps. First the Rutherford scattering cross section, differential, in the center of mass frame is calculated, \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \), where, \( \Theta \) is the c.o.m scattering angle. Then using transformation from the center of mass to the laboratory frame, \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \), the differential scattering cross section is calculated in the laboratory frame, for two particles of equal mass and elastic scattering, as \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \), where, \( \theta \) is the laboratory frame scattering angle. On the other hand, in the relativistic quantum mechanics of Dirac, the differential scattering cross section can be calculated in the laboratory frame directly, \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \). Taking non-relativistic limit has yielded, \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \) the classical non-relativistic differential cross-section, \( \frac{k^2}{E_{lab} \cos \theta \sin \phi} \). In this paper, we extend the analysis up to \((v/c)^2\) order.

II.

Starting from the Dirac equation of a spin half particle, of rest mass \( m_0 \) and charge \( q \), in the presence of electromagnetic field, \( A^\mu \), is given by \((i\hbar \gamma^\mu \partial_\mu - qe^\mu A_\mu - m_0c)\psi(x) = 0 \) and considering a scattering between a hypothetical proton and an electron, assuming the hypothetical proton’s mass being equal to that of the electron i.e. setting \( m_0 = M_0 \) we obtained the differential scattering cross section \( \sigma \) as in our previous paper, \( \sigma \), as

\[
\frac{d\sigma}{d\Omega_f} \bigg|_{lab} = \frac{4}{(4\pi\epsilon_0)^2} \frac{1}{m_0 c^2 E_i v_i} \int_{m_0 c^2}^{E_i + mc^2} \frac{[p_f |d\sigma_f|E_f]}{(p_f - p_i)^2 + \delta(2m_0 c^2 - \frac{2E_i E_f}{c} + 2\vec{p}_i \vec{p}_f + 2m_0(E_i - E_f))}
\]

Resorting to non-relativistic limit keeping terms up to \((v/c)^2\) order, tentamounts to,

\[
E_i = m_0 c^2 + \frac{v_i^2}{2} + \frac{3}{8}m_0 v_i^2 \beta_i^2,
\]

\[
\frac{E_i}{c} E_f = m_0 c^2 + \frac{1}{2}m_0^2 (v_i^2 + v_f^2) + \frac{3}{8}m_0^2 (v_i^2 + v_f^2) (\beta_i^2 + \beta_f^2),
\]

\[
E_i - E_f = \frac{1}{2}m_0 (v_i^2 - v_f^2) + \frac{3}{8}m_0 (v_i^2 \beta_i^2 - v_f^2 \beta_f^2),
\]

\[
[p_f] = m_0 v_f (1 + \frac{1}{2} \beta_f^2),
\]

\[
\vec{p}_i \vec{p}_f = m_0^2 v_i v_f \sin \theta (1 + \frac{1}{2} \beta_f^2),
\]

\[
dE_f = m_0 v_f \sin \theta (1 + \frac{1}{2} \beta_f^2)
\]

one achieves,

\[
\frac{d\sigma}{d\Omega_f} \bigg|_{lab} = \frac{4}{(4\pi\epsilon_0)^2} \frac{1}{m_0 c^2 E_i c} \frac{|p_f|d\sigma_f|E_f|}{(p_f - p_i)^2 + \delta(2m_0 c^2 - \frac{2E_i E_f}{c} + 2\vec{p}_i \vec{p}_f + 2m_0(E_i - E_f))}
\]

\[
d\sigma = \delta(2m_0 c^2 - \frac{2E_i E_f}{c} + 2\vec{p}_i \vec{p}_f + 2m_0(E_i - E_f))\]

\[
(p_f - p_i)^2 = 2m_0^2 c^2 - 2\vec{p}_i \vec{p}_f = 2|m_0^2 c^2 - \frac{E_i E_f}{c} - \vec{p}_i \vec{p}_f| = m_0^2 [-v_i^2 - v_f^2 + 2v_i v_f \cos \theta (1 + \frac{1}{2} (\beta_i^2 + \beta_f^2)) - \frac{3}{4}m_0^2 (v_i^2 + v_f^2) (\beta_i^2 + \beta_f^2) + m_0^3 (v_i^2 - v_f^2) + \frac{3}{4}m_0^3 (v_i^2 \beta_i^2 - v_f^2 \beta_f^2)]
\]

\[
(p_f - p_i)^2 |_{v_i \rightarrow v_i \cos \theta (1 + \frac{1}{2} \beta_i^2 (1 + \cos \theta^2)) = 0} = -m_0^2 v_i^2 \sin^2 \theta [1 - \frac{1}{4} \beta_i^2 (1 + \cos \theta^2) (1 + \cos \theta^2 - 3)]
\]

\[
[2m_0^2 E_f E_i + m_0 (p_i, p_f) (E_f - E_i - m_0 c^2) + m_0^3 c^2 [2(E_f - E_i) + m_0 c^2]] |_{v_i \rightarrow v_i \cos \theta (1 + \frac{1}{2} \beta_i^2 (1 + \cos \theta^2)) = 0} = 2m_0^2 c^2 [1 + \frac{1}{2} \beta_f (\beta_f + \beta_i \cos \theta)]
\]

\[
|2m_0^2 E_f E_i + m_0 (p_i, p_f) (E_i - E_f - m_0 c^2) + m_0^3 c^2 [2(E_f - E_i) + m_0 c^2]| |_{v_f \rightarrow v_f \cos \theta (1 + \frac{1}{2} \beta_f^2 (1 + \cos \theta^2)) = 0} = 2m_0^2 c^2 [1 + \beta_i^2 \cos^2 \theta]
\]
Hence, the differential scattering cross-section, in the NR, for two non-identical particles of equal mass and equal but opposite charges, in the laboratory frame i.e. when one particle is at rest initially is given by, up to \((\frac{v}{c})^2\) or, \(\beta^2\), order

\[
\frac{d\sigma}{d\Omega}|_{\text{lab}} = 4\left(\frac{e^2}{4\pi \epsilon_0}\right)^2 \frac{1}{m_0 c^2 E_i v_i} \int_{m_0 c^2}^{E_i + m_0 c^2} \frac{|p_f| dE_f}{(p_f - p_i)^2} \delta(2m_0 c^2 - 2E_i E_f/c + 2p_i \cdot p_f + 2m_0(E_i - E_f)) \\
\frac{2m_0^2 E_f E_i + m_0(p_i \cdot p_f)(E_i - E_f - m_0 c^2) + m_0^3 c^2 [2(E_f - E_i) + m_0 c^2]}{m_0^2 v_f(1 + \frac{1}{2} \beta_i^2) m_0 v_f dE_f(1 + \frac{3}{2} \beta_i^2)} \\
[1 + \beta_i^2 (1 + \cos^2 \theta)(\cos^2 \theta - 3)] \frac{1}{2m_0 v_f} \delta(v_f - v_i \cos \theta(1 - \frac{1}{4} \beta_i^2(1 + \cos^2 \theta))) 2m_0^4 c^4 [1 + \beta_i^2 \cos^2 \theta] \\
= 4\left(\frac{e^2}{4\pi \epsilon_0}\right)^2 \frac{\cos \theta}{m_0^2 v_1^4 \sin^4 \theta} (1 - \frac{1}{4} \beta_i^2(1 + \cos^2 \theta))(1 - \frac{1}{2} \beta_i^2(1 + \beta_i^2 \cos^2 \theta)[1 + \frac{\beta_i^2 (1 + \cos^2 \theta)(\cos^2 \theta - 3)}{\sin^2 \theta}]) (1 + \frac{5}{4} \beta_i^2 \cos^2 \theta) \\
= 4\left(\frac{e^2}{4\pi \epsilon_0}\right)^2 \frac{\cos \theta}{2E_{\text{lab}}^2 \sin^4 \theta} (1 - \frac{1}{4} \beta_i^2(1 + \cos^2 \theta) + \beta_i^2[-\frac{1}{2} + \cos^2 \theta + \frac{5}{4} \cos^2 \theta + (1 + \cos^2 \theta)(\cos^2 \theta - 3)]) \\
= (\frac{e^2}{4\pi \epsilon_0})^2 \frac{\cos \theta}{E_{\text{lab}}^2 \sin^4 \theta} [1 - \beta_i^2(\frac{2}{\sin^2 \theta} + \frac{1}{4} - \frac{3}{2} \cos^2 \theta)] \\
= \frac{k^2 \cos \theta}{E_{\text{lab}}^2 \sin^4 \theta} [1 - \beta_i^2(\frac{2}{\sin^2 \theta} + \frac{1}{4} - \frac{3}{2} \cos^2 \theta)]
\]

III. ACKNOWLEDGEMENT

The reference where this is done, has not reached the author. Hopefully, nothing new has been presented in the paper.