notes on probe-D-branes

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1 Wilsonian renormalisation group of probe-D-branes

The DBI action:

\[ S_{\text{DBI}} = \int dr L_{\text{DBI}} = \int d^{p+1} x dr L_{\text{DBI}} = -N_q \int d^{p+1} x dr r^p \sqrt{1 - A_0'^2} \]  

(1.1)

The constant of motion

\[ \frac{\delta S_{\text{DBI}}}{\delta A_0} = \frac{\delta L_{\text{DBI}}}{\delta A_0'} = \frac{N_q r^p A_0'}{\sqrt{1 - A_0'^2}} = -d \]  

(1.2)

and therefore

\[ \frac{\delta S_B}{\delta A_0} = - \frac{\delta S_{\text{DBI}}}{\delta A_0} = d \]  

(1.3)

so that when varied and evaluated on-shell \( S = S_B + S_{\text{DBI}} \) vanishes. Then

\[ S = S_B[\rho] + S_{\text{DBI}}[\rho] \]

\[ = S_B[\rho - \delta \rho] + S_{\text{DBI}}[\rho - \delta \rho] + \int_{\rho - \delta \rho}^{\rho} d\rho' \partial_{\rho'} S_B + \int_{\rho - \delta \rho}^{\rho} d\rho' \int d^{p+1} x \frac{\delta S_B}{\delta A_0} A_0' + \int_{\rho - \delta \rho}^{\rho} d\rho dr L_{\text{DBI}} \]  

(1.4)

The Wilsonian renormalisation group equation is

\[ \partial_{\rho} S_B = - \int d^{p+1} x \left[ \frac{\delta S_B}{\delta A_0} \frac{\partial A_0}{\partial \rho} + L_{\text{DBI}} \right] \]

\[ = \int d^{p+1} x \frac{1}{N_q \rho^p} \left[ N_q^2 \rho^2 \left( \frac{\delta S_B}{\delta A_0} \right)^2 \right] \sqrt{1 - A_0'^2} \]  

(1.5)

Using \( \sqrt{1 - A_0'^2} = \frac{N_q \rho^p}{\sqrt{N_q^2 \rho^2 + \left( \frac{\delta S_B}{\delta A_0} \right)^2}} \), the RG flow equation is

\[ \partial_{\rho} S_B = N_q \rho^p \int d^{p+1} x \sqrt{1 + \frac{1}{N_q^2 \rho^2} \left( \frac{\delta S_B}{\delta A_0} \right)^2} \]  

(1.6)
Formally, we can expand this to get

\[
\partial \rho S_B = \int d^{p+1} x N_q \rho^p \left[ 1 + \frac{1}{2} \rho^{2p} \left( \frac{\delta S_B}{\delta A_0} \right)^2 + \frac{1}{8} \rho^{-4p} \left( \frac{\delta S_B}{\delta A_0} \right)^4 + \ldots \right]
\]

\[
= \int d^{p+1} x N_q \rho^p \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k \rho^{-2kp} \left( \frac{\delta S_B}{\delta A_0} \right)^{2k}
\]

(1.7)

Now we can write at \( \rho_0 \)

\[
S_{B_{\text{sub}}}^{\rho_0} = S_B[\rho_0] - S_{c.t.}[\rho_0] = \frac{1}{2} \int d^{p+1} x \delta A_0 = \frac{1}{2} \int d^{p+1} x dA_0 - \frac{N_q}{p+1} \int d^{p+1} x \delta A_0 (\sqrt{-g} \delta A_0)_{0}(1.8)
\]

so that

\[
\frac{\delta}{\delta A_0} (\delta A_0) = \frac{\delta}{\delta A_0} (dA_0)
\]

(1.9)

and

\[
A_0 = \frac{2\sqrt{-g}(\rho_0)}{(p+1)(d-\delta_0)}
\]

(1.10)

We can now make all terms in \( S_B \) run and write out explicitly all generally possible counter-terms.

\[
S_B = N_q \int d^{p+1} x \left[ \frac{\sqrt{-g} \alpha}{p+1} + \frac{1}{2} \delta A_0 - \sqrt{-g} \sum_{n=2}^{\infty} \frac{\lambda_n}{n} A_0^n \right]
\]

(1.11)

with

\[
\alpha(\rho_0) = 1, \quad \delta(\rho_0) = \delta_0, \quad \sqrt{-g} \lambda_n(\rho_0) = 0, \text{ at } \rho_0 \to \infty,
\]

(1.12)

set by the minimal-subtraction values of holographic renormalisation counter-terms. The zeroth term corresponds to the volume renormalisation and higher orders to multi-trace deformations. At orders of \( A_0^0, A_0^1 \) and \( A_0^2 \) we find from (1.6)

\[
\partial_{\rho} (\sqrt{-g} \alpha) = (p+1) \sqrt{\rho^{2p} + \delta^2}
\]

(1.13)

\[
\partial_{\rho} \delta = -2 \sqrt{-g} \frac{\delta \lambda_2}{\sqrt{\rho^{2p} + \delta^2}}
\]

(1.14)

\[
\partial_{\rho} (\sqrt{-g} \lambda_2) = c_1 \lambda_2^2 + c_2 \lambda_3
\]

(1.15)

1.1 RG equation in the IR with zero temperature

Consider the \( \rho \to 0 \) regime of \( \sqrt{-g} = \rho^{p+1} \), where

\[
\partial_{\rho} S_B = V_p \frac{\delta S_B}{\delta A_0} = N_q V_p \left[ \delta - \sum_{n=1}^{\infty} \sqrt{-g} \lambda_{n+1} A_0^n \right]
\]

(1.16)

We get

\[
\partial_{\rho} (\sqrt{-g} \alpha) = (p+1) \delta
\]

(1.17)

\[
\partial_{\rho} \delta = -2 \sqrt{-g} \lambda_2
\]

(1.18)

\[
\partial_{\rho} (\sqrt{-g} \lambda_n) = n \sqrt{-g} \lambda_{n+1}, \text{ for } n \geq 2.
\]

(1.19)
We find

\[ S_B[\rho \to 0] = \frac{1}{2(p + 1)} V_{p}(1 + e^{A_{0} \rho}) \sqrt{-g} \alpha \]  

(1.20)
and using (1.16) we find

\[ \partial_{\rho} (\sqrt{-g}\alpha) = 0 \Rightarrow \sqrt{-g} \alpha \bigg|_{\rho \to 0} = C_{\text{const}} + O(\rho) \]  

(1.21)

and

\[ \partial(\rho \to 0) = 0 \Rightarrow S_{B}^{\text{ren}} \to 0 \]  

(1.22)

Writing \( S_B = \frac{1}{2} V_{p}dZ(\rho)A_{0} \) and using (1.16) we find

1.2 RG equation in the IR with non-zero temperature

Use \( \rho = r_{H} + u \) and take \( u \to 0 \). If \( \alpha(\rho) \) is analytic at \( u = 0 \) then

\[ \partial(u) = \frac{r_{H}^{p+1/2}\alpha(r_{H})}{2\sqrt{p + 1}} \frac{1}{\sqrt{u}} + O(\sqrt{u}) \]  

(1.23)
2 Thermodynamics

$$\Omega_{\text{fun}}(\rho) = -S_{\text{DBI on shell}} = S_B(\rho) \quad (2.1)$$