Symmetry of the conductivity matrix

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1. The Green’s function equation

Let us first consider the conductivity matrix on a non-compact manifold in \( d = 2 \) dimensions. The differential equation that describes the behaviour of \( \alpha \),

\[
\partial_i \left[ Z \sqrt{\gamma} \gamma^{ij} (E_j + \partial_j \alpha) \right] = 0,
\]

(1)
can be formally solved by writing

\[
\alpha(\vec{x}) = \int d^2y G(\vec{x} - \vec{y}) \frac{\partial}{\partial y^i} \left[ Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j,
\]

(2)
which leads to the Green’s function equation

\[
\partial \frac{\partial}{\partial x^i} \left[ Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} G(\vec{x} - \vec{y}) \right] = -\delta^{(2)}(\vec{x} - \vec{y}).
\]

(3)
On a torus, the right-hand-side needs to be modified so that the integral over the expression vanishes. Hence, we can write down a new Green’s function equation

\[
\partial \frac{\partial}{\partial x^i} \left[ Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} G(\vec{x} - \vec{y}) \right] = -\delta^{(2)}(\vec{x} - \vec{y}) + \frac{1}{\int d^2 x' \sqrt{\gamma(x')}} \sqrt{\gamma(\vec{x})}.
\]

(4)
Note that this extension of the Green’s function equation is consistent with the defining differential equations as the addition is purely a function of \( x \). Hence, we can formally define \( G(x - y) = \tilde{G}(x - y) + R(x) \), so that

\[
\partial \frac{\partial}{\partial x^i} \left[ Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} R(x) \right] = \frac{1}{\int d^2 x' \sqrt{\gamma(x')}} \sqrt{\gamma(x)},
\]

(5)
\[
\partial \frac{\partial}{\partial x^i} \left[ Z \sqrt{\gamma} \gamma^{ij} \frac{\partial}{\partial x^j} \tilde{G}(\vec{x} - \vec{y}) \right] = -\delta^{(2)}(\vec{x} - \vec{y}).
\]

(6)
This implies that

\[
\alpha(\vec{x}) = \int d^2y \tilde{G}(\vec{x} - \vec{y}) \frac{\partial}{\partial y^i} \left[ Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j = \int d^2y G(\vec{x} - \vec{y}) \frac{\partial}{\partial y^i} \left[ Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j,
\]

(7)
as the following divergence integral over a torus vanishes (the metric and \( Z \) are single-valued):

\[
R(x) \int d^2y \frac{\partial}{\partial y^i} \left[ Z(\vec{y}) \sqrt{\gamma(\vec{y})} \gamma^{ij}(\vec{y}) \right] E_j = R(x) \int d^2y \sqrt{\gamma} \nabla_i \left( Z(\vec{y}) E^i \right) = 0.
\]

(8)

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2. Symmetry of the conductivity matrix

The next important thing is to understand whether $G(x - y)$ is symmetric under the interchange of $x$ and $y$. Instead of this statement, we will prove a somewhat weaker statement, which will be sufficient to show that the conductivity matrix is symmetric. Consider the integral

$$I = \int d^2y \sqrt{\gamma(y)} \left\{ G(y - x) \nabla_i \left[ Z(y) \nabla^i G(y - x') \right] - G(y - x') \nabla_i \left[ Z(y) \nabla^i G(y - x) \right] \right\}$$

$$= \int d^2y \nabla_i \left\{ \sqrt{\gamma(y)} G(y - x) Z(y) \nabla^i G(y - x') - \sqrt{\gamma(y)} G(y - x') Z(y) \nabla^i G(y - x) \right\}$$

$$= 0,$$  \hspace{1cm} (9)

where all covariant derivatives act w.r.t. $y$. Since the integrand is a total derivative, and we are integrating over a compact torus without a boundary (and $G(x-y)$ is single valued by construction), the integral automatically vanishes.

Consider now another integral, which also clearly vanishes,

$$I' = \frac{\partial^2}{\partial x^i \partial x'^j} I = 0.$$  \hspace{1cm} (10)

By using Eq. (4), we find that

$$I' = \frac{\partial^2}{\partial x^i \partial x'^j} \left[ G(x - x') - G(x' - x) + \frac{1}{V_2} \int d^2y \sqrt{\gamma(y)} \left[ G(y - x) - G(y - x') \right] \right]$$

$$= \frac{\partial^2}{\partial x^i \partial x'^j} \left[ G(x - x') - G(x' - x) \right] = 0.$$  \hspace{1cm} (11)

(12)

Hence,

$$\frac{\partial^2}{\partial x^i \partial y^j} G(x - y) = \frac{\partial^2}{\partial x^i \partial y^j} G(y - x).$$  \hspace{1cm} (13)

The solution for the conserved “auxiliary” current can be written as

$$e^2 J^i = Z \sqrt{\gamma} \gamma^{ij} (E_j + \partial_j \alpha)$$

$$= \left[ Z(\tilde{x}) \sqrt{\gamma(\tilde{x})} \gamma^{ij}(\tilde{x}) - \int d^2y Z(\tilde{x}) Z(\tilde{y}) \sqrt{\gamma(x)} \gamma^{ij}(\tilde{x}) \gamma^{jk}(\tilde{y}) \gamma^{kl}(\tilde{y}) \frac{\partial^2}{\partial x^k \partial y^l} G(\tilde{x} - \tilde{y}) \right] E_j,$$  \hspace{1cm} (14)

and the conductivity matrix is given by

$$e^2 \sigma^{ij} = \frac{1}{V_2} \int d^2x Z(\tilde{x}) \sqrt{\gamma(\tilde{x})} \gamma^{ij}(\tilde{x})$$

$$- \frac{1}{V_2} \int d^2x d^2y Z(\tilde{x}) Z(\tilde{y}) \sqrt{\gamma(x)} \gamma^{ij}(\tilde{x}) \gamma^{jk}(\tilde{y}) \gamma^{kl}(\tilde{y}) \frac{\partial^2}{\partial x^k \partial y^l} G(\tilde{x} - \tilde{y}).$$  \hspace{1cm} (15)

To show that $\sigma^{ij}$ is symmetric, consider

$$e^2 \left( \sigma^{ij} - \sigma^{ji} \right) = -\frac{1}{V_2} \int d^2x d^2y Z(\tilde{x}) Z(\tilde{y}) \sqrt{\gamma(x)} \gamma^{ij}(\tilde{x}) \gamma^{ji}(\tilde{y}) \frac{\partial^2}{\partial x^i \partial y^j} \left[ G(\tilde{x} - \tilde{y}) - G(\tilde{y} - \tilde{x}) \right]$$

$$= 0,$$  \hspace{1cm} (16)

due to the symmetry of the metric tensor $\gamma_{ij}$ and Eq. (13).

The analysis in other dimensions is trivial due to our ability to eliminate $Z$ from (1) with conformal transformations and the fact that we didn’t use any special properties of two dimensional spaces in the proof.