Cosmology/Gravity

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1 Boundary theory coupled to gravity

Bulk action

\[ S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + 2\Lambda^{(5)} \right) - \int d^5x L_m, \]  

(1.1)

where \( \Lambda^{(5)} = -\frac{d(d-1)}{2L^2} = -\frac{6}{L^2} \). The stress-energy tensor is

\[ T^{\mu\nu} = \frac{1}{8\pi G_5} \left[ K^{\mu\nu} - K^{\gamma\mu\nu} - \frac{3}{L} \gamma^{\mu\nu} + \frac{L}{2} \left( R^{\mu\nu} - \frac{1}{2} R^{\gamma^{\mu\nu}} \right) + \ldots \right]. \]  

(1.2)

Introducing \( \Lambda \) as the four-dimensional cosmological constant, we find

\[ R^{\mu\nu} - \frac{1}{2} R^{\gamma^{\mu\nu}} - \Lambda^{\mu\nu} + \ldots - \frac{2}{L} 8\pi G_5 T^{\mu\nu} = -\frac{2}{L} \left( K^{\mu\nu} - \gamma^{\mu\nu} K \right) + \left( \Lambda + \frac{6}{L^2} \right) \gamma^{\mu\nu}. \]  

(1.3)

Setting the LHS Einstein’s equation to zero, with an effective \( G_4 = 2G_5/L \), we get the identity

\[ K^{\mu\nu} = -\frac{1}{L} \left( 1 + \frac{L^2 \Lambda}{6} \right) \gamma^{\mu\nu}. \]  

(1.4)
2 Dilaton gravity

Theory

\[ S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - 2\partial_\mu\phi\partial^\mu\phi - 2\Lambda^{(5)} e^{\eta\phi} \right) \]  

(2.1)

Equations of motion

\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \frac{1}{2} \eta \Lambda^{(5)} e^{\eta\phi} = 0 \]  

(2.2)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda^{(5)} g_{\mu\nu} e^{\eta\phi} - 2\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu} \partial_\lambda\phi\partial^\lambda\phi = 0 \]  

(2.3)

Solution with set \( \Lambda^{(5)} = -6 \) is

\[ ds^2 = -f(r)dt^2 + \left( \frac{r}{r_h} \right)^{\frac{16}{8+3\eta^2}} \left( dx^2 + dy^2 + dz^2 \right) + \frac{dr^2}{f(r)}, \]

where \( f(r) = \left( \frac{8 + 3\eta^2}{64 - 6\eta^2} \right)^2 \left( \frac{r}{r_h} \right)^{\frac{16}{8+3\eta^2}} \left[ 1 - \left( \frac{r_h}{r} \right)^{\frac{32-3\eta^2}{8+3\eta^2}} \right] \)  

(2.4)

\[ \phi = -\frac{6\eta}{8 + 3\eta^2} \log \left( \frac{r}{r_h} \right) \]  

(2.5)

Look for a brane embedding

\[ t(r) \]  

(2.6)

A set of normalised tangent vectors

\[ T^\mu = \sqrt{\frac{f}{f^2 (\partial_t t)^2 - 1}} \left( \frac{\partial_t}{\partial r}, 0, 0, 0, 1 \right) \]  

(2.7)

\[ X^\mu = r^{-\frac{s}{8 + 3\eta^2}} (0, 1, 0, 0, 0) \]  

(2.8)

\[ Y^\mu = r^{-\frac{s}{8 + 3\eta^2}} (0, 0, 1, 0, 0) \]  

(2.9)

\[ Z^\mu = r^{-\frac{s}{8 + 3\eta^2}} (0, 0, 0, 1, 0) \]  

(2.10)

so that \( T^2 = -1 \) and \( X^2 = Y^2 = Z^2 = 1 \). A normal is

\[ n_\mu = nf(r) \left( -1, 0, 0, 0, \frac{\partial t}{\partial r} \right) \]  

(2.11)

where

\[ n^\mu n_\mu = 1 \]  

(2.12)

implies

\[ n = \frac{1}{\sqrt{f \left( f^2 (\partial_t t)^2 - 1 \right)}} \]  

(2.13)
hence
\[ n_\mu = \sqrt{\frac{f}{f^2 (\partial_r t)^2 - 1}} \left( -1, 0, 0, 0, \frac{\partial t}{\partial r} \right) \] (2.14)

The full set of vectors is
\[ T^\mu = \sqrt{\frac{f}{f^2 (\partial_r t)^2 - 1}} \left( \frac{\partial t}{\partial r}, 0, 0, 0, 1 \right) \] (2.15)
\[ \bar{X}^\mu = r^\frac{s}{8 + 3\eta^2} \left( 0, 1, 0 \right) \] (2.16)
\[ n^\mu = \sqrt{\frac{f}{f^2 (\partial_r t)^2 - 1}} \left( 1, 0, 0, f \frac{\partial t}{\partial r} \right) \] (2.17)

Induced metric
\[ g^{(\text{ind})}_{\mu\nu} \equiv \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \] (2.18)

Extrinsic curvature
\[ K_{\mu\nu} = - \left( \delta^\lambda_\mu - n_\mu n_\lambda \right) \nabla_\lambda n_\nu \] (2.19)

Junction condition (check)
\[ K_{\mu\nu} = -\gamma_{\mu\nu} \] (2.20)

Solution
\[ \frac{\partial t}{\partial r} = \pm \frac{(8 + 3\eta^2) r}{f \sqrt{(8 + 3\eta^2)^2 r^2 - 64 f}} \] (2.21)

The induced metric \( g^{(\text{ind})}_{\mu\nu} = \gamma_{\mu\nu} \) is given by the line element
\[ ds^2_\gamma = -\frac{64}{(8 + 3\eta^2)^2 r^2 - 64 f(r)} dr^2 + \left( \frac{r}{r_h} \right)^\frac{16}{8 + 3\eta^2} (dx^2 + dy^2 + dz^2) \] (2.22)

Solve
\[ \tau = C + 8 \int \frac{dr}{\sqrt{(8 + 3\eta^2)^2 r^2 - 64 f(r)}} \] (2.23)
\[ a(\tau)^2 = \left( \frac{r(\tau)}{r_h} \right)^\frac{16}{8 + 3\eta^2} \] (2.24)

so that we find the induced FRW metric
\[ ds^2_\gamma = -dr^2 + a(\tau)^2 (dx^2 + dy^2 + dz^2) \] (2.25)
At $\eta = 0$ we find the standard radiation-dominated (CFT) result of Gubser. For $\eta > 0$ and at large $r$, 

$$f(r) \rightarrow \frac{(8 + 3\eta^2)^2}{64 - 6\eta^2} \frac{r^2}{r_h^2} \left( \frac{r}{r_h} \right)^{\frac{16}{8 + 3\eta^2}}$$

(2.26)

and we find

$$r = \exp \left\{ \frac{8 + 3\eta^2}{8} (\tau - \text{const.}) \right\}$$

(2.27)

and

$$a(\tau) = Ce^{\tau}.$$ 

(2.28)

At non-zero $\eta$, for $r \approx r_h$

$$a$$

(2.29)

3 Probe fields

Consider a probe scalar field $\phi$ with an action

$$S = -\frac{K}{2} \int_{\mathcal{M}} d^5x \sqrt{-g} \nabla^\mu \phi \nabla_\mu \phi + \ldots,$$

(3.1)

which satisfied the equation of motion in five dimensions. The boundary action is then

$$S = -\frac{K}{2} \int_{\mathcal{M}} d^4x n_\mu \phi \partial_\mu \phi = -\frac{k}{2} \int_{\mathcal{M}} d^5x \sqrt{-g} \nabla_\mu (\phi \nabla^\mu \phi)$$

$$= -\frac{K}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} n_\mu \phi \partial_\mu \phi,$$

(3.2)

since $\nabla_\mu \phi = \partial_\mu \phi$.

Using the foliation $t = t(r)$ and the normal $n_\mu = nf(r) \left( -1, 0, 0, 0, \frac{\partial t}{\partial r} \right)$, the boundary action is then

$$S = -\frac{K}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} \phi g^{\mu\nu} n_\mu \partial_\nu \phi$$

$$= -\frac{K}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} n(r)f(r)\phi \left( -g^{tt} \frac{\partial}{\partial t} + g^{rr} \frac{\partial}{\partial r} \frac{\partial}{\partial r} \right) \phi,$$

(3.3)

which for our theory gives

$$S = -K \int_{\partial\mathcal{M}} d^4x \frac{24}{(8 + 3\eta^2)^2 r^2} \frac{(8 + 3\eta^2)^2}{(8 + 3\eta^2)^2 r^2 - 32f(r)\phi \partial_r \phi}{\sqrt{(8 + 3\eta^2)^2 r^2 - 64f(r)}}$$

(3.5)

We wish to impose the Dirichlet boundary condition on the hypersurface $t(r)$, i.e. $\phi = \text{const.}$, which means

$$\partial_t \phi(t, x^i, r) \bigg|_{\partial\mathcal{M}} = 0,$$

(3.6)
and
\[
- f(r)^2 \frac{\partial t}{\partial r} + \frac{\partial}{\partial t} \phi(t, x^i, r) \bigg|_{\partial M} = 0
\]
(3.7)
\[
\Rightarrow \left[ - \frac{(8 + 3\eta^2) rf}{\sqrt{(8 + 3\eta^2)^2 r^2 - 64f}} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right] \phi(t, x^i, r) \bigg|_{\partial M} = 0.
\]
(3.8)

At \( \eta = 0 \), this gives
\[
\left[ - \frac{r^4 - r_h^4}{r^2 h} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right] \phi(t, x^i, r) \bigg|_{\partial M} = 0
\]
(3.9)

Consider the bulk solution decomposed as
\[
\phi(t, \vec{x}, r) = \int \frac{d^4k}{(2\pi)^4} e^{i\omega t - i\vec{k} \cdot \vec{x}} \varphi_k(r)
\]
(3.10)
\[
\frac{\partial \phi}{\partial r} = \int \frac{d^4k}{(2\pi)^4} e^{i\omega t - i\vec{k} \cdot \vec{x}} \left( i\omega \frac{\partial}{\partial r} \varphi_k + \varphi_k \frac{\partial \varphi_k}{\partial r} \right) = \int \frac{d^4k}{(2\pi)^4} e^{i\omega t - i\vec{k} \cdot \vec{x}} \left( \frac{i\omega (8 + 3\eta^2) r}{f\sqrt{(8 + 3\eta^2)^2 r^2 - 64f}} \varphi_k + \varphi_k \frac{\partial \varphi_k}{\partial r} \right)
\]
(3.11)

Hence
\[
S = - K \int \frac{d^4k d^4p \delta \omega dr}{(2\pi)^8} e^{i(k^\alpha + p^\alpha)\ell(r) - i(k + p) \cdot \vec{x}} \frac{24}{r^{8+3\eta^2}} \frac{r^{8+3\eta^2} r}{(8 + 3\eta^2)^2 r^2 - 32f} \frac{1}{f\sqrt{(8 + 3\eta^2)^2 r^2 - 64f}} \left( \varphi_p \varphi_k + \varphi_p \frac{\partial \varphi_k}{\partial r} \right)
\]
(3.12)
\[
= - K \int \frac{d^4k d^4p \delta dr}{(2\pi)^8} e^{i(k^\alpha + p^\alpha)\ell(r)} \frac{24}{r^{8+3\eta^2}} \frac{r^{8+3\eta^2} r}{(8 + 3\eta^2)^2 r^2 - 32f} \frac{1}{f\sqrt{(8 + 3\eta^2)^2 r^2 - 64f}} \left( \varphi_p \varphi_k + \varphi_p \frac{\partial \varphi_k}{\partial r} \right)
\]
(3.13)

**3.1 Conformal case**

At \( \eta = 0 \), we have
\[
\frac{\partial t}{\partial r} = \frac{r}{f\sqrt{r^2 - f}} = \frac{1}{r_h^2} \frac{1}{1 - (r_h/r)^4}
\]
(3.15)
we find
\[
t = \frac{r_0 + r}{r_h^2} + \frac{1}{4r_h} \sum_{n=0}^3 \left( i^n \log \left[ 1 - i^n \frac{r_h}{r} \right] \right)
\]
(3.16)
Then

\[ S = -K \int \frac{d^4k dp^0 dr}{(2\pi)^5} e^{i(k_0 + \rho \phi)\epsilon(r)} \frac{r(r_h^4 + r^4)}{2r_h^2}\left( \frac{ik_0 r^4}{r_h^2 (r^4 + r_h^4)} \varphi_{\rho_0, -k} \cdot \varphi_{k_0, k} + \varphi_{\rho_0, -k} \frac{\partial \varphi_{k_0, k}}{\partial r} \right) \]

(3.17)

and furthermore

\[ S = -\frac{K}{2} \int \frac{d^4k dp^0 dr}{(2\pi)^5} e^{i(k_0 + \rho \phi)\epsilon(r)} \frac{r(r_h^4 + r^4)}{2r_h^2}\left( \frac{ik_0 r^4}{r_h^2 (r^4 + r_h^4)} \varphi_{\rho_0, -k} \cdot \varphi_{k_0, k} + \varphi_{\rho_0, -k} \frac{\partial \varphi_{k_0, k}}{\partial r} \right) \]

(3.18)

and using \( z = r_h/r, z_0 = r_h/r_0 \) and \( T = r_h/\pi, \) as well as \( t^0 = k_0/(2\pi T), p^0 = p_0/(2\pi T), \)

\[ S = -\frac{\pi^3 T^5 K}{2} \int \frac{d^3k (2\pi)^3}{(2\pi)^3} \frac{1}{d^3t} dp^0 dz e^{2i(t^0 + \rho \phi)\frac{z_0 + z}{\pi}} \frac{1}{1 + z} \frac{1}{1 + i z} \frac{1}{z^5} \left( \frac{2i t^0}{z^2 (1 - z^2)} \varphi_{\rho_0, -k} \cdot \varphi_{k_0, k} - \varphi_{\rho_0, -k} \frac{\partial \varphi_{k_0, k}}{\partial z} \right) \]

(3.19)

4 Fluid/gravity

Work out the foliation procedure in Eddington-Finkelstein coordinates! Consider the five-dimensional black brane metric

\[ ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx^2 + dy^2 + dz^2), \]

where \( f(r) = 1 - \left( \frac{r_h}{r} \right)^4. \)

(4.1)

Change coordinates to the Eddington-Finkelstein coordinate \( v, \)

\[ t = v - \frac{1}{4r_h} \sum_{i=0}^3 \left( i^k \log \left[ 1 - i^k \frac{r}{r_h} \right] \right), \]

(4.2)

so that

\[ ds^2 = -r^2 f(r) dv^2 + 2dv dr + r^2 (dx^2 + dy^2 + dz^2). \]

(4.3)

We know the metric solution at first order. Perturb

\[ n_{\mu} = n_{\mu}^{(0)} + \epsilon n_{\mu}^{(1)} \]

(4.4)

so

\[ n_{\mu} n^\nu = n_{\mu}^{(0)} n^\nu_{(0)} + \epsilon \left( n_{\mu}^{(0)} n^\nu_{(1)} + n_{\mu}^{(1)} n^\nu_{(0)} \right) \]

(4.5)
and \( K_{\mu\nu} = - (\delta^\lambda_\mu - n_\mu n^\lambda) \nabla_\lambda n_\nu \) leads to

\[
K_{\mu\nu} = K_{(0)\mu\nu} + \epsilon K_{(1)\mu\nu}.
\]

(4.6)

First-order metric takes the form

\[
ds^2 = \sum_{n=1}^{6} A_n,
\]

(4.7)

where

\[
A_1 = -2u_a dx^a dr, \quad A_2 = -r^2 f_0(br) u_a u_b dx^a dx^b, \quad (4.8)
\]

\[
A_3 = r^2 \Delta_{ab} dx^a dx^b, \quad A_4 = 2r^2 b F_0(br) \sigma_{ab} dx^a dx^b, \quad (4.9)
\]

\[
A_5 = \frac{2}{3} r u_a \partial_c u^c dx^a dx^b, \quad A_6 = -ru_c \partial_c (u_a u_b) dx^a dx^b \quad (4.10)
\]

and \( f_0 \) and \( F_0 \) are expanded to first order in derivatives of \( b \) and \( u^\mu \).

Use the foliation

\[
t(x^a, br) = t_0(r) + \epsilon (x^a \partial_a b_0 + b_1) r t'_0(r) + \epsilon t_1(r) \partial_a u^a + \epsilon t_2(r) u^a \partial_a b.
\]

(4.11)

set of unnormalised tangent vectors

\[
R^\mu = \left( \frac{\partial t}{\partial r}, 0, 0, 0, 1 \right) \quad (4.12)
\]

\[
X^\mu = (0, 1, 0, 0, 0) \quad (4.13)
\]

\[
Y^\mu = (0, 0, 1, 0, 0) \quad (4.14)
\]

\[
Z^\mu = (0, 0, 0, 1, 0) \quad (4.15)
\]

Thus

\[
0 = g_{\mu\nu} R^\mu n^\nu = \frac{\partial t}{\partial r} n_0 + n_4 \implies n_4 = -\frac{\partial t}{\partial r} n_0 \quad (4.16)
\]

so

\[
n_\mu = n \left( -1, 0, 0, 0, \frac{\partial t}{\partial r} \right) \quad (4.17)
\]

5 Probe scalar in WKB approximation

5.1 Conformal case

Consider the conformal case with the metric

\[
ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \left( dx^2 + dy^2 + dz^2 \right),
\]

where \( f(r) = 1 - \left( \frac{r_h}{r} \right)^4 \).  \quad (5.1)
The scalar two-point function in the large mass $m \gg 1$ approximation scales as

$$\langle O(x)O(y) \rangle \sim \exp \left\{ -m \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right\} \equiv e^{-S}. \quad (5.2)$$

Let us compute an equal-time correlator, which implies that we are fixing the position of the brane $t(r)$ at some bulk position $\rho = r_h\sqrt{2}\tau$, in terms of the boundary time. Choosing the proper time $\tau = x$, the exponent is

$$S = m \int dx \sqrt{r^2 - r^2 f t'^2 + \frac{1}{r^2} f'^2 r^2}. \quad (5.3)$$

Since we want an equal time correlator we will set $t' = 0$. $S$ possesses a conserved quantity

$$H = r' \frac{\partial L}{\partial r'} - L = -\frac{r^2}{\sqrt{r^2 + r^2 f'^2}}. \quad (5.4)$$

Let us focus only on late-time behaviour, so that $\rho \gg r_h$ and $f(r) \approx 1$. Looking for a geodesic between $x = \pm \ell/2$ at $r = \rho$ we find

$$x = \pm \sqrt{\frac{4 + \ell^2 \rho^2}{4\rho^2} - \frac{1}{r^2} + O(r_h^4)}. \quad (5.5)$$

The action the becomes

$$S = 2m \int_0^{\rho} \frac{dr}{2\rho} \frac{\sqrt{r^2 - \frac{4\rho^2}{2 + r^2 \rho^2}}}{\sqrt{r^2 - \frac{4\rho^2}{2 + r^2 \rho^2}}} = \log \left[ \frac{1}{4} \left( \ell \rho + \sqrt{4 + \ell^2 \rho^2} \right)^2 \right], \quad (5.6)$$

hence for $\ell^2 \rho^2 \gg 1$,

$$e^{-S} \sim \frac{1}{(\ell \rho)^{2m}}. \quad (5.7)$$

Assuming $\Delta \sim m \gg 1$ and knowing that the scale factor scales as

$$a(\tau) \propto \sqrt{\tau}, \quad (5.8)$$

the equal time scalar correlator is

$$\langle O(\tau, \vec{x})O(\tau, \vec{y}) \rangle \sim \frac{1}{|\vec{x} - \vec{y}|^{2\Delta}} a(\tau)^{2\Delta}. \quad (5.9)$$

### 5.2 Non-conformal case

The metric is

$$ds^2 = -f(r)dt^2 + \left( \frac{r}{r_h} \right)^{\frac{16}{8 + 3\eta^2}} (dx^2 + dy^2 + dz^2) + \frac{dr^2}{f(r)},$$

where

$$f(r) = \left( \frac{8 + 3\eta^2}{64 - 6\eta^2} \right)^2 \left( \frac{r}{r_h} \right)^{\frac{16}{8 + 3\eta^2}} \left[ 1 - \left( \frac{r_h}{r} \right)^{\frac{32 - 3\eta^2}{8 + 3\eta^2}} \right]. \quad (5.10)$$
Again we are interested in \( r \gg r_h \), so
\[
d s^2 = -f(r) d t^2 + \left( \frac{r}{r_h} \right)^{\frac{16}{8+3\eta^2}} (d x^2 + d y^2 + d z^2) + \frac{d r^2}{f(r)},
\]
where \( f(r) = \frac{(8 + 3\eta^2)^2 r_h^2}{64 - 6\eta^2} \left( \frac{r}{r_h} \right)^{\frac{16}{8+3\eta^2}} \). (5.11)

With \( t' = 0 \) we get with \( \alpha = 8 + 3\eta^2 \)
\[
S = m \int dx \left( \frac{r}{r_h} \right)^{\frac{16}{\alpha}} + \frac{64 - 6\eta^2}{\alpha r_h^2} \left( \frac{r}{r_h} \right)^{\frac{16}{\alpha}} r^2 \] (5.12)
and
\[
H = -\frac{(r/r_h)^{\frac{16}{\alpha}}}{\sqrt{(r/r_h)^{\frac{16}{\alpha}} + \frac{64 - 6\eta^2}{\alpha r_h^2} (r_h/r)^{\frac{16}{\alpha}} r^2}} \] (5.13)

Hence
\[
\frac{d r}{d x} = \frac{r_h \sqrt{\alpha}}{H \sqrt{64 - 6\eta^2}} \left( \frac{r}{r_h} \right)^{\frac{16}{\alpha}} \left( \frac{r}{r_h} \right)^{\frac{16}{\alpha}} - H^2 \] (5.14)

We find with a new variable \( u = r/r_h \),
\[
x = \pm \frac{\sqrt{2\alpha (32 - 3\eta^2)}}{H (16 - \alpha)} u^{\frac{16 - \alpha}{\alpha}} \sqrt{u^{16/\alpha} - H^2} \quad 2 F_1 \left[ 1, -\frac{8 - \alpha}{16}, \frac{\alpha}{16}; \frac{u^{16/\alpha}}{H^2} \right] \] (5.15)

We need to fix \( H \).....

Further
\[
S = m \sqrt{\frac{64 - 6\eta^2}{8 + 3\eta^2}} \int_{u_{min}}^{u_{opt}} \frac{d u}{\sqrt{u^{16/(8+3\eta^2)} - H^2}}
= -\frac{m}{H^2} \sqrt{\frac{64 - 6\eta^2}{8 + 3\eta^2}} \left[ u \sqrt{u^{16/(8+3\eta^2)} - H^2} \quad 2 F_1 \left[ 1, \frac{16 + 3\eta^2}{16}, \frac{24 + 3\eta^2}{16}, \frac{u^{16/(8+3\eta^2)}}{H^2} \right] \right]_{u_{min}}^{u_{opt}}
\] (5.16)

6 Notes

The gravitational action has a restricted time domain because the brane moves outwards from the horizon or some radial position where the cosmological evolution in the model begins. Hence the action has the form
\[
S_{bulk} = \int_{\mathcal{M}} d^5 x \mathcal{L} = \int_{r_h}^{\infty} d r \int_{-\infty}^{\infty} d^3 x \int_{t_0}^{T(r)} d t \mathcal{L}.
\] (6.1)
This gives the usual bulk equations of motion which enable us to solve the hyper-surface embedding equation and find \( t(r) \). In AdS-Schwarzschild this is

\[
T(r) = \frac{r}{r_h^2} + \frac{1}{4r_h^3} \sum_{n=0}^{3} \left( i^n \log \left[ 1 - i^n \frac{r_h}{r} \right] \right) - T_0. \tag{6.2}
\]

Notice that this expression diverges as \( r \to r_h \). We can thus choose for the boundary to start at some \( r_0 > r_h \) at time \( t = 0 \). Thus

\[
T = \frac{r_0}{r_h^2} + \frac{1}{4r_h^3} \sum_{n=0}^{3} \left( i^n \log \left[ 1 - i^n \frac{r_h}{r_0} \right] \right). \tag{6.3}
\]

Consider a probe scalar field \( \phi \) with an action

\[
S = -\frac{K}{2} \int_{\mathcal{M}} d^5x \sqrt{-g} \nabla_\mu \phi \nabla^\mu \phi + \ldots, \tag{6.4}
\]

which satisfied the equation of motion in five dimensions. The boundary action is then

\[
S = -\frac{K}{2} \int_{\mathcal{M}} d^5x \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = -\frac{k}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-g} \nabla_\mu \left( \phi \nabla^\mu \phi \right)
\]

\[
= -\frac{K}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} n_\mu \phi \partial^\mu \phi = -\frac{K}{2} \int_{r_0}^{\infty} dr \int d^3x \sqrt{-\gamma} n_\mu \phi \partial^\mu \phi, \tag{6.5}
\]

since \( \nabla_\mu \phi = \partial_\mu \phi \) and having used \( t = T(r) \).