Cosmological constant

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1 Strongly coupled hidden sector

Assume that our universe has a strongly coupled hidden sector on top of the visible (standard model) sector. They may be weakly coupled. The theory is defined in an evolving universe with a metric \( g_{ab} \),

\[
S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} [S_{\text{vis}} + S_{\text{hid}} + S_{\text{int}}]. \tag{1.1}
\]

Furthermore, we assume that the cosmological constant of the boundary has a zero cosmological constant!

We will assume that the hidden sector has a holographic dual and that the four-dimensional hyper-surface is embedded into a five dimensional bulk space time \( G_{\mu\nu} \) with a metric

\[
ds^2 = -e^{2\lambda(r)} dt^2 + e^{2\nu(r)} dr^2 + \left( \frac{r}{L} \right)^\alpha d\mathbf{x}^2. \tag{1.2}
\]

To find the embedding \( t(r) \), we define the four un-normalised tangent vectors

\[ R^\mu = \left( \frac{\partial t}{\partial r}, 0, 0, 0, 1 \right), \quad X^\mu = (0, 1, 0, 0, 0), \tag{1.3} \]
\[ Y^\mu = (0, 0, 1, 0, 0), \quad Z^\mu = (0, 0, 0, 1, 0). \tag{1.4} \]

The normalised normal vector to the hyper-surface is

\[
n_\mu = \pm \sqrt{\frac{e^{2\lambda}}{e^{2(\lambda-\nu)}/L^2 - 1}} \left( -1, 0, 0, 0, t' \right). \tag{1.5}
\]

The induced metric can be written as

\[
\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \tag{1.6}
\]

and the extrinsic curvature as

\[
K^{\mu\nu} = -\frac{1}{L} \gamma^{\mu\nu}. \tag{1.7}
\]
The junction equation (1.7) allows us to solve for \( \partial t / \partial r \),

\[
\frac{\partial t}{\partial r} = \pm \frac{r e^{2\nu - \lambda}}{\sqrt{r^2 e^{2\nu} - \frac{1}{4} L^2 \alpha^2}}. \tag{1.8}
\]

The induced metric’s line element is thus

\[
ds^2 = -\frac{L^2 \alpha^2 e^{2\nu}}{4r^2 e^{2\nu} - L^2 \alpha^2} \dr^2 + \left( \frac{r}{L} \right)^\alpha \, d\vec{x}^2. \tag{1.9}
\]

The metric has no dependence on the function \( \lambda(r) \).

We can now look for the time coordinate \( \tau \) and the FRW scale factor \( a(\tau) \) of the boundary space-time. The time coordinate is

\[
\tau = \tau_0 \pm \int dr \frac{L \alpha e^\nu}{\sqrt{r^2 e^{2\nu} - L^2 \alpha^2}}. \tag{1.10}
\]

In the late-time regime we have \( r \gg 1 \). Now, if \( \lim_{r \to \infty} r^2 e^{2\nu} = 0 \) then \( \tau \) would become imaginary. If \( \lim_{r \to \infty} r^2 e^{2\nu} = \mathcal{O}(1) \), as in the AdS-Schwarzschild, the integral depends on the details of the function \( \nu \). In that case

\[
\nu = -\frac{1}{2} \log \left( \frac{r}{L} \right)^2 \left( 1 - \left( \frac{r_h}{r} \right)^4 \right) \tag{1.11}
\]

If, however, \( r^2 e^{2\nu} \gg \frac{1}{4} L^2 \alpha^2 \), we find that

\[
\tau = \tau_0 + \frac{L \alpha}{2} \log r. \tag{1.12}
\]

This result does not depend on the details of two functions specifying the metric.

Finally, we are able to write down the late-\( \tau \) hyper-surface metric

\[
ds^2 = -d\tau^2 + a(\tau)^2 d\vec{x}^2, \tag{1.13}
\]

where the scale factor is

\[
a(\tau) = e^{\frac{1}{2} \pi \tau}. \tag{1.14}
\]

The Hubble time is

\[
H = \frac{\dot{a}}{a} = \frac{1}{2L}. \tag{1.15}
\]

which can be written in terms of the vacuum energy density, \( H^2 = 8\pi G_4 \rho_V / 3 \). Furthermore, \( \rho_V = 3/(32\pi LG_5) \) and \( \Lambda = 8\pi G_4 \rho_V \), which gives us an effective cosmological constant on the brane, which is universal for all theories [of some type]. The cosmological constant is

\[
\Lambda = \frac{3}{4L^2}. \tag{1.16}
\]
and is provided in the dual description solely by the dynamics of the strongly coupled hidden sector. In reality $\Lambda \approx 10^{-52} m^{-2}$ or $\Lambda \sim 10^{-121} M_{pl}^2 = 10^{-121} 10^{38} GeV^2 = 10^{-83} GeV^2$ in $c = h = 1$ units. Thus $L^2 \sim 10^{84} GeV^{-2}$ and $L^4 \sim 10^{166} GeV^{-4}$. Assume $1/\ell_s = M_{pl}$, so

$$\frac{L^4}{\ell_s^4} \sim 10^{166} \cdot 10^{76} = 10^{242}$$

(1.17)

or

$$L \sim 10^{26} m$$

(1.18)

The diameter of the sphere which is the observable universe is $8.8 \times 10^{26} m$.

In the standard AdS/CFT example with D3 branes the supergravity description is applicable in the regime

$$\frac{L^4}{\ell_s^4} \sim g_s N \sim g_{YM}^2 N = \lambda \gg 1.$$  

(1.19)

Hence, $\Lambda \sim \frac{1}{\ell_s^2 \sqrt{\lambda}}$

$$\Lambda \sim \frac{1}{L^2} \ll \frac{1}{\ell_s^2}.$$  

(1.20)

This is not a very strong bound, but at least we know that the cosmological constant needs to be much smaller that the string scale.

Gubser states that the cut-off of the theory can be measured as the energy of a fundamental string stretched form the Planck brane to the horizon of AdS$_5$ when measure in time $\tau$. This is equivalent to separating one D3 brane from the rest so that the string tension give the mass of the heaviest W boson. Now, since at large $\tau$ requires large $r$, by UV/IR the dual theory should be able to access the extreme UV degrees of freedom. Hence, we require

$$\Lambda_{\text{cut-off}} \sim \frac{L}{\alpha'} \sim \frac{L}{\ell_s^2},$$

(1.21)

so for supergravity in the bulk to be a good approximation

$$\Lambda \sim \frac{1}{L^2} \sim \frac{1}{\ell_s^4 \Lambda_{\text{cut-off}}} \ll \frac{1}{\ell_s^2} \Rightarrow 1 \ll \ell_s^2 \Lambda_{\text{cut-off}}^2$$

(1.22)

Rough estimate of the string scale near the Planck scale gives

$$\Lambda \ll M_{pl}^2,$$

(1.23)

which is much better than the QFT calculation giving

$$\Lambda \sim G_4 M_{pl}^4 = h c M_{pl}^2.$$  

(1.24)

The Ricci tensor is given by

$$R_{ab} = \frac{3}{2L^2} \text{diag} \left[ -1, e^{\sqrt{2}/L}, e^{\sqrt{2}/L}, e^{\sqrt{2}/L} \right].$$

(1.25)
and the Ricci scalar is given by
\[ R = 6 \frac{L^2}{c^2}. \] (1.26)

The Einstein’s equation without higher-curvature terms should be satisfied in the late universe with small curvatures. If we use the holographic stress-energy tensor expression, from which the embedding equation was derived,
\[ R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \mathcal{O}(R^2) = \frac{2}{L} 8\pi G_5 T_{vis}^{\mu\nu} + \frac{2}{L} 8\pi G_5 T_{hid}^{\mu\nu}, \] (1.27)
should contain higher-curvature terms. However, in order for each term to be sub-leading, we require
\[ |R| \gg |R^2| \implies 1 \gg \frac{1}{L^2} \implies L^2 \gg 1. \] (1.28)

The validity of the Einstein-Hilbert action on the boundary thus imposes a stronger bound on the effective cosmological constant
\[ \Lambda \sim \frac{1}{L^2} \ll 1. \] (1.29)

2 Cosmological laboratory

Given the bulk metric
\[ ds^2 = -e^{2\lambda(r)} dt^2 + e^{2\nu(r)} dr^2 + \left( \frac{r}{L} \right)^{2\beta} d\vec{x}^2, \] (2.1)
the induced metric is
\[ ds^2 = - \left( \frac{\beta^2 L^2 e^{2\nu}}{r^2 e^{2\nu} - \beta^2 L^2} \right) dr^2 + \left( \frac{r}{L} \right)^{2\beta} d\vec{x}^2. \] (2.2)

Hence the boundary time is
\[ \tau = \tau_0 \pm \int dr \frac{\beta L e^{\nu}}{\sqrt{r^2 e^{2\nu} - \beta^2 L^2}}, \] (2.3)
and the scale factor
\[ a(\tau) = \left( \frac{r(\tau)}{L} \right)^{\frac{\beta}{2}}. \] (2.4)

Our goal is to reconstruct the bulk metric from our choice of \( a(\tau) \). We assume that the foliation function \( r(\tau) \) is invertible, so that
\[ \frac{dr(r)}{dr} = \frac{L \beta e^{\nu(r)}}{\sqrt{r^2 e^{2\nu(r)} - \beta^2 L^2}} \implies \frac{dr(\tau)}{d\tau} = \frac{\sqrt{r(\tau)^2 e^{2\nu(\tau)} - \beta^2 L^2}}{L \beta e^{\nu(\tau)}}. \] (2.5)
and hence
\[ \frac{d \log r(\tau)}{d \tau} = \frac{\sqrt{r(\tau)^2 e^{2\nu(\tau)} - \beta^2 L^2}}{\beta L r(\nu(\tau))}. \] (2.6)

We know that
\[ r(\tau) = L a(\tau)^{1/\beta}, \] (2.7)
which means that we can solve for
\[ e^{2\nu(\tau)} = \frac{\beta^2}{a(\tau)^{2/\beta} \left( 1 - L^2 \left( \frac{d \log a(\tau)}{d \tau} \right)^2 \right) a(\tau)^{2/\beta} \left( 1 - \left( \frac{L}{\beta} \frac{d \log a(\tau)}{d \tau} \right)^2 \right)}. \] (2.8)

Hence, we know both \( r(\tau) \) and \( \nu(\tau) \), which are required for the metric, assuming we are able to invert \( r(\tau) \) to find \( \tau(r) \). The function \( \lambda(r) \) is left undetermined. The bulk metric can be written in terms of new coordinates \((t, \tau, \vec{x})\),
\[ ds^2 = -e^{2\lambda(\tau)} dt^2 + \frac{\beta^2 L^2 \left( \frac{d \log a(\tau)}{d \tau} \right)^2}{\beta^2 - L^2 H(\tau)^2} d\tau^2 + a(\tau)^2 d\vec{x}^2. \] (2.9)

or
\[ ds^2 = -e^{2\lambda(\tau)} dt^2 + \frac{\beta^2 L^2 a(\tau)^2}{\beta^2 a(\tau)^2 - L^2 H(\tau)^2} d\tau^2 + a(\tau)^2 d\vec{x}^2. \] (2.10)

In terms of the Hubble parameter,
\[ H \equiv \frac{\dot{a}}{a}, \] (2.11)
the metric is
\[ ds^2 = -e^{2\lambda(\tau)} dt^2 + \frac{L^2 H(\tau)^2}{1 - L^2 H(\tau)^2} d\tau^2 + a(\tau)^2 d\vec{x}^2. \] (2.12)

We can think of \( \lambda \) as parametrising a family of different metrics, which all give FRW on the boundary. We should set \( \beta = 1 \), most likely without the loss of generality, so that the metric is
\[ ds^2 = -e^{2\lambda(\tau)} dt^2 + \frac{L^2 H(\tau)^2}{1 - L^2 H(\tau)^2} d\tau^2 + a(\tau)^2 d\vec{x}^2. \] (2.13)

To simplify the metric, we could make the standard choice of \( \lambda = -\nu \),
\[ ds^2 = -\frac{a(\tau)^2/\beta}{\beta^4} \left( \beta^2 - L^2 H(\tau)^2 \right) dt^2 + \frac{\beta^2 L^2 H(\tau)^2}{\beta^2 - L^2 H(\tau)^2} d\tau^2 + a(\tau)^2 d\vec{x}^2. \] (2.14)

The simplest choice we can make is to take \( \beta = 1 \), and \( e^{2\lambda} = (r/L)^2 \). The bulk metric is then
\[ ds^2 = a(\tau)^2 \left( -dt^2 + d\vec{x}^2 \right) + \frac{L^2 H(\tau)^2}{1 - L^2 H(\tau)^2} d\tau^2. \] (2.15)