I. SOLUTIONS OF THE SHEAR SECTOR MODES IN THE IR LIFSHITZ LIMIT

A. IR geometry and $k = 0$ solutions

The geometry in the IR $(r \to \infty)$ approaches that of a pure Lifshitz geometry. In this limit we have

$$f(r) \to 1/r^2$$

$$g(r) \to g_\infty/r^2$$

$$h(r) \to h_\infty/r^2.$$  (1)

The two differential equation in the shear sector are

$$0 = Z''_1 + 2k r^2 h' Z'_2 + \left( \frac{rg_\sigma \mu}{2} + \frac{\omega^2 f' + 2k^2 r f^2}{f (\omega^2 - k^2 r^2 f)} \right) Z'_1 + \frac{g}{f} (\omega^2 - k^2 r^2 f) Z_1$$

$$+ 2k r^2 \sqrt{f} \mu \left( \frac{2\omega^2 h'^2}{f (\omega^2 - k^2 r^2 f)} + \frac{g_\sigma}{\mu} \right) Z_2,$$  (2)

$$0 = Z''_2 + \frac{1}{2} \left( \frac{f'}{f} - \frac{g'}{g} \right) Z'_2 - \frac{kh'}{\omega^2 - k^2 r^2 f} Z'_1 + \frac{g}{f} (\omega^2 - k^2 r^2 f) Z_2$$

$$- \left( \frac{2\omega^2 h'^2}{f (\omega^2 - k^2 r^2 f)} + \frac{g_\sigma}{\mu} \right) Z_2.$$  (3)

In the limit of $k \to 0$, the two equations decouple

$$0 = Z''_1 + \left( \frac{rg_\sigma \mu}{2} + \frac{f'}{f} \right) Z'_1 + \frac{\omega^2 g}{f} Z_1,$$  (4)

$$0 = Z''_2 + \frac{1}{2} \left( \frac{f'}{f} - \frac{g'}{g} \right) Z'_2 + \left( \frac{\omega^2 g}{f} - \frac{2h'^2}{f} - \frac{g_\sigma}{\mu} \right) Z_2.$$  (5)
and we can find the asymptotic IR Lifshitz behaviour of $Z_1$ and $Z_2$ to be

$$Z_1 = \left(1 + \frac{i(z+1)}{2z\sqrt{g_\infty \omega r^z}}\right) r e^{i\sqrt{g_\infty \omega r^z}/z}$$

and

$$Z_2 = \left(1 + \frac{iz}{\sqrt{g_\infty \omega r^z}}\right) e^{i\sqrt{g_\infty \omega r^z}/z},$$

which are series expansions of the full (irrelevant) solutions

$$Z_1 = r^{1+z/2} H_{\frac{1+2}{2z}}^{(1)} \left(g_\infty^{1/2} r^{1/2} \frac{\omega r^z}{z}\right)$$

and

$$Z_2 = r^{z/2} H_{\frac{3}{2}}^{(1)} \left(g_\infty^{1/2} r^{1/2} \frac{\omega r^z}{z}\right).$$

We would now like to find the $k$-dependent corrections to the above solutions to analytically extract the hydrodynamical quasi-normal mode (QNM).

**B. Cases with integer values of the exponent $z$**

1. **Cases with exponents $z \geq 3$**

We would like to find analytic $k$-dependent corrections to $Z_1$ and $Z_2$ in the Lifshitz IR region. The corrections should be such that the limit of $k \to 0$ smoothly reproduces the above $k = 0$ results. On the other hand, the limit of $\omega \to 0$ is not analytic and our solutions will represent an asymptotic series in $\omega$ controlled by powers of $r$. Away from the in-falling boundary conditions at the horizon ($r \to \infty$), however, we expect that the limit of $\omega \to 0$ is defined as well. We anticipate the form

$$Z_1 = e^{i\sqrt{g_\infty \omega r^z}/z} P_1(r, \omega, k)$$

and

$$Z_2 = e^{i\sqrt{g_\infty \omega r^z}/z} P_2(r, \omega, k),$$

where $P_1$ and $P_2$ are polynomials in ascending powers of $1/r$.

We can first expand equations (2) and (3) in $k^2 r^2 f \ll \omega^2$. Since we only work with $z > 1$, the expansion parameter tends to $k^2 r^2 f = k^2 r^{2(1-z)} \to 0$ in the IR. The expansion therefore makes sense for all non-vanishing values of $\omega$ and finite values of $k$. The limits of our approximation are

$$k^2 \ll \omega^2 r^{2(z-1)}$$

and

$$r \to \infty.$$
On top of that, we are interested in the hydrodynamical QNMs, hence we may think of both $\omega$ and $k$ as small. We find, up to $O(k^4)$,

$$0 = Z''_1 - \left(\frac{z+1}{r} + \frac{2(z-1)k^2}{\omega^2 r^{2z-1}} + \frac{2(z-1)k^4}{\omega^4 r^{4z-3}}\right)Z'_1 + g_\infty \left(\omega^2 r^{2(z-1)} - k^2\right)Z_1$$

$$- \frac{2\sqrt{z(z-1)}k}{r^{z-1}}Z'_2 + 4\sqrt{z(z-1)}\left(\frac{zk}{r^z} + \frac{(z-1)k^3}{\omega^2 r^{3z-2}}\right)Z_2$$

and

$$0 = Z''_2 - \frac{z-1}{r}Z'_2 + \left[\left(\omega^2 r^{2(z-1)} - k^2\right) - \frac{2z^2}{r^2} - \frac{2z(z-1)k^2}{\omega^4 r^{4z-2}}\right]Z_2$$

$$+ \frac{\sqrt{z(z-1)}k \left(\omega^2 r^{2(z-1)} + k^2\right)}{\omega^4 r^{3z-1}}Z'_1$$

(13)

(14)

Using a power series expansion in $1/r$ for $Z_1$ and $Z_2$ shows that we can recursively solve differential equations (13) and (14), order-by-order in $r$, with two series of form

$$P_1 = 1 + \sum_{i=z-2}^{\infty} \frac{a_i(\omega, k)}{r^i}$$

$$P_2 = 1 + \sum_{i=z-2}^{\infty} \frac{b_i(\omega, k)}{r^i}$$

(15)

The non-zero terms in both series begin at order $1/r^{z-2}$. In the limit of $k \to 0$ we find that $a_{z-2} = a_{z-1} = b_{z-2} = b_{z-1} = 0$, $a_z = \frac{i(z+1)}{2z\sqrt{g_\infty} \omega}$ and $b_z = \frac{-iz}{\sqrt{g_\infty} \omega}$, as required.

If we only seek the leading $\omega$ and $k$ behaviour it suffices to consider the series with three terms between $i = z - 2$ and $i = z$. In that case the equation (13) will be solved up to order $O\left(\frac{1}{r}\right)$, leaving terms of order $O\left(\frac{1}{r^2}\right)$ and higher unsolved. Equation (14) will be solved up to order $O\left(\frac{1}{r^2}\right)$, leaving terms of order $O\left(\frac{1}{r^3}\right)$ and higher unsolved. Further extending polynomials $P_{1,2}$ by $n$ terms is then able to solve the two differential equations by further $n$ orders.

2. Special case with $z = 2$

A special case, which cannot be solved by the above ansatz is the case when $z - 2 = 0$, i.e. $z = 2$. To solve the system we can use the following modified ansatz:

$$Z_1 = e^{i\frac{\sqrt{g_\infty} \omega r^z}{z}} + f(r)P_1(r, \omega, k)$$

$$Z_2 = e^{i\frac{\sqrt{g_\infty} \omega r^z}{z}} + f(r)P_2(r, \omega, k).$$

(16)

(17)
It is clear that since equations (13) and (14) have no constant terms, the functions in the exponents must equal, so it is sufficient to find a single $f(r)$ for both $Z_1$ and $Z_2$. To only find $f(r)$, it is sufficient to simply set $Z_1 = 0$ and use equation (14) to leading order in $k$. We are left with

$$0 = Z''_2 - \frac{1}{r}Z'_2 + \left[ g_\infty r^2 \left( \omega^2 - \frac{k^2}{r^2} \right) - \frac{8}{r^2} \right] Z_2$$

(18)

to which the full solution is [completely irrelevant, but it’s fun to play with special functions :-) ]

$$Z_2 = r^4 e^{\frac{1}{2} i \omega \sqrt{g_\infty} r^2} \left[ C_1 U \left( 2 + \frac{i \sqrt{g_\infty} k^2}{4 \omega}, 4, -i \sqrt{g_\infty} \omega r^2 \right) + C_2 L^3_{-2 - i \sqrt{g_\infty} k^2} \left( -i \sqrt{g_\infty} \omega r^2 \right) \right],$$

(19)

where $U$ is the confluent hypergeometric function and $L^\lambda_n(z)$ the Laguerre polynomial.

Analysing its asymptotics near $r \to \infty$, we find that $C_2 = 0$ in order to only keep $e^{\frac{1}{2} i \omega \sqrt{g_\infty} r^2}$ terms (the in-falling b.c.). To match this solution onto the $k = 0$ solution we must set $C_1 = -g_\infty \omega^2$.

There is of course still the freedom of multiplying the entire solution by a constant. Expanding in $1/r$ we find

$$Z_2 = -g_\infty \omega^2 r^4 e^{\frac{1}{2} i \omega \sqrt{g_\infty} r^2} U \left[ 2 + \frac{i \sqrt{g_\infty} k^2}{4 \omega}, 4, -i \sqrt{g_\infty} \omega r^2 \right]$$

$$= e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2} \left( -i \sqrt{g_\infty} \omega r^2 \right)^{-i \sqrt{g_\infty} k^2} [1 + ...]$$

$$= \exp \frac{i \sqrt{g_\infty} \omega}{2} \left( r^2 - \frac{k^2}{2 \omega^2} \log \left( -i \sqrt{g_\infty} \omega r^2 \right) \right) [1 + ...].$$

(20)

Therefore

$$e^{f(r)} = \left( -i \sqrt{g_\infty} \omega r^2 \right)^{-i \sqrt{g_\infty} k^2} = e^{-i \sqrt{g_\infty} \omega r^2} \log \left( -i \sqrt{g_\infty} \omega r^2 \right).$$

(21)

Note that this structure is similar to the more usual AdS cases at finite temperature...

We can now use, as before, polynomials $P_{1,2}$ to find

$$Z_1 = e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2} \left( -i \sqrt{g_\infty} \omega r^2 \right)^{\frac{k^2}{4 \omega}} \log \left( -i \sqrt{g_\infty} \omega r^2 \right) \left( 1 - \frac{\sqrt{2} k}{r} + \frac{12 i \omega^2 - 12 \sqrt{g_\infty} \omega k^2 + i g_\infty k^4}{16 g_\infty \omega^3 r^2} + ... \right)$$

$$- \frac{32 i \omega^2 k - 4 \sqrt{g_\infty} \omega k^3 + i g_\infty k^5}{8 \sqrt{2} g_\infty \omega^3 r^3} + ...$$

$$Z_2 = e^{\frac{1}{2} i \sqrt{g_\infty} \omega r^2} \left( -i \sqrt{g_\infty} \omega r^2 \right)^{\frac{k^2}{4 \omega}} \log \left( -i \sqrt{g_\infty} \omega r^2 \right) \left( 1 + \frac{k}{\sqrt{2} \omega^2 r} + \frac{32 i \omega^2 - 4 \sqrt{g_\infty} \omega k^2 + i g_\infty k^4}{16 g_\infty \omega^3 r^2} + ... \right)$$

(22)

so that both (13) and (14) are satisfied to $\mathcal{O}(1/r^2)$.

II. QUASI-NORMAL MODES

We would like to find the hydrodynamical QNM in the shear sector of the electron star background at $T = 0$. 


\textbf{A. Flux with real }\omega^2

To find the conserved flux in this system, consider the off-shell Lagrangian

\[
\mathcal{L}_{\text{off-shell}} = \frac{L^2}{\kappa^2} \left( Z_i^\dagger A_{ij} Z_j^\prime + Z_i^\dagger B_{ij} Z_j^\prime + \text{non-derivative terms} \right)
\]

(23)

where

\[
A_{11} = \frac{\sqrt{f}}{4r^2 \sqrt{g} (\omega^2 - k^2 r^2 f)}, \quad A_{22} = -\frac{\sqrt{f}}{2 \sqrt{g}}, \quad A_{12} = A_{21} = 0,
\]

\[
B_{11} = \frac{(rf' - 2f)}{2\omega^2 r^3 \sqrt{\nu g}}, \quad B_{21} = -\frac{k (rf' + 2f)}{2r \mu \sqrt{g} (\omega^2 - k^2 r^2 f)}, \quad B_{12} = B_{22} = 0.
\]

(24)

This Lagrangian is invariant under simultaneous global \(U(1)\) transformations of both \(Z_1\) and \(Z_2\). The reason for this is the cross-term \(Z_1^\dagger B_{21} Z_2^\prime\). Assuming that \((r, \omega^2, k) \in \mathbb{R}\), the flux can then be found to be

\[
\mathcal{F} = 2i \left[ -Z_1 A_{11} Z_1^\prime + Z_1 A_{11} Z_1^{\dagger*} + Z_2 A_{22} Z_2^\prime - Z_2 A_{22} Z_2^{\dagger*} + \frac{1}{2} B_{21} (Z_1^* Z_2 - Z_2^* Z_1) \right].
\]

(27)

\(\mathcal{F}\) is conserved along the radial direction, i.e. \(\partial_r \mathcal{F} = 0\).

Now, in the UV part of the geometry the fields can be expanded as

\[
Z_1 = Z_1^{(0)} + r^2 Z_1^{(2)} + r^3 Z_1^{(3)} + \ldots
\]

\[
Z_2 = Z_2^{(0)} + r Z_2^{(1)} + \ldots,
\]

(28)

where \(Z_2^{(1)}\) is related to the vev of the QFT current \(J_\mu\), while \(Z_1^{(2)}\) is completely determined by the sources of the \(T_{\mu\nu}\) components of \(Z_1^{(0)}\). The vev of \(T_{\mu\nu}\) comes in at the order of \(r^3\). The value of the flux at the AdS boundary is

\[
\lim_{r \to 0} \mathcal{F}(r) = 2i \lim_{r \to 0} \left( Z_1 A_{11} Z_1^{\dagger*} - Z_2 A_{22} Z_2^{\dagger*} \right) + 2i A_{22}(0) \left( Z_2^{(0)*} Z_2^{(1)} - Z_2^{(0)} Z_2^{(1)*} \right) + \frac{1}{2} B_{21}(0) \left( Z_1^{(0)*} Z_2 - Z_1^{(0)} Z_2^{(0)*} \right)
\]

(29)

which along with the limiting values

\[
\lim_{r \to 0} A_{11} = -\lim_{r \to 0} \frac{\sqrt{f}}{4r^2 \sqrt{g} (\omega^2 - k^2 r^2 f)} = \frac{c}{4 \sqrt{g} (\omega^2 - c^2 k^2)}
\]

\[
\lim_{r \to 0} A_{22} = \frac{\sqrt{f}}{2 \sqrt{g}} = -\frac{c}{2}
\]

\[
\lim_{r \to 0} B_{21} = \lim_{r \to 0} \frac{k (rf' + 2f)}{2r \mu \sqrt{g} (\omega^2 - k^2 r^2 f)} = \frac{3c \dot{M}}{2 \mu} \frac{k}{\omega^2 - c^2 k^2}
\]

(30)
gives the conserved flux

\[
F = \mathcal{I} \left[ \frac{\omega^2 - c^2 k^2}{c^2} \left( \lim_{r \to 0} \frac{1}{r} \left( Z_1^{(0)} Z_1^{(2)*} - Z_1^{(2)*} Z_1^{(0)} \right) + \frac{3}{2} \left( Z_1^{(0)} Z_1^{(3)*} - Z_1^{(3)*} Z_1^{(0)} \right) \right) + Z_2^{(0)} Z_2^{(1)*} - Z_2^{(1)*} Z_2^{(0)} + \frac{3 M k}{2 \hat{\mu} (\omega^2 - c^2 k^2)} \left( Z_1^{(0)*} Z_2^{(0)} - Z_1^{(0)} Z_2^{(0)*} \right) \right].
\] (31)

To impose the Dirichlet boundary conditions at the boundary we need to fix \( Z_1^{(0)} \) and \( Z_2^{(0)} \) to some constants. However, to find only the QNMs, without the full Green’s functions, it is particularly useful to set \( Z_1^{(0)} = Z_2^{(0)} = 0 \). Generally, the values of \( Z_1^{(0)} \) and \( Z_2^{(0)} \) can be thought of as functions of \( \omega \) and \( k \) at some fixed physical parameters \( \hat{M}, \hat{Q}, \hat{\mu}, \) etc. describing the star geometry. Given some propagating modes that satisfy \( Z_1^{(0)} = Z_2^{(0)} = 0 \), we can see that the flux vanishes away from the light-cone \( (\omega^2 = c^2 k^2) \) for such \( \omega(k) \). Therefore

For a quasinormal mode \( \tilde{\omega}(k) \) \quad \rightarrow \quad F (\tilde{\omega}(k)) = 0 \quad (32)

It is interesting to note that the flux actually diverges unless we set \( Z_1^{(0)} = 0 \) or alternatively if \( Z_1^{(0)} Z_1^{(2)*} - Z_1^{(2)*} Z_1^{(0)} \) vanishes.

We would like to use this fact to find QNMs from the IR part of the geometry. The question we need to answer is therefore in what other cases can \( F = 0 \)? We can always set \( Z_1^{(0)} \) and \( Z_2^{(0)} \) to be real. Then the flux vanishes if \( Z_1^{(2)}, Z_1^{(3)} \) and \( Z_2^{(1)} \) are real as well. This is something we would, however, not generically expect to be true.

### B. Flux with complex frequency

We should look for the flux of \( \omega \in \mathbb{C} \) fluctuations to find the value of \( F \) on the QNMs. The off-shell action is

\[
S^{(2)} = \frac{L^2}{\kappa^2} \int d^4k dr \left\{ Z_i'(-k) A_{ij}(k) Z_j'(k) + Z_i(-k) B_{ij}(k) Z_j'(k) + \cdots \right\}
\] (33)

Because only \( A_{11}, A_{22}, B_{11} \) and \( B_{21} \) are non-zero the symmetry of this action is

\[
Z_i(k) \rightarrow e^{i\alpha} Z_i(k) \\
Z_i(-k) \rightarrow e^{-i\alpha} Z_i(-k)
\] (34)
We are using \(-k\) for \((-\omega, -k)\). The Nöther current (flux) is then

\[
\mathcal{F} = i \left\{ \left[ Z'_1(-k)Z_1(k) - Z_1(-k)Z'_1(k) \right] [A_{11}(k) + A_{11}(-k)] + \\
+ \left[ Z'_2(-k)Z_2(k) - Z_2(-k)Z'_2(k) \right] [A_{22}(k) + A_{22}(-k)] + \\
+ Z_1(-k)Z_1(k) [B_{11}(k) - B_{11}(-k)] + \\
+ Z_1(k)Z_2(-k)B_{21}(k) - Z_1(-k)Z_2(k)B_{21}(-k) \right\}
\]

Now \(A_{11}, A_{22}\) and \(B_{11}\) are invariant under \(k \to -k\), whereas \(B_{21}(-k) = -B_{21}(k)\).

\[
\mathcal{F} = i \left\{ 2A_{11}(k) \left[ Z'_1(-k)Z_1(k) - Z_1(-k)Z'_1(k) \right] + 2A_{22}(k) \left[ Z'_2(-k)Z_2(k) - Z_2(-k)Z'_2(k) \right] + \\
+ B_{21}(k) [Z_1(-k)Z_2(k) + Z_1(k)Z_2(-k)] \right\}
\]

Imagine that \(\mathcal{F}(\omega, k)\) is a polynomial defined over the complex plane of which zeroes we denote by \(\tilde{\omega}_i(k)\). From our construction above I claim that these are the QNMs of the electron star system. Hence

\[
\mathcal{F}(\omega, k) = \prod_{i=1}^{\infty} (\omega - \tilde{\omega}_i(k))
\]

**III. EXTERIOR OF THE STAR**

Outside the star the geometry is that of the Reissner-Nordström-AdS. We have \(\hat{\sigma} = \hat{\rho} = \hat{\rho} = 0\) and

\[
f = \frac{c^2}{r^2} - \dot{M}r + \frac{r^2\dot{Q}^2}{2}, \quad g = \frac{c^2}{r^4 f}, \quad h = \mu - r\dot{Q}.
\]

Also, as everywhere along the geometry,

\[
\mu(r) = \frac{h(r)}{\sqrt{f(r)}}.
\]

Equations (2) and (3) become

\[
0 = Z''_1 + 2kr^2 h' Z'_2 + \frac{\omega^2 f'}{f (\omega^2 - k^2 r^2 f)} Z'_1 + \frac{g}{f} \left( \frac{\omega^2 f'}{f (\omega^2 - k^2 r^2 f)} \right) Z_1
\]

\[
+ 2kr^2 \sqrt{f} \mu \left( \frac{2\omega^2 h'^2}{f (\omega^2 - k^2 r^2 f)} \right) Z_2,
\]

\[
0 = Z''_2 + \frac{1}{2} \left( \frac{f'}{f} - \frac{g'}{g} \right) Z'_2 - \frac{kh'}{\omega^2 - k^2 r^2 f} Z'_1 + \frac{g}{f} \left( \frac{\omega^2 - k^2 r^2 f}{\omega^2 - k^2 r^2 f} \right) Z_2
\]

\[
- \frac{2\omega^2 h'^2}{f (\omega^2 - k^2 r^2 f)} Z_2
\]
IV. SMALL STAR LIMIT

The easiest case to tract analytically is the limit when the star becomes small. Fermionic excitations in this scenario were analysed in [1].

The profile of the star is characterised by three functions \( \hat{\sigma}, \hat{\rho}, \hat{p} \). They all reach their maximum value in the IR at \( r \to \infty \) limit, where the geometry is pure Lifshitz. They monotonically decrease with decreasing \( r \) and reach \( \hat{\sigma} = \hat{\rho} = \hat{p} = 0 \) at the boundary of the star \( (r = r_s) \). The small star limit is characterised by

\[
\lambda^2 \equiv h_\infty^2 - \hat{m}^2 \ll 1
\]  

(42)

where

\[
\lambda^2 = \frac{6^{4/3} \hat{m}^{2/3} (1 - \hat{m}^2)^{2/3}}{(2\hat{m}^4 - 7\hat{m}^2 + 6)^{2/3}} \frac{1}{\beta^{2/3}}.
\]

Therefore at an arbitrary \( \hat{m} \), the small star limit is achieved by taking large \( \beta \). The exponent \( z \) becomes

\[
z = \frac{1}{1 - \hat{m}^2} + \frac{\lambda^2}{(1 - \hat{m}^2)^2} + \ldots
\]  

(43)

The correction to the Lifshitz geometry inside the star is

\[
f = \frac{1}{r^{2z}} \left( 1 + f_1 \frac{1}{r^{|\alpha|}} + \ldots \right)
\]

\[
g = \frac{g_\infty}{r^{2z}} \left( 1 + g_1 \frac{1}{r^{|\alpha|}} + \ldots \right)
\]

\[
h = \frac{h_\infty}{r^{2z}} \left( 1 + h_1 \frac{1}{r^{|\alpha|}} + \ldots \right)
\]

(44)

where

\[
|\alpha| = \frac{\hat{m} \sqrt{3} (2 - \hat{m}^2) 1}{\sqrt{1 - \hat{m}^2}} - 1 - \frac{1}{2 (1 - \hat{m}^2)} + \ldots
\]  

(45)

and

\[
g_\infty = \frac{6 - 7\hat{m}^2 + 2\hat{m}^4}{6 (1 - \hat{m}^2)^2} + \frac{(6 - 7\hat{m}^2 + 2\hat{m}^4) (1 + 4\hat{m}^2)}{12\hat{m}^2 (1 - \hat{m}^2)^3} \lambda^2 + \ldots
\]  

(46)

Corrections to the pure Lifshitz geometry inside the star therefore become exponentially suppressed for \( r > 1 \) when \( \lambda \ll 1 \). It is shown in [1] that \( f_1, g_1 \) and \( h_1 \) can be normalised in such a way that to leading order in \( \lambda \) the boundary of the star is at \( r_s = 1 \), while the correction to the pure Lifshitz geometry remains exponentially suppressed.