Nontrivial zeros of the Riemann zeta function

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ABSTRACT

The Riemann hypothesis, stating that all nontrivial zeros of the Riemann zeta function have real parts equal to $\frac{1}{2}$, is one of the most important conjectures in mathematics. In this paper we prove the Riemann hypothesis by adding an extra unbounded term to the traditional definition, extending its validity to $\operatorname{Re} z > 0$. This is then analysed in both halves of the critical strip ($0 < \operatorname{Re} z < \frac{1}{2}$, $\frac{1}{2} < \operatorname{Re} z < 1$). A contradiction is obtained when it is assumed that $\zeta(z) = 0$ in either of these halves.

1. INTRODUCTION

In 1859 Bernhard Riemann published an article titled, *On the number of primes less than a given quantity*. In that work he speculated that all complex valued nontrivial zeros of the zeta function have a real part equal to $\frac{1}{2}$. And this became known as the Riemann hypothesis. Ever since then, mathematicians have endeavoured to prove it. In 1900 David Hilbert added this problem to his list of 23 most important problems of the twentieth century. And since 2000 it has remained as one of six millennium problems.

The Riemann zeta function with real part, $\operatorname{Re} z > 1$ is traditionally defined by the infinite sum

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C} \cap \{\operatorname{Re} z > 1\}. \tag{1}$$

In this work we start with an integral form of the zeta function, which is analytically continued to the imaginary axis but excludes the only pole at $z = 1$. This form is given by (Heymann, 2020, p8)

$$\zeta(z) = \frac{z}{z-1} - z\int_1^{\infty} \{x\} x^{-z-1} dx, \quad z \neq 1, \ \operatorname{Re} z > 0. \tag{2}$$

We further rely on Riemann's functional equation, which implies symmetry on the positions of nontrivial zeros about the critical line ($\operatorname{Re} z = \frac{1}{2}$), and allows the zeta function to be analytically continued to the whole complex plane. This is given by (Riemann, 1859)

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z) \tag{3}$$

where

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \quad \operatorname{Re} z > 0$$

is the gamma function extending the factorial function to the complex plane. From equation (3) it is straightforward to determine the trivial zeros at $z = -2n,\ n \in \mathbb{N} - \{0\}$. All other zeros are known to lie only within the critical strip defined by $0 < \operatorname{Re} z < 1$ (Heymann, 2020).

In this work we prove the Riemann hypothesis in two stages. The first is to write the zeta function in terms of the infinite sum shown in equation (1), which is extended to $\operatorname{Re} z > 0$ by the addition of an extra divergent term. This is used to aid the second stage where we use the functional formula to obtain a contradiction when we assume $\zeta(z) = 0$ in both halves of the critical strip ($0 < \operatorname{Re} z < \frac{1}{2}$, $\frac{1}{2} < \operatorname{Re} z < 1$). This leaves only the critical line where zeros may be found.

2. PROOF OF THE RIEMANN HYPOTHESIS

The first task is to prove the following theorem:

Theorem 1: The Riemann zeta function for $\operatorname{Re} z > 0$, in terms of the traditional infinite sum is given by

$$\zeta(z) = \lim_{N \to \infty} \left(\sum_{n=1}^{N} n^{-z} - \frac{N^{1-z}}{1-z} \right), \quad \operatorname{Re} z > 0. \tag{4}$$

Proof:

We begin with the following known relationship

$$\zeta(z) = \frac{z}{z-1} - z \int_1^\infty \{x\} x^{-z-1} dx, \quad \operatorname{Re} z > 0. \tag{2}$$

We then rewrite the integral as

$$\int_1^\infty \{x\} x^{-z-1} dx = \lim_{N \to \infty} \int_1^N \{x\} x^{-z-1} dx$$

where $\{x\} = x - \lfloor x \rfloor$, while noting that $N \to \infty$. Integrating by parts gives

$$\int_1^N \{x\} x^{-z-1} dx = \left[\frac{\{x\} x^{-z}}{-z} \right]_1^N - \int_1^N \frac{x^{-z}}{-z} \{x\}' \, dx .$$

And by differentiating the fractional part of *x*, this becomes

$$\int_1^N \{x\} x^{-z-1} dx = \left[\frac{\{x\} x^{-z}}{-z} \right]_1^N - \int_1^N \frac{x^{-z}}{-z} \left(1 - \sum_{n \in \mathbb{Z}} \delta(x-n) \right) dx .$$

The first term on the right hand side vanishes due to the real part of z being positive and $\{1\} = 0$ at the lower limit. By directly evaluating the integral on the right hand side, the integral on the left becomes

$$\begin{aligned} \int_1^N \{x\} x^{-z-1} dx &= -\left[\frac{x^{1-z}}{-z(1-z)} \right]_1^N - \frac{1}{z} \sum_{n=2}^N n^{-z} \\ &= \frac{N^{1-z} - 1}{z(1-z)} - \frac{1}{z} \sum_{n=2}^N n^{-z}. \end{aligned}$$

Note that the lower limit on the sum starts at $n = 2$. This is due to the lower limit on the integral being 1, where the discontinuity in $\{x\}$ appearing there is not included. The first discontinuity is therefore seen at $x = 2$ corresponding to $n = 2$ in the sum.

Substituting back into equation (2) we get

$$\zeta(z) = \frac{z}{z-1} - \frac{N^{1-z} - 1}{1-z} + \sum_{n=2}^N n^{-z} = \frac{z}{z-1} + \frac{1}{1-z} - \frac{N^{1-z}}{1-z} + \sum_{n=2}^N n^{-z}.$$

Here we note that the sum starting at $n = 2$ is just the sum starting at $n = 1$ minus 1. Therefore, using

$$\sum_{n=2}^N n^{-z} = \sum_{n=1}^N n^{-z} - \frac{z-1}{z-1}$$

we have

$$\begin{aligned} \zeta(z) &= \sum_{n=1}^N n^{-z} - \frac{z-1}{z-1} + \frac{z}{z-1} - \frac{1}{z-1} - \frac{N^{1-z}}{1-z} = \sum_{n=1}^N n^{-z} - \frac{z-1}{z-1} + \frac{z-1}{z-1} - \frac{N^{1-z}}{1-z} \\ &= \sum_{n=1}^N n^{-z} - \frac{N^{1-z}}{1-z}, \quad N \to \infty, \quad \operatorname{Re} z > 0. \end{aligned}$$

This shows the validity of equation (4) extending the traditional definition to $\operatorname{Re} z > 0$. Moreover, we see that this reduces to the original definition for $\operatorname{Re} z > 1$ due to the vanishing second term.

□

The next stage is to complete the proof using equation (4). For this we adopt the following shorthand notation,

$$S_z = \sum_{n=1}^{\infty} n^{-z}$$

$$C_z = \lim_{N\to\infty} \frac{N^{1-z}}{1-z}$$

$$F_z = \frac{\zeta(z)}{\zeta(1-z)} = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z).$$

We are now in a position to proceed with the second stage to prove the Riemann hypothesis.

Theorem 2, the Riemann hypothesis: All of the nontrivial zeros of the Riemann zeta function contained within the critical strip, $0 < \operatorname{Re} z < 1$, have real parts equal to $\frac{1}{2}$.

Proof:

From the shorthand expressions defined above it is known that within the critical strip, F_z is bounded and nonzero. Equation (4) is also written as $\zeta(z) = S_z - C_z$. From this and the functional equation, we get

$$F_z = \frac{S_z - C_z}{S_{1-z} - C_{1-z}}, \quad \operatorname{Re} z > 0. \tag{5}$$

Also, from the expression for C_z it is straightforward to deduce

$$\frac{C_z}{C_{1-z}} = \frac{zN^{1-2z}}{1-z} \to 0, \quad N \to \infty, \quad \operatorname{Re} z > \frac{1}{2}$$

$$\left|\frac{C_z}{C_{1-z}}\right| = \left|\frac{zN^{1-2z}}{1-z}\right| \to \infty, \quad N \to \infty \quad 0 < \operatorname{Re} z < \frac{1}{2}$$

while noting the divergence of both C_z and C_{1-z} independently. It follows that this quotient is bounded and nonzero only when $\operatorname{Re} z = \frac{1}{2}$.

The next task is to write equation (5) in terms of C_z / C_{1-z}. For this we start with the functional equation,

$$\begin{aligned}
& F_z\left(S_{1-z} - C_{1-z}\right) = S_z - C_z \\
& \Rightarrow F_z S_{1-z} - F_z C_{1-z} = S_z - C_z \\
& \Rightarrow F_z S_{1-z} - S_z = F_z C_{1-z} - C_z \\
& \Rightarrow \frac{F_z S_{1-z} - S_z}{C_{1-z}} = F_z - \frac{C_z}{C_{1-z}} \\
& \Rightarrow \frac{C_z}{C_{1-z}} = F_z - \frac{F_z S_{1-z} - S_z}{C_{1-z}}.
\end{aligned}$$

Therefore, we have

$$\frac{C_z}{C_{1-z}} = F_z\left(1-\frac{S_{1-z}}{C_{1-z}}\right)+\frac{S_z}{C_{1-z}}. \tag{6}$$

Now let $\zeta(z)=0$. This gives us $S_z = C_z$ and $S_{1-z} = C_{1-z}$. Therefore, equation (6) becomes

$$\frac{C_z}{C_{1-z}} = \frac{S_z}{C_{1-z}}. \tag{7}$$

Because $S_{1-z} = C_{1-z}$, we may also substitute the denominator on the right hand side of equation (7),

$$\frac{C_z}{C_{1-z}} = \frac{S_z}{S_{1-z}} = R = \frac{zN^{1-2z}}{1-z}, \quad N \to \infty. \tag{8}$$

This allows us to write

$S_z = RS_{1-z}$ and $C_z = RC_{1-z}$
$\Rightarrow S_z - C_z = RS_{1-z} - RC_{1-z}$
$\Rightarrow S_z - C_z = R\left(S_{1-z} - C_{1-z}\right)$

From here we arrive at

$$\frac{S_z - C_z}{S_{1-z} - C_{1-z}} = R. \tag{9}$$

Assuming continuity F_z and the zeta function, from equation (5) this implies that $R = F_z$ whenever $\zeta(z)=0$. Since $|F_z| > 0$ and is bounded, the only way that this can be true is if the same applies to R, and this is only the case when $\operatorname{Re} z = \frac{1}{2}$.

One might be tempted to apply the above analysis without assuming $\zeta(z)=0$, and we would still arrive at equation (9). However, it is known that $R = F_z$ is not true everywhere in the critical strip. At the very least we could say that equation (8) is invalidated and the right hand side of equation (9) would not correspond to F_z. Fortunately, equation (5), remains valid irrespective of whether $\zeta(z)=0$ or not, and the only place where the right hand side of equation (9) could equate to F_z is at $\operatorname{Re} z = \frac{1}{2}$.

Away from the critical line equation (9) translates to

$$2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) = 0, \quad \operatorname{Re} z > \frac{1}{2} \tag{10}$$

and

$$2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) \to \infty, \quad \operatorname{Re} z < \frac{1}{2}. \tag{11}$$

But because none of the factors on the left hand side here vanish for $\frac{1}{2} < \operatorname{Re} z < 1$ and F_z remains bounded everywhere within the critical strip, equations (10, 11) show the expected contradiction to the assumption that $\zeta(z) = 0$. Therefore, the only place where zeros can exist within the critical strip, is on the critical line. This concludes the proof of the Riemann hypothesis.

□

Before concluding, we make a few remarks about phase. From equation (9), we note that

$$\frac{zN^{1-2z}}{1-z} = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z) \tag{12}$$

whenever $\zeta(z) = 0$. The right hand side would certainly fix the phase, and one may question whether this would generate an undesirable contradiction. The phase of the left hand side is given by the following,

$$\frac{zN^{1-2z}}{1-z} = \frac{\sqrt{\sigma^2 + t^2}\exp(i\theta)N^{1-2\sigma}\exp(-i2t\ln N)}{\sqrt{(1-\sigma)^2 + t^2}\exp(i\phi)}$$

where $z = \sigma + it$, $\theta = \tan^{-1}(t/\sigma)$ and $\phi = \tan^{-1}(t/(1-\sigma))$. Therefore, the phase of the left hand side of equation (12) is given by

$$\arg\left[\frac{zN^{1-2z}}{1-z}\right] = \tan^{-1}(t/\sigma) - \tan^{-1}(t/(1-\sigma)) - 2t\ln N.$$

In the third term on the right hand side here, we note that $\ln N \to \infty$. The phase is therefore infinitely sensitive to changes in $\operatorname{Im} z = t$. Therefore, any value of t can be made to fit the phase belonging to the right hand side of equation (12). Moreover, this in no way provides a straightforward way to compute the imaginary parts of the nontrivial zeros, nor does it invalidate the proof of the Riemann hypothesis.

3. CONCLUSION

In this work we have demonstrated the Riemann hypothesis to be true and that the real parts of all nontrivial zeros are $\frac{1}{2}$. By writing the zeta function in a form that includes the traditional sum, extended to $\operatorname{Re} z > 0$, it has been shown that no nontrivial zeros can exist away from the critical line, implying that the only occurrences of zeros within the critical strip are at $\operatorname{Re} z = \frac{1}{2}$ as hypothesized.

REFERENCES

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