A Simple Proof of Fermat's Last Theorem for the Cube

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Abstract

A simple proof of Fermat's Last Theorem (FLT) for the cube is obtained using the binomial expansion formula. The difference of two natural numbers having equal natural powers necessarily has a representation according to the formula of an incomplete binomial. It is proved that the cube of a natural number cannot be represented as an incomplete binomial, which means a simple proof of FLT for $n = 3$ is obtained.

KEYWORDS: Fermat, binomial, binomial expansion, simple proof.

1 Introduction

Fermat's Last Theorem (FLT), formulated by the French mathematician Pierre de Fermat in 1637, is described as follows [1]:

For any [natural number] $n > 2$, the equation $a^n + b^n = c^n$ has no solutions in natural numbers a, b, c.

According to Fermat's notes, he wrote: "It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain" [1]. Fermat's Last Theorem was proved in 1994 by Andrew Wiles, using complex mathematical apparatus based on elliptic curves, which were unknown in Fermat's time [2]. In this connection, the search for a simple proof of FLT continues to this day, which shows the relevance of the problem.

It is known that Euler, using complex numbers, presented a rather complicated proof of FLT for the cube [1]. A review of works [3, 4, 5, 6] devoted to FLT shows that until now there was no simple proof of FLT for $n = 3$, therefore obtaining a simple proof of FLT for $n = 3$ is also an urgent task. From the review of these works it also follows that the ideas and methods applied to prove FLT for $n = 3$ in this work, as well as the obtained results, are new.

2 Difference of Equal Powers and Binomial Expansion

2.1 Lemma 1

Before presenting the proof of FLT for $n = 3$, we show a feature of the difference of equal powers of two natural numbers.

1. Any natural power of a natural number $a > 1$ can be represented as a natural power of the sum of two natural numbers, $a^n = (b + d)^n$, $a, b, d, n \in \mathbb{N}$;
2. Any natural power of a natural number represented as $(b + d)^n$ can be expanded using the binomial formula;
3. The difference of equal powers of two natural numbers $A = c^n - b^n$ can be represented as $A = (b + d)^n - b^n$;
4. From point 3 it follows that the difference of equal powers of two natural numbers does not correspond to the expansion by the binomial formula, since when expanding by the binomial formula, the two numbers b^n will cancel.

Based on points 3 and 4, we can formulate the following lemma.

Lemma 1. *Let $a, b, c, n \in \mathbb{N}$, $a < b < c$, and $a^n = c^n - b^n$. Then there exists a natural number $d = c - b$ such that*

$$a^n = S + d^n,$$

where $S = \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k$.

In other words, if the difference of two natural nth powers is itself a natural nth power, then this power is representable as the sum of all its middle terms and the last term of the binomial expansion of $(b+d)^n$.

Proof. By assumption,

$$a^n = c^n - b^n.$$

Set $d = c - b$. Then $d \in \mathbb{N}$ and $c = b + d$. Consequently,

$$a^n = (b+d)^n - b^n. \tag{1}$$

By Newton's binomial formula

$$(b+d)^n = b^n + \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k + d^n.$$

Subtracting b^n, we obtain

$$a^n = \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k + d^n.$$

Denoting $S = \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k$, we have

$$a^n = S + d^n.$$

□

Definition 1. If from the binomial expansion of any natural number of the form $(b+d)^n$, consisting of $n+1$ terms, we subtract the first term b^n (or the last term d^n), then the resulting expression consisting of n terms is called an incomplete binomial expansion.

Note. For convenience, in what follows we will assume that only the first term is missing in the incomplete binomial.

From Lemma 1 it follows that the difference of two natural numbers having equal natural powers necessarily has a representation in the form of an incomplete binomial expansion $c^n - b^n = S + d^n$, in which, compared to the ordinary binomial expansion, the first term b^n is absent.

Thus, if equation (1) has a solution in natural numbers, then a^n must be representable by the formula of an incomplete binomial in natural numbers.

In other words, to find a simple proof of Fermat's Last Theorem we must prove that a natural power greater than 2 of any natural number cannot be represented by the formula of an incomplete binomial expansion in natural numbers.

2.2 Lemma 2

Since in the formula $a^n = c^n - b^n$ the number a satisfies the condition $a < b < c$, there is a possibility to represent this formula as

$$(s+d)^n = (b+d)^n - b^n, \tag{2}$$

where $d = 1, 2, \ldots$ is the interval between numbers.

Lemma 2. *Let $a, b, c, n \in \mathbb{N}$, $a < b < c$ and $a^n + b^n = c^n$. Then there exists a natural d and a positive $s > 0$ (not necessarily natural), such that:*

$$\begin{cases} c = b + d \\ a = s + d \end{cases}$$

$s = b + a - c$.

Proof. Set $d = c - b$. Then $d \in \mathbb{N}$ and $c^n = (b+d)^n$, since $b < c$.

By assumption $a^n = c^n - b^n = (b+d)^n - b^n$.

Expand $(b+d)^n$ by Newton's binomial

$$(b+d)^n = b^n + \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k + d^n = b^n + S + d^n,$$

where $S = \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k} d^k > 0$ (all summands are positive for $b, d \geq 1$).

Then $a^n = S + d^n$, therefore $a = (S + d^n)^{1/n} > (d^n)^{1/n}$.

It follows that $s = a - d > 0$.

Then $a = s + d$ and $c = b + d$ — both numbers a and c are expressed through the same d. □

2.3 Difference of Cubes

From FLT it follows that if it were false, then the following equality would have a natural solution for $a < b < c$,

$$c^3 - b^3 = a^3. \tag{3}$$

Since the cube of a natural number can be represented as $c^3 = (b+d)^3$, which has the following binomial expansion

$$(b+d)^3 = b^3 + 3b^2 d + 3bd^2 + d^3, \quad b, d \in \mathbb{N},$$

which can be represented as

$$b^3 + 3b^2d + 3bd^2 + d^3 - b^3 = a^3,$$

or

$$a^3 = 3b^2d + 3bd^2 + d^3. \tag{4}$$

Formula (4) means that if FLT were false, then the cube of a natural number could be represented as an incomplete binomial expansion. Further, taking into account that the number a can be represented as the sum of two numbers, its cube can be represented as $a^3 = (s + d)^3$.

After this, we perform the binomial expansion of the number $(s + d)^3$,

$$(s + d)^3 = s^3 + 3s^2d + 3sd^2 + d^3, \quad s, d \in \mathbb{N}.$$

Substituting into (4) we obtain

$$s^3 + 3s^2d + 3sd^2 + d^3 = 3b^2d + 3bd^2 + d^3,$$

hence

$$s^3 = (3b^2d + 3bd^2 + d^3) - (3s^2d + 3sd^2 + d^3). \tag{5}$$

$$s^3 = (c^3 - b^3) - (a^3 - s^3). \tag{6}$$

Formula (6) is another representation of the FLT formula.

Formula (5) means that if the cube of a natural number (a^3) is representable as an incomplete binomial in accordance with formula (4), then the cube of s, which is the first summand of the number a ($s + d = a$), must equal the difference of two incomplete binomials. Moreover, the difference of the differences of two incomplete binomials must equal the number that is the smaller summand of the smaller of the two incomplete binomials.

Thus, we will investigate the cubes of four numbers n_i, n_j, n_k, n_e, which correspond to the numbers s, a, b, c. More precisely, we will investigate the difference of cubes $n_j^3 - n_i^3$ and $n_e^3 - n_k^3$, which correspond to the differences of cubes $a^3 - s^3$ and $c^3 - b^3$, and then the difference of differences $(n_e^3 - n_k^3) - (n_e^3 - n_k^3)$. Here the numbers of each pair differ by h, and the pairs can be at distances $L \geq h$.

Note that essentially $h = d$, however in formula (5) the number d is an unknown quantity, while h is a given quantity; to emphasize this we decided to use different notations.

Let us conduct an investigation in accordance with the following equation corresponding to the difference of two incomplete binomials

$$\left(n_e^3 - n_k^3\right) - \left(n_j^3 - n_i^3\right) = \left((n_k + h)^3 - n_k^3\right) - \left((n_i + h)^3 - n_i^3\right) = Q$$

Further, we express all three numbers n_j, n_k, n_e through n_i, which is the smallest,

$$\left((n_i + h + h + L)^3 - (n_i + h + L)^3\right) - \left((n_i + h)^3 - n_i^3\right) = Q. \qquad (7)$$

To investigate the differences of incomplete binomial expansions, we must obtain the complete set of differences of incomplete binomials.

For this, we first find the differences of adjacent cubes, therefore we take $h = 1$, then we obtain the set of incomplete binomials of adjacent cubes, expressed by the formula

$$M = \{m \mid m = (b + h)^3 - b^3, h = 1\}.$$

After this, based on the elements of the set M, we investigate the difference of binomials, varying the step of incomplete binomials L, and as a result we obtain the complete set of differences of incomplete binomials.

Taking this into account, we represent formula (7) in the form

$$\left((n_i + 2 + L)^3 - (n_i + 1 + L)^3\right) - \left((n_i + 1)^3 - n_i^3\right) = Q. \qquad (8)$$

Any difference of incomplete binomials will be a multiple of 6 (the proof is given in Appendix A), therefore considering that we are looking for the case where the difference of incomplete binomials equals the cube of a natural number, we will investigate the case $Q = (6t)^3$.

Note. Appendix B provides a proof of the validity of taking $h = 1$ and obtaining the complete set of differences of incomplete binomials by the above method.

Further, considering the above and (5), we take $n_i^3 = s^3 = (6t)^3$, and represent equation (7) in the form

$$\left((6t + 2 + L)^3 - (6t + 1 + L)^3\right) - \left((6t + 1)^3 - (6t)^3\right) = Q,$$

or

$$(6t + 2 + L)^3 - (6t + 1 + L)^3 - (6t + 1)^3 + (6t)^3 = Q. \qquad (9)$$

Next, we prove that under the condition

$$(6t+2+L)^3 \leq ((6t+1+L)^3+(6t+1)^3),$$

for all $t \geq 1$ the inequality $Q < (6t)^3$ holds.

Note that the left side of the inequality corresponds to c^3, and the right side corresponds to b^3+a^3.

Introducing the notation $6t = m$, we represent formula (9) in the form

$$Q = (m+2+L)^3 - (m+1+L)^3 - (m+1)^3 + m^3. \tag{10}$$

Let us show the correspondence of parameters: $m = s,\ m+1 = a,\ m+1+L = b,\ m+2+L = c$. If FLT were false, then we would have $Q = m^3$.

Now we derive a formula for L.

2.4 Step Between Incomplete Binomials

Lemma 3. *Let $m \in \mathbb{N}$. Then there exists a maximum integer $L_{\max}$ for which the inequality holds*

$$(m+L+2)^3 \leq (m+L+1)^3 + (m+1)^3. \tag{11}$$

Moreover, for all $L \leq L_{\max}$ the inequality holds, and for all $L > L_{\max}$ it does not hold.

Proof. From (11) we obtain

$$3(m+L+1)^2 + 3(m+L+1) + 1 \leq (m+1)^3.$$

Set $y = m+L+1$.

Then

$$3y^2 + 3y + 1 \leq (m+1)^3.$$

The positive root of the corresponding quadratic equation

$$3y^2 + 3y + 1 = (m+1)^3$$

equals

$$y_+ = -\frac{1}{2} + \sqrt{\frac{4(m+1)^3 - 1}{12}}.$$

Therefore

$$y \le y_+,$$

that is,

$$L \le y_+ - m - 1.$$

Hence

$$L_{\max} = \left\lfloor \sqrt{\frac{4(m+1)^3 - 1}{12}} - m - \frac{3}{2} \right\rfloor . \tag{12}$$

□

2.5 Difference of Differences of Incomplete Binomials

Consider the expression

$$Q = \left[(m+L+2)^3 - (m+L+1)^3\right] - \left[(m+1)^3 - m^3\right], \quad m = 6t.$$

The parameter L is used only to isolate the boundary configuration of cube comparison. Therefore, in what follows we only consider the value

$$Q = Q(L_{\max}),$$

corresponding to the maximum L for which inequality (9) holds.

It is precisely this boundary case that is related to the initial configuration of cubes; for $L > L_{\max}$ inequality (9) is violated, and the corresponding values of Q no longer relate to the situation under consideration.

Lemma 4. *Let $m \ge 6$, $m = 6t$ and*

$$L_{\max} = \left\lfloor \sqrt{\frac{4(m+1)^3 - 1}{12}} - m - \frac{3}{2} \right\rfloor ,$$

and the quantity Q is defined as

$$Q = (m+L+2)^3 - (m+L+1)^3 - (m+1)^3 + m^3.$$

Then $Q < m^3$.

Proof. Set $y = m + L_{\max} + 1$.

Then

$$Q = (y+1)^3 - y^3 - (m+1)^3 + m^3. \tag{13}$$

$$(y+1)^3 - y^3 < (m+1)^3.$$

From the formula for $L_{\max}$ we have

$$L_{\max} \le \sqrt{\frac{4(m+1)^3 - 1}{12}} - m - \frac{3}{2},$$

then

$$y = m + L_{\max} + 1 \le \sqrt{\frac{4(m+1)^3 - 1}{12}} - \frac{1}{2}.$$

Consequently

$$y + \frac{1}{2} \le \sqrt{\frac{4(m+1)^3 - 1}{12}}.$$

Squaring,

$$\left(y + \frac{1}{2}\right)^2 \le \frac{4(m+1)^3 - 1}{12}.$$

Multiply by 12:

$$12y^2 + 12y + 3 < 4(m+1)^3 - 1,$$

that is,

$$3y^2 + 3y + 1 \le (m+1)^3.$$

Consequently

$$3y^2 + 3y + 1 < (m+1)^3.$$

But

$$(y+1)^3 - y^3 = 3y^2 + 3y + 1,$$

therefore

$$(y+1)^3 - y^3 < (m+1)^3. \tag{14}$$

From inequality (13) it follows that

$$(y+1)^3 - y^3 - (m+1)^3 < 0.$$

Adding m^3, we obtain

$$(y+1)^3 - y^3 - (m+1)^3 + m^3 < 0.$$

Since the left side equals Q, we have

$$Q < m^3.$$

□

3 Main Theorems

Theorem 1. *For any natural numbers $a,\ b,\ d$, the inequality*

$$a^3 \neq (b+d)^3 - b^3$$

holds.

Proof. Suppose contrary: let there exist natural numbers $a,\ b,\ d$ for which

$$a^3 = (b+d)^3 - b^3.$$

As shown earlier, from the existence of such a solution it follows that the corresponding parameters can be represented as

$$m = 6t,$$

and the equality

$$Q = (6t)^3$$

arises, where the quantity Q is defined by the expression

$$Q = \left[(m+L+2)^3 - (m+L+1)^3\right] - \left[(m+1)^3 - m^3\right].$$

The parameter L is introduced as a shift for cube comparison. Consider the inequality

$$(m+L+2)^3 \leq (m+L+1)^3 + (m+1)^3.$$

Since the function

$$(m+L+1)^3 + (m+1)^3 - (m+L+2)^3$$

is strictly decreasing in L, the set of natural L satisfying this inequality has the form

$$\{1,\ 2,\ \dots, L_{\max}\}.$$

Consequently, there exists a maximum natural number $L_{\max}$ for which the inequality holds, and for all $L > L_{\max}$ it is already violated.

From (10) for $L = L_{\max}$ we obtain

$$(m + L_{\max} + 2)^3 - (m + L_{\max} + 1)^3 - (m + 1)^3 \leq 0.$$

Adding m^3, we obtain

$$Q(L_{\max}) \leq m^3 = (6t)^3.$$

Earlier it was also proved that for all $t \geq 1$ and $m \geq 6$

$$Q(L_{\max}) \leq (6t)^3. \tag{15}$$

But from the existence of a hypothetical solution, the equality follows

$$Q = (6t)^3.$$

Together with (15) this gives a contradiction

$$(6t)^3 = Q < (6t)^3,$$

which is impossible.

Therefore the original equation

$$a^3 = (b + d)^3 - b^3$$

has no solutions in natural numbers.

This means that the cube of a natural number cannot be represented as an incomplete binomial. □

Theorem 2. *There are no natural numbers $s, d,\ x$ satisfying*

$$(s + d)^3 = 3(s + x)^2 d + 3(s + x)d^2 + d^3. \tag{16}$$

Proof. From Theorem 1 it follows that formula (4) has no natural solution, consequently in formula (16) the number $s + x$ cannot be a natural number. □

4 Conclusion

If a hypothetical solution of FLT for the cube existed, then this cube would be representable in the form of an incomplete binomial (Lemma 1 and Lemma 2), then there would exist a cube of a natural number equal to the difference of two incomplete binomials (formula 4). Moreover, this number would have to equal the number that is the smaller summand of the smaller of the two incomplete binomials, which is one of the two summands of the smaller cube.

The value of such differences coincides with

$$Q = \left[(m+L+2)^3 - (m+L+1)^3\right] - \left[(m+1)^3 - m^3\right],$$

where $m = 6t$ (formula 10 and Lemma 4). It has been established (Lemma 3) that the value of the maximum step for $m \geq 6$ should be

$$L = \left\lfloor \sqrt{\frac{4(m+1)^3 - 1}{12}} - m - \frac{3}{2} \right\rfloor .$$

After this, it was proved that for all natural $t \geq 1$ and $m \geq 6$, the difference Q is less than m^3 (Lemma 4), consequently the difference of differences of incomplete binomials cannot be the cube of a natural number.

Further, the equation in Theorem 1 is proved: the difference of two natural cubes is not equal to the cube of a natural number, which means that the cube of a natural number cannot be represented as an incomplete binomial.

Theorem 2, based on Theorem 1, proves that the equation $a^3 = 3b^2d + 3bd^2 + d^3$ represented as $(s+d)^3 = 3(s+x)^2 d + 3(s+x)d^2$ has no natural solution.

Since FLT was proved in 1994, we have refrained from proving FLT by our method for $n > 3$, although our method can be applied for any natural power of a natural number.

Obviously, with an increase in the exponent n, the number of terms of the incomplete binomial expansion with different powers, over which the removed first term of the binomial is distributed, increases significantly, so we can assert that $s + x$ cannot be a natural number for all $n > 3$, that is, no natural number raised to a power $n > 2$ can be represented as an incomplete cube.

For example, the representation of the fifth power of a natural number as an incomplete binomial has the form

$$(s+d)^5 = 5(s+x)^4 d + 10(s+x)^3 d^2 + 10(s+x)^2 d^3 + 5(s+x)d^4 + d^5,$$

so we can assume that x cannot be a natural number to compensate for the missing term s^5.

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A Lemma on Multiplicity of Difference of Incomplete Binomials by 6

Lemma 1. *The difference of two incomplete cubic binomial expansions with the same step h is always divisible by 6.*

Proof. Consider two incomplete cubic binomial expansions with the same step h:

$$(b+h)^3 - b^3 - h^3 = 3b^2h + 3bh^2,$$
$$(s+h)^3 - s^3 - h^3 = 3s^2h + 3sh^2.$$

Their difference equals

$$\left(3b^2h + 3bh^2\right) - (3s^2h + 3sh^2).$$

Factor out the common factor $3h$:

$$3h\left((b^2 - s^2) - h(b-s)\right).$$

Set

$$x = b - s.$$

Then

$$b^2 - s^2 = (b-s)(b+s) = x(b+s),$$

and therefore

$$\left(3b^2h + 3bh^2\right) - \left(3s^2h + 3sh^2\right) = 3hx(b+s+h).$$

It remains to show that the number $hx(b+s+h)$ is even.

If h is even or x is even, then the product hx is even, and the whole expression is even. If h and x are odd, then

$$b + s + h = (s + x) + s + h = 2s + x + h$$

is even, since $2s$ is even and $x + h$ is even.

Consequently, in all cases the number

$$hx(b+s+h)$$

is even. Therefore,

$$3hx(b+s+h),$$

is divisible by both 2 and 3, and hence by 6. □

B Lemma on the Validity of Taking h = 1

Denoting $n_i^3 = m^3$ and $6t = m$, we represent formula (7) in the form

$$Q = \left[(m+L+2)^3 - (m+L+1)^3\right] - \left[(m+1)^3 - m^3\right].$$

Then using

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1,$$

we obtain

$$Q = \left[3(m+L)^2 + 3(m+L) + 1\right] - \left[3m^2 + 3m + 1\right].$$

After simplification this will have the form

$$Q = 3L(2m + L + 1).$$

We will investigate different differences of incomplete binomials, which form a set, so we introduce the following notation

$$B = \{3L(2m + L + 1) : L,\ m \in \mathbb{N}\}.$$

By analogy, based on formula (6) we obtain another set

$$A = \{3hL(2m + L + h) : h,\ L,\ m \in \mathbb{N}\}.$$

Lemma 1. *$A = B$.*

Proof. First, we show that $B \subseteq A$.

If in the general formula of set A we set $h = 1$, then we obtain set B, consequently any element of set B belongs to set A, that is $B \subseteq A$.

Now we prove that $A \subseteq B$.

Let $N = 3hL(2m + L + h) \in A$.

As already established, any such number is a multiple of 6. Therefore there exists a natural number

$$k = \frac{hL(2m + L + h)}{2} \in \mathbb{N}$$

such that $N = 6k$.

Now set

$$L = 1, \quad m = k - 1.$$

Then

$$3L(2m + L + 1) = 3 \cdot 1 \cdot (2(k - 1) + 1 + 1) = 3(2k) = 6k = N.$$

Thus, the number N has the form of an element of set B. Consequently, $A \subseteq B$.

Therefore, $A = B$. $\square$