Fermat's Last Theorem for Cubes

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ABSTRACT: The article presents a simple proof of Fermat's Last Theorem (FLT) for a cube, obtained on the basis of the binomial expansion. The difference of two natural numbers having equal natural degrees certainly has a representation according to the incomplete binomial formula. It is proved that the cube of a natural number cannot be represented as an incomplete binomial, which means a simple proof of the FLT for n = 3 is obtained.

KEY WORDS: binomial formula, incomplete binomial expansion, proof.

1 INTRODUCTION

Fermat's Last Theorem (FLT) are formulated as follows: for any natural number $n > 2$, the formula

(1.1) $a^n + b^n = c^n$

has no solutions in natural numbers a, b, c [1].

Fermat wrote: “It is impossible to decompose a cube into two cubes, a biquadrate into two biquadrates, and in general no degree greater than a square into two degrees with the same exponent. I found a truly wonderful proof of this, but the margins of the book are too narrow for him "[1]. The Last Fermat's Theorem was proved in 1994 by Andrew Wiles using complex mathematical tools based on elliptic curves that were not known in Fermat's time [2]. In this regard, the search for a simple proof of the FLT continues at the present time, which shows the relevance of the problem.

It is known that Euler, using complex numbers, presented a rather complicated proof of the FLT for a cube [1]. A review of papers [3, 4, 5, 6] devoted to FLT shows that until now there has been no simple proof of FLT for n = 3, therefore, obtaining a proof of simple FLT for n = 3 is also an urgent problem. It also follows from the survey of these works that the ideas and methods used to prove the FLT for n = 3 in this work, as well as the results obtained, are new.

2 DIFFERENCE OF EQUAL DEGREES AND BINOMINAL DECOMPOSITION

2.1 Difference of equal powers of two natural numbers

Before presenting the proof of the FLT for $n = 3$, we show that the difference of equal powers of two natural numbers is singular.

1) Any natural power of a natural number a >1 can be represented as a natural power of the sum of two natural numbers, $a^n = (b + d)^n$, $a, b, d, n \in$ N;
2) Any natural power of a natural number represented in the form $(b + d)^n$ can be expanded by the binomial formula;
3) The difference of equal powers of two natural numbers $A = c^n - b^n$ can be represented as $A = (b + d)^n - b^n$;
4) From point 3 it follows that the difference of equal degrees of two natural numbers does not correspond to the decomposition by the binomial formula, since the decomposition by the binomial formula will reduce two numbers b^n.

Based on items 3 and 4, the following theorem can be formulated:

Theorem 2.1. The difference of two natural numbers having equal natural degrees $c^n - b^n = A$, which can be represented as $(b + d)^n - b^n = A$, where $c^n > b^n$, $c, b, d, n \in$ N, certainly has a representation in the form of an incomplete binomial expansion, in which, in comparison with the usual binomial expansion, the term b^n is absent.

The proof of Theorem 2.1 is elementary, and it is described in the above items 2 and 3 of the proof. From Theorem 2.1 it follows that the difference of two natural numbers having equal natural degrees has a strict representation, namely, it corresponds to the decomposition according to the binomial formula, in which b^n is absent.

To clarify the term “incomplete binomial expansion”, we will accept the following definitions.

Definition 2.1. If one element b^n (or d^n) is subtracted from the binomial expansion of any natural number of the form $(b + d)^n$, consisting of $n + 1$ elements, then the resulting expression consisting of n elements is called an incomplete binomial expansion.

For completeness of information, below we give the definition of the term "redundant binomial expansion".

Definition 2.2. If we add one element b^n (or d^n) to the binomial expansion of any natural number of the form $(b + d)^n$, consisting of $n + 1$ elements, then the resulting expression consisting of $n + 2$ elements is called a redundant binomial expansion.

Note: Excess binomial decomposition is not considered in this work, the definition is provided for information only.

Since in the formula $a^n = c^n - b^n$ the number a corresponds to the condition $a < b < c$, while representing this formula in the form $(s + d)^n = (b + d)^n - b^n$, where the interval between the numbers d can have different natural values, i.e. $d = 1, 2, \ldots$, the following question may arise:

Question 1. Is it correct for all-natural numbers a, b, c, n, d to represent the number a^n, calculated by the formula $a^n = c^n - b^n$, by the formula $a^n = (s + d)^n$?

Answer to question 1: it is obvious that if $a,\ n > 1$, then $a > d$, since $a^n = (b+d)^n - b^n = S_{bd} + d^n$, where S_{bd} is a natural number, depending on the numbers b, d, n, or

(2.1) $a = \sqrt[n]{S_{bd} + d^n} > d.$

It follows from formula (2.1) that the number a corresponds to the condition $d < a < b$. In this regard, a^n can be represented as $a^n = (s+d)^n$.

2.2 Incomplete binomial decomposition

Below we show an incomplete binomial expansion for the general case.

Let a natural power of a natural number $a^n = (s+d)^n$ be given. We want to represent this number in the form of an incomplete binomial expansion, where the element s^n is absent, for this we proceed as follows.

First, we perform the binomial expansion of the number $(s+d)^n$, then we obtain

(2.2) $(s+d)^n = \sum_{k=0}^{n}\binom{n}{k}s^{n-k}d^k = \binom{n}{0}s^n + \binom{n}{1}s^{n-1}d + \cdots + \binom{n}{k}s^{n-k}d^k + \cdots + \binom{n}{n}d^n,$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!} = C_n^k$ are binomial coefficients.

Next, from formula (2.2), we remove the first term $\binom{n}{0}s^n$, after that, to compensate for the removed s^n, we add the number x to the numbers s, then we get the following formula

(2.3) $(s+d)^n = \binom{n}{1}(s+x)^{n-1}d + \cdots + \binom{n}{k}(s+x)^{n-k}d^k + \cdots + \binom{n}{n}d^n.$

Formula (2.3) is a representation of the number $(s+d)^n$ in the form of an incomplete binomial expansion.

Thus, the representation of the natural power of a natural number in the form of an incomplete binomial is equivalent to the distribution of the first term $\binom{n}{0}s^n$ of the complete binomial using the number x to other terms of the binomial.

Further, for convenience, we represent formula (2.2) in the following brief form

(2.4) $(s+d)^n = s^n + SD$, or $a^n = s^n + SD$,

where SD is an incomplete binomial decomposition of the number $(s+d)^n$.

Formula (2.3) can also be represented in a compact form

(2.5) $(s+d)^n = SD(x),$

where $SD(x)$ is the incomplete binomial decomposition of the number $(s+x)^n$.

2.3 Difference of squares of two natural numbers

The reason that the difference between the squares of natural numbers can be the square of a natural number, although it is expressed by the formula of incomplete binomial decomposition, is explained by the fact that, in this case, by the formula of incomplete binomial decomposition, it is possible to obtain the set of all odd numbers and most of the even numbers.

Using the formula for the binomial decomposition of the square of the sum of natural numbers, we show what has been said:

(2.6) $(b+d)^2 - b^2 = b^2 + 2bd + d^2 - b^2$.

The left side will be denoted by the letter A,

(2.7) $A = b^2 + 2bd + d^2 - b^2$, or $A = 2bd + d^2$.

If we take $d = 1$, then we have

(2.8) $A = 2b + 1$.

From formula (2.8) it follows that for the interval $d = 1$, the difference of the squares of neighboring natural numbers forms the set of all odd numbers. If the interval is $d = 2$, then the difference of adjacent squares forms a set, the elements of which are the majority of even numbers greater than 2, since the difference formula will have the form

(2.9) $A = 4b + 4$.

From what has been said it follows that the difference of the squares of some neighboring natural numbers can be equal to the square of the natural number, although the difference of the squares will not correspond to the decomposition by the binomial formula.

Based on the regularity of the difference of equal powers of two natural numbers, we formulate the following theorem.

Theorem 2.2. The natural power of a natural number a^n cannot be represented as an incomplete binomial expansion in integers for $a > 1,\ n > 2$.

Theorem 2.2 together with Theorem 2.1 means that $c^n - b^n \neq a^n$, that is, the difference of two natural numbers having equal natural degrees cannot be equal to the natural power of a natural number.

3 A SIMPLE PROOF OF FLT FOR n = 3

3.1 Patterns of cubes of natural numbers

It follows from the FLT that if it is not true, then the following equality has a natural solution,

(3.1) $c^3 - b^3 = a^3$.

Since the cube of a natural number can be represented as $c^3 = (b + d)^3$, which has the following binomial expansion

(3.2) $(b + d)^3 = b^3 + 3b^2d + 3bd^2 + d^3$,

we represent formula (3.1) in the form

$b^3 + 3b^2d + 3bd^2 + d^3 - b^3 = a^3$, or

(3.3) $a^3 = 3b^2d + 3bd^2 + d^3$.

Formula (3.3) means that if the FLT is incorrect, then the cube of a natural number can be represented as an incomplete binomial expansion.

Further, given that the number a can be represented as the sum of two natural numbers, its cube can be represented as $a^3 = (s + d)^3$. After that, we perform the binomial expansion of the number $(s + d)^3$,

(3.4) $(s + d)^3 = s^3 + 3s^2d + 3sd^2 + d^3$.

Note that formula (3.3) is equivalent to removing from formula (3.4) the first term of the binomial expansion s^3, then increasing the numbers s on the right-hand side of formula (3.4) by x to compensate for s^3, i.e. formula (3.3) is equivalent to the following formula

(3.5) $a^3 = 3(s + x)^2d + 3(s+x)d^2 + d^3$.

From formula (3.5) it follows that when the cube of a natural number is represented as an incomplete binomial, the removed term of the binomial s^3 is distributed to other terms of the binomial using the number x.

Example 1. Given $8^3 = 512$ represented as $8^3 = (6 + 2)^3$, it is required to represent 8^3 as an incomplete binomial. Using formula (3.5), we write $8^3 = 3(6 + x)^2 2 + 3(6+x)2^2 + 2^3$.

Next, by the selection method, we find x, $x \approx 2{,}2196$, after that, using the incomplete binomial formula, we calculate 8^3,

$8^3 = 3(6 + 2{,}2196)^2 2 + 3(6+2{,}2196)2^2 + 2^3 = 405{,}3709 + 98{,}6352 + 8 \approx 512{,}006$.

Then, based on formulas (3.3) and (3.4), taking into account $a^3 = (s + d)^3$, we obtain the following equality

$s^3 + 3s^2d + 3sd^2 + d^3 = 3b^2d + 3bd^2 + d^3$ or

(3.6) $s^3 = 3b^2d + 3bd^2 + d^3 - (3s^2d + 3sd^2 + d^3)$.

Formula (3.6) means that if the FLT is incorrect, then the difference between the incomplete binomial decompositions of the cubes of two natural numbers will be equal to the cube of the natural number included in the subtracted incomplete binomial (s^3).

Thus, we will further prove that formulas (3.5) and (3.6) have no natural solution. If natural numbers are used in formula (3.6), then we obtain the inequality

(3.7) $s^3 \neq 3b^2d + 3bd^2 + d^3 - (3s^2d + 3sd^2 + d^3)$.

It should be noted that if we consider cubes of arbitrary natural numbers, then the number of cubes of natural numbers, which are the difference of incomplete binomials of cubes of two natural numbers, is infinitely large.

Example 2. The difference of incomplete binomial expansions of the numbers $(36 + 1)^3$ and $(35 + 1)^3$ will be equal to the cube 6,

$((36 + 1)^3 - 36^3) - ((35 + 1)^3 - 35^3) = 3997 - 3781 = 216 = 6^3$.

To prove the FLT based on formula (3.6), we must consider the positive difference of any natural numbers located at a distance l ($l = 1,2, \dots$), from each other, which are equal to the incomplete binomial decomposition of the cube of natural numbers. To do this, we compose a set of incomplete binomial decompositions of cubes of natural numbers differing by 1, i.e. elements of this set are calculated by the formula $q = 3s^2d + 3sd^2 + d^3, d = 1, s = 1, 2, \dots$

(3.8) $Q = \{q | q = (s + 1)^3 - s^3;\ s \in \mathbb{N}\}$.

Further, on the basis of the first set, we compose the second set of natural numbers, each element of which corresponds to the positive difference of any two elements of the first set and is calculated by the formula

(3.9) $G = \{g | g = [(s + l + 1)^3 - (s + l)^3] - [(s + 1)^3 - s^3];\ l \in \mathbb{N}\}$.

Note that the expression $g = [(s + l + 1)^3 - (s + l)^3] - [(s + 1)^3 - s^3]$ corresponds to the formula $g = 3d(s + l)^2 + 3d^2(s + l) - (3ds^2 + 3d^2s),\ d = 1, s = 1, 2, \dots, l = 1, 2, \dots$.

The second set includes the positive differences of all incomplete binomial decompositions of cubes of natural numbers. To show the relationship of cubes and their incomplete binomials, as well as the difference of incomplete binomials, Tables 1 and 2 are compiled.

An example of elements of the set Q is the numbers shown in column 4 of Table 1, and an example of elements of the set G is the numbers shown in columns 5, 6, 7 of the same table.

Note that the incomplete binomial decomposition is also the difference of two natural numbers to the power of n located at a distance d from each other, so they are equal to the difference of the cubes of natural numbers differing by the number d.

Note: In Tables 1 and 2 Incomplete binomial expansions, denoted by the letters SDi, correspond to elements of the set Q, and the difference of incomplete binomials, denoted by $SDj - SDi$, correspond to elements of the set G.

For convenience, we represent formula (3.6) in the following compact form

(3.9) $s^3 = BD - SD,$

where BD is an incomplete binomial decomposition of the number $(b + d)^3$; $SD -$ incomplete binomial decomposition of the number $(s + d)^3$.

Note that the differences of incomplete binomial expansions located at different distances from each other can be calculated in a different way: 1) First, incomplete binomial expansions of cubes of natural numbers located at different distances from each other are calculated, i.e. first, incomplete binomial decompositions of cubes of natural numbers are calculated for different d; 2) Then the differences of neighboring incomplete binomial decompositions of cubes of natural numbers are calculated for each d.

Table 1. Difference of incomplete binomial expansions for $d = 1$

$\boldsymbol{a = s + d}$	$\boldsymbol{a^3}$	$\boldsymbol{s^3}$	$\boldsymbol{SD}$	$\boldsymbol{SDj - SDi}$		
				$\boldsymbol{l = 1}$	$\boldsymbol{l = 2}$	$\boldsymbol{l = 3}$
1	2	3	4	5	6	7
1=0+1	1	0	1			
				6		
2=1+1	8	1	7		18	
				12		36
3=2+1	27	8	19		30	
				18		54
4=3+1	64	27	37		42	
				24		72
5=4+1	125	64	61		54	
				30		90
6=5+1	216	125	91		66	
				36		108
7=6+1	343	216	127		78	
				42		126
8=7+1	512	343	169		90	
				48		144
9=8+1	729	512	217		102	
				54		
10=9+1	1000	729	271			

The numbers shown in column 5 of Table 1 are part of the elements of the set G, which are equal to the difference of neighboring incomplete binomials calculated for $d = 1$ and $l = 1$, and the numbers given on columns 6 and 7, which are equal to the difference of incomplete binomials for $l = 2$ and $l = 3$, respectively, are elements of subsets of the set G. An example of the second method calculating the difference of adjacent incomplete binomial decompositions of cubes of natural numbers for each d is Table 2.

The numbers shown in column 5 of Table 2 are part of the elements of the set G, which are equal to the difference of neighboring incomplete binomials calculated for $d = 1$ and $l = 1$, and the numbers given on columns 7 and 9, which are equal to the difference of incomplete binomials for $d = 2$ and $d = 3$, respectively, are elements of subsets of the set G.

Based on the above, it can be argued that considering the difference of incomplete binomial expansions for $d = 1$ and $l = 1$ is equivalent to considering the difference of incomplete binomial expansions for any natural d and l.

Table 2. Difference of incomplete binomials for different d for $l = 1$

$\boldsymbol{a = s + d}$	$\boldsymbol{a^3}$ $\boldsymbol{d = 1}$	$\boldsymbol{s^3}$	$\boldsymbol{SD,}$ $\boldsymbol{d = 1}$	$\boldsymbol{SDj - SDi}$	$\boldsymbol{SD,}$ $\boldsymbol{d = 2}$	$\boldsymbol{SDj - SDi}$	$\boldsymbol{SD,}$ $\boldsymbol{d = 3}$	$\boldsymbol{SDj - SDi}$
1	2	3	4	5	6	7	8	9
a=0+d	1	0	1					
				6				
a=1+d	8	1	7		26		63	
				12		30		54
a=2+d	27	8	19		56		117	
				18		42		72
a=3+d	64	27	37		98		189	
				24		54		90
a=4+d	125	64	61		152		279	
				30		66		108
a=5+d	216	125	91		218		387	
				36		78		126
a=6+d	343	216	127		296		513	
				42		90		144
a=7+d	512	343	169		386		657	
				48		102		
a=8+d	729	512	217		488			
				54				
a=9+d	1000	729	271					

As follows from Tables 1 and 2, in two ways, equal values of the difference of incomplete binomials are obtained, i.e. the same result. Taking into account the above, we give a proof of the FLT for a cube with $d = 1$, $b = s + l$, $l \in \mathbb{N}$.

3.2 Simple proof of the FLT for n = 3

Further, for $n = 3$, we prove that the Diophantine equation $a^3 = c^3 - b^3$, which has the representation $(s + d)^3 = (b + d)^3 - b^3$, has no natural solution. Note that the proof of the FLT for $n = 3$ means that the cube of a natural number cannot be represented as an incomplete binomial expansion using only natural numbers.

Formula (3.5) can be represented as

(3.10) $s^3 = 3d(s + l)^2 + 3d^2(s + l) - (3ds^2 + 3d^2s)$,

where $b = s + l$; l is a natural number.

Considering that $(s + l)^2 = s^2 + 2 \cdot s \cdot l + l^2$, we simplify the formula after the following transformations:

$s^3 = 3d(s^2 + 2 \cdot s \cdot l + l^2) + 3d^2(s + l) - (3ds^2 + 3d^2s)$;

$s^3 = 3ds^2 + 6dsl + 3dl^2 + 3d^2s + 3d^2l - 3ds^2 - 3d^2s$;

$s^3 = 6dsl + 3dl^2 + 3d^2l$;

(3.11) $s^3 = 6dl\left(s + \frac{l}{2} + \frac{d}{2}\right)$.

For $d = 1$, we obtain the equality

(3.12) $s = \frac{s^3}{6l} - \frac{l+1}{2}$.

By the condition of the problem, all the numbers of the formula must be integers, and the ratio $\frac{s^3}{6l}$ for natural s, $d = 1$ and l will be an integer if $s = 6lk$, therefore, based on formula (3.12), we can write $6lk = \frac{(6lk)^3}{6l} - \frac{l+1}{2}$. This implies $6lk = (6l)^2k^3 - \frac{l+1}{2}$, or

(3.13) $k = 6lk^3 - \frac{l+1}{2 \cdot 6l}$.

Obviously, this equality does not have an integer solution, since for any natural number k we obtain the following inequality,

(3.14) $k < 6lk^3 - \frac{l+1}{12l}$.

This inequality is valid for any natural numbers k and l.

Since we are using the first method, where $d = 1$ and $l = 1$, we can write

$$k < 6k^3 - \frac{1}{6}.$$

Note that the ratio $\frac{s^3}{6l}$ for natural $s, d = 1$ and l will also be an integer if $s^3 = 6lk$. In this case, there should be $lk = 6^2$, this is possible in the following three cases: 1) $k = 1$, $l = 6^2$; 2)$k = 6^2$, $l = 1$; 3)$k = 6$, $l = 6$. For these three cases, using formula (3.12), we obtain

$$s = \frac{s^3}{6l} - \frac{l+1}{2},$$

1) $6 \neq \frac{6^3}{6^3} - \frac{6+1}{2};\ 6 \neq 1 - \frac{7}{2};$ 2) $6 \neq \frac{6^3}{6} - \frac{1+1}{2};\ 6 \neq 6^2 - 1;$ 3) $6 \neq \frac{6^3}{6^2} - \frac{6+1}{2};\ 6 \neq 6 - \frac{7}{2}$.

As can be seen, in all three cases we have obtained an inequality, which is a proof of the validity of inequality (3.7).

If $lk \neq 6^2$, while $s^3 = 6lk$, then it should be $s^3 = (6t)^3$ and $s = 6t$, in this case $lk = 6^2t^3$, as it should be $6lk = (6t)^3$.

It follows that $l = 6^2$ and $k = t^3$ (or $k = 6^2$ and $l = t^3$). For $l = 6^2$ and $s = 6t$ by formula (3.12) we obtain

$6t = \frac{(6t)^3}{6^3} - \frac{6^2+1}{2}$ or $6t = t^3 - \frac{37}{2}$.

From here we find t,

(3.15) $t = \frac{t^3}{6} - \frac{37}{12}$.

Obviously, equation (3.15) has no natural solution.

Thus, we have proved that formula (3.6) has no natural solution, which means that an elementary proof of the PTF for n = 3 has been obtained.

CONCLUSION

The difference of two natural numbers having equal natural degrees certainly has a representation in the form of an incomplete binomial decomposition, therefore, if the FLT is incorrect, then the natural power of a natural number must be represented as an incomplete binomial in integers. It is proved that the square of a natural number can be represented as an incomplete binomial in integers, and the cube of a natural number cannot be represented as an incomplete binomial. This is due to the fact that when the square of a natural number is represented as an incomplete binomial, the removed term (the first term of the binomial is the square of a natural number) is added to only one member of the incomplete binomial having degree 1, and when the cube of a natural number is represented as an incomplete binomial, the removed term (the first term of the binomial is the cube of a natural number) is distributed to several members of the incomplete binomial, with different powers greater than 1.

With an increase in the exponent n, the number of terms of the incomplete binomial expansion with different degrees, on which the distant binomial term will be distributed (the first binomial term is a natural number to the power $n > 3$), also increases strongly. It follows that if the cube of a natural number cannot be distributed into terms of a polynomial in the form of integers, then it is all the more impossible to distribute the degree of a natural number for $n > 3$.

Thus, we can assert that the proof of the FLT for $n = 3$ follows from the proof of the FLT as a whole.

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