Some remarks concerning the factorization of mirror composite numbers and its relationship with Goldbach conjecture.

Óscar E. Chamizo Sánchez. Redonda Kingdom University, Faculty of Sciences, Department of Mathematics.

Abstract:

In this paper we present the concept of mirror composite numbers. Mirror composite numbers are composite numbers of the form $2n-p$ for some $n$ positive natural number and $p$ prime. We shall show that the factorization of these numbers have interesting properties in order to face the Goldbach conjecture by the divide et impera method.

Definitions:

From now on, $m$ and $n$ are positive natural numbers, $p$ and $q$ are prime numbers.

All prime numbers $p>=5$ are of the form $6m+1$ or $6m-1$. A prime of the form $6m+1$ is a right prime; a prime of the form $6m-1$ is a left prime.

A mirror composite number is a composite number of the form $2n-p$ for some $n$ and some prime $p>=5$.

Given a mirror composite $2n-p$, if $p=6m+1$, i.e., if $p$ is a right prime, $2n-p$ is a right mirror composite (r.m.c.).

Given a mirror composite $2n-p$, If $p=6m-1$, i.e., if $p$ is a left prime, $2n-p$ is a left mirror composite (l.m.c.).

Lemma 1.

Fixed $n$, if $3$ is a factor of some l.m.c (respectively r.m.c.), $3$ is a factor of every l.m.c. (r.m.c) and $3$ is not a factor of any r.m.c. (l.m.c)

Proof:

The difference between two l.m.c. (r.m.c.) is $6n$. If $3|m$, $3|m±6n$. On the other hand, if $3|2n-(6m+1)$, then $3∤2n-(6m+1)$ and viceversa.

Lemma 2.

Fixed $n$, if $q≠3$ is a prime factor of two different l.m.c. (respectively r.m.c.), the difference between them is a multiple of $6q$ so the minimum gap between two consecutive occurrences of factor $q$ is $6q$ for all l.m.c. (r.m.c.)

Proof:

If $q | 2n-(6x-1)$ and $q | 2n-(6y-1)$ exists $z$ such that $zq=6(x-y)$, so $z$ is multiple of 6, given that $q$ is a prime and $q ≠ 2,3$.

If $q | 2n-(6x+1)$ and $q | 2n-(6y+1)$ exists $z$ such that $zq=6(x-y)$, so $z$ is multiple of 6, given that $q$ is a prime and $q ≠ 2,3$. 
Goldbach conjecture states that for all $n$ and all $p$ such that $3 \leq p \leq 2n-3$, some $2n-p$ is a prime, i.e., not every $2n-p$ is composite.

Let's suppose for the sake of contradiction that exists $n$ such that every $2n-p$ is composite. Then, 3 consecutive odd numbers, $2n-3$, $2n-5$ and $2n-7$ are composite, so one and only one of them must be multiple of 3.

**Case A: $3 \mid 2n-7$:**

$3 \mid 2n-7 \Rightarrow 3 \mid 2n-(6m+1)$ for all $m$ (Lemma 1). So every right mirror composite is a multiple of 3 and no left mirror composite is a multiple of 3. So all element of the sequence:

$$2n-3, 2n-5, 2n-11, 2n-17, 2n-23, \ldots 2n-q,$$

where $q$ is a left prime $5 \leq q \leq 2n-3$, must be factorized. There are $i$ consecutive primes $p_i$ from $p_1=5$ to $p_k$, where $p_k$ is the largest prime less than $\sqrt{2n}$, available for that factorization.

Now, given the correlative sequence of odd numbers $2n-3$, $2n-5$, $2n-7$, $2n-9$, $2n-11$, $2n-13$, $2n-15$, $2n-a\ldots$, let be $2n-a_i$ the number containing the first occurrence of prime factor $p_i$ in that sequence. Notice that:

- For each $p_i$, $a_i$ is unique.
- $3 \leq a_i \leq 2p_i+1$.
- For some $i$, $a_i=3$; for some $i$, $a_i=5$; for some $i$, $a_i=11 \ mod p_i$; for some $i$, $a_i=17 \ mod p_i$; for some $i$, $a_i=23 \ mod p_i$ and so on.
- $2n-q$, i.e., $2n-(6m-1)$, is composite if and only if exists $i$ such that $6m-1 \equiv a_i \ mod p_i$ (Lemma 2).

Now, let's state conditions in order to find some $2n-q$ with $q=6m-1$ and $q$ inside the interval $\sqrt{2n} < q \leq 2n-3$ that can not be factorized:

1) $q$ is a prime, i.e., $q$ is not multiple of any $p_i$, so $6m-1 \neq 0 \ mod p_i$ for all $i$.

2) There is no $p_i$ factor available for $2n-q$, so $6m-1 \neq a_i \ mod p_i$ for all $i$.

<table>
<thead>
<tr>
<th>Prime condition for $6m-1$</th>
<th>No factor available condition for $2n-(6m-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6m \not\equiv 1 \ mod 5$</td>
<td>$6m \not\equiv (a_1+1) \ mod 5$</td>
</tr>
<tr>
<td>$6m \not\equiv 1 \ mod 7$</td>
<td>$6m \not\equiv (a_2+1) \ mod 7$</td>
</tr>
<tr>
<td>$6m \not\equiv 1 \ mod 11$</td>
<td>$6m \not\equiv (a_3+1) \ mod 11$</td>
</tr>
</tbody>
</table>
6m \not\equiv 1 \mod 13 \quad 6m \not\equiv (a_k+1) \mod 13
\ldots \ldots \ldots \ldots \ldots
6m \not\equiv 1 \mod p_k \quad 6m \not\equiv (a_k+1) \mod p_k

Hence for each $p_i$, there are at least $p_i - 2$ remainders moduli $p_i$ that fullfill the conditions. That amounts up to a minimum of 3.5.9.11...(p_k-2) different systems of linear congruences whith prime moduli, each one of them has a different and unique solution, not every one outside the aforementioned interval.

For now, it will be enough to notice that at least $p_i - 2$ remainders fullfill the conditions for each $p_i$ to conclude (Pigeonhole strong form principle) that at least exists some (in fact, a lot of) 6m that fullfills the conditions for all $p_i$. Hence, exists some 2n-q that can not be factorized, so 2n-q is prime and the conjecture holds for all 2n such that $3|2^{n}-7$, i.e., for all $2n \equiv 1 \mod 3$.

**Case B: $3|2^{n-5}$:**

\[3|2^{n-5} \Rightarrow 3|2^{n-(6m-1)} \text{ for all } m \text{ (Lemma 1).} \]

So every left mirror composite is a multiple of 3 and no right mirror composite is a multiple of 3…

Following the same thought process than before, with q a right prime of the form 6m+1, it’s straightforward to conclude that the conjecture holds for all $2n$ such that $3|2^{n-5}$, i.e., for all $2n \equiv 2 \mod 3$.

**Case C: $3|2^{n-3}$:**

Matter of forward research.

Oscar E. Chamizo Sánchez. latinrodrigo@gmail.com
PA3.

References:
Christian Goldbach, *Letter to L. Euler, June 7 (1742)*.