Infinite Continued Fractions

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Abstract: Transcendental numbers expressed by infinite continued fractions.

Identity 0.1.

$$\frac{\sqrt{2}\pi^2}{16} = \cfrac{1}{1 + \cfrac{1^4}{8 - \cfrac{3^4}{34 + \cfrac{5^4}{24 - \cfrac{7^4}{130 + \cfrac{9^4}{40 - \cfrac{11^4}{\cdots}}}}}}}$$

Identity 0.2.

$$\sqrt{2} = 2 - \cfrac{1}{1 + \cfrac{1}{2 - \cfrac{1}{1 + \cfrac{2}{4 - \cfrac{2}{1 + \cfrac{3}{6 - \cfrac{3}{\cdots}}}}}}}$$

Identity 0.3.

$$\arctan\left(\frac{1}{2}\right) = \cfrac{1}{1 + \cfrac{1}{1 - \cfrac{1}{5 + \cfrac{2}{1 - \cfrac{2}{9 + \cfrac{3}{1 - \cfrac{3}{\cdots}}}}}}}$$

Identity 0.4.

$$\frac{1}{\ln(2)} = 2 - \cfrac{1}{1 + \cfrac{1 \cdot 1}{2 - \cfrac{1 \cdot 2}{1 + \cfrac{2 \cdot 2}{4 - \cfrac{2 \cdot 3}{1 + \cfrac{3 \cdot 3}{6 - \cfrac{3 \cdot 4}{\cdots}}}}}}}$$

Identity 0.5.

$$\zeta(2) = \frac{5}{4} + \cfrac{1}{2 + \cfrac{1^2}{2 - \cfrac{1^2}{6 + \cfrac{2^2}{2 - \cfrac{2^2}{10 + \cfrac{3^2}{2 - \cfrac{3^2}{\cdots}}}}}}}$$

Identity 0.6.

$$\ln(16) = \frac{7}{3} + \cfrac{1}{2 + \cfrac{1^2}{4 - \cfrac{1^2}{2 + \cfrac{2^2}{8 - \cfrac{2^2}{2 + \cfrac{3^2}{12 - \cfrac{3^2}{\cdots}}}}}}}$$

Identity 0.7.

$$\frac{24 \ln(2) - 17}{6 \ln(2) - 5} = \cfrac{1}{2 + \cfrac{1 \cdot 1}{4 - \cfrac{1 \cdot 2}{2 + \cfrac{2 \cdot 2}{8 - \cfrac{2 \cdot 3}{2 + \cfrac{3 \cdot 3}{12 - \cfrac{3 \cdot 4}{\cdots}}}}}}}$$

2

Identity 0.8.

$$\frac{3-4\ln(2)}{1-\ln(2)} = \cfrac{1}{1+\cfrac{1^2\cdot 1}{2^2-\cfrac{1^2\cdot 2}{1+\cfrac{2^2\cdot 2}{4^2-\cfrac{2^2\cdot 3}{1+\cfrac{3^2\cdot 3}{6^2-\cfrac{3^2\cdot 4}{\ldots}}}}}}}$$

Identity 0.9.

$$\frac{6-8\ln(2)}{2\ln(2)-1} = \cfrac{1}{1-\cfrac{1^2\cdot 1}{2^2+\cfrac{1^2\cdot 2}{1-\cfrac{2^2\cdot 2}{4^2+\cfrac{2^2\cdot 3}{1-\cfrac{3^2\cdot 3}{6^2+\cfrac{3^2\cdot 4}{\ldots}}}}}}}$$

Identity 0.10.

$$\frac{34-48\ln(2)}{12\ln(2)-7} = \cfrac{1}{2-\cfrac{1\cdot 1}{4+\cfrac{1\cdot 2}{2-\cfrac{2\cdot 2}{8+\cfrac{2\cdot 3}{2-\cfrac{3\cdot 3}{12+\cfrac{3\cdot 4}{\ldots}}}}}}}$$

Identity 0.11.

$$\frac{10-12\ln(2)}{3} = \cfrac{1}{2-\cfrac{1^2}{4+\cfrac{1^2}{2-\cfrac{2^2}{8+\cfrac{2^2}{2-\cfrac{3^2}{12+\cfrac{3^2}{\ldots}}}}}}}$$

Identity 0.12.

$$\frac{1 - 2\ln(2)}{\ln(2) - 1} = \cfrac{1}{1 - \cfrac{1 \cdot 1}{2 + \cfrac{1 \cdot 2}{1 - \cfrac{2 \cdot 2}{4 + \cfrac{2 \cdot 3}{1 - \cfrac{3 \cdot 3}{6 + \cfrac{3 \cdot 4}{\cdots}}}}}}}$$

Identity 0.13.

$$\zeta(2) = 1 + \cfrac{1}{2 - \cfrac{1^2}{2 + \cfrac{1^2}{6 - \cfrac{2^2}{2 + \cfrac{2^2}{10 - \cfrac{3^2}{2 + \cfrac{3^2}{\cdots}}}}}}}$$

Identity 0.14.

$$\frac{4}{\pi} = 2 - \cfrac{1}{1 + \cfrac{1 \cdot 1}{4 - \cfrac{1 \cdot 3}{1 + \cfrac{2 \cdot 3}{8 - \cfrac{2 \cdot 5}{1 + \cfrac{3 \cdot 5}{12 - \cfrac{3 \cdot 7}{\cdots}}}}}}}$$

Identity 0.15.

$$\ln(2) = 1 - \cfrac{1}{2 + \cfrac{1 \cdot 1}{1 - \cfrac{1 \cdot 3}{6 + \cfrac{2 \cdot 3}{1 - \cfrac{2 \cdot 5}{10 + \cfrac{3 \cdot 5}{1 - \cfrac{3 \cdot 7}{\cdots}}}}}}}$$

Identity 0.16.

$$\frac{3\zeta(2)}{2} = 2 + \cfrac{1}{1^2 + \cfrac{1^2 \cdot 1}{1 - \cfrac{1^2 \cdot 3}{3^2 + \cfrac{2^2 \cdot 3}{1 - \cfrac{2^2 \cdot 5}{5^2 + \cfrac{3^2 \cdot 5}{1 - \cfrac{3^2 \cdot 7}{\dots}}}}}}}$$

Identity 0.17.

$$\frac{\zeta(2)}{2} = \frac{5}{8} + \cfrac{1}{2^2 + \cfrac{1^2 \cdot 1}{1 - \cfrac{1^2 \cdot 3}{6^2 + \cfrac{2^2 \cdot 3}{1 - \cfrac{2^2 \cdot 5}{10^2 + \cfrac{3^2 \cdot 5}{1 - \cfrac{3^2 \cdot 7}{\dots}}}}}}}$$

Identity 0.18.

$$\frac{9\pi}{4} = 8 - \cfrac{1}{1 + \cfrac{1^2 \cdot 1}{4^2 - \cfrac{1^2 \cdot 3}{1 + \cfrac{2^2 \cdot 3}{8^2 - \cfrac{2^2 \cdot 5}{1 + \cfrac{3^2 \cdot 5}{12^2 - \cfrac{3^2 \cdot 7}{\dots}}}}}}}$$

Identity 0.19.

$$\frac{256}{9\pi} = 8 + \cfrac{1}{1 - \cfrac{1^2 \cdot 1}{4^2 + \cfrac{1^2 \cdot 3}{1 - \cfrac{2^2 \cdot 3}{8^2 + \cfrac{2^2 \cdot 5}{1 - \cfrac{3^2 \cdot 5}{12^2 + \cfrac{3^2 \cdot 7}{\dots}}}}}}}$$

Identity 0.20.

$$\frac{60\ln(2)-37}{5}=\cfrac{1}{1+\cfrac{1^3}{3\cdot 2^2-\cfrac{1^3}{1+\cfrac{2^3}{3\cdot 4^2-\cfrac{2^3}{1+\cfrac{3^3}{3\cdot 6^2-\cfrac{3^3}{\cdots}}}}}}}$$

Identity 0.21.

$$6\ln(2)=5-\cfrac{1}{1+\cfrac{1^2}{6-\cfrac{1^2}{1+\cfrac{2^2}{12-\cfrac{2^2}{1+\cfrac{3^2}{18-\cfrac{3^2}{\cdots}}}}}}}$$

Identity 0.22.

$$\frac{3\zeta(2)}{2}=\frac{20}{9}+\cfrac{1}{3+\cfrac{1^2}{1-\cfrac{1^2}{9+\cfrac{2^2}{1-\cfrac{2^2}{15+\cfrac{3^2}{1-\cfrac{3^2}{\cdots}}}}}}}$$

Identity 0.23.

$$\frac{720\ln(2)-504}{60\ln(2)-47}=\cfrac{1}{1+\cfrac{1^2\cdot 1}{3\cdot 2^2-\cfrac{1^2\cdot 2}{1+\cfrac{2^2\cdot 2}{3\cdot 4^2-\cfrac{2^2\cdot 3}{1+\cfrac{3^2\cdot 3}{3\cdot 6^2-\cfrac{3^2\cdot 4}{\cdots}}}}}}}$$

Identity 0.24.

$$\frac{24\ln(2)-14}{3} = \cfrac{1}{1+\cfrac{1^4}{2^3-\cfrac{1^4}{1+\cfrac{2^4}{4^3-\cfrac{2^4}{1+\cfrac{3^4}{6^3-\cfrac{3^4}{\cdots}}}}}}}$$

Identity 0.25.

$$\frac{24\ln(2)-14}{3} = \cfrac{1}{1+\cfrac{1^3}{2\cdot 2^2-\cfrac{1^3}{1+\cfrac{2^3}{2\cdot 4^2-\cfrac{2^3}{1+\cfrac{3^3}{2\cdot 6^2-\cfrac{3^3}{\cdots}}}}}}}$$

Identity 0.26.

$$\frac{1680\ln(2)-1066}{105} = \cfrac{1}{1+\cfrac{1^5}{2^4-\cfrac{1^5}{1+\cfrac{2^5}{4^4-\cfrac{2^5}{1+\cfrac{3^5}{6^4-\cfrac{3^5}{\cdots}}}}}}}$$

Identity 0.27.

$$\frac{1680\ln(2)-1066}{105} = \cfrac{1}{1+\cfrac{1^4}{2\cdot 2^3-\cfrac{1^4}{1+\cfrac{2^4}{2\cdot 4^3-\cfrac{2^4}{1+\cfrac{3^4}{2\cdot 6^3-\cfrac{3^4}{\cdots}}}}}}}$$

Identity 0.28.

$$\frac{504 - 720\ln(2)}{60\ln(2) - 37} = \cfrac{1}{1 - \cfrac{1^2 \cdot 1}{3 \cdot 2^2 + \cfrac{1^2 \cdot 2}{1 - \cfrac{2^2 \cdot 2}{3 \cdot 4^2 + \cfrac{2^2 \cdot 3}{1 - \cfrac{3^2 \cdot 3}{3 \cdot 6^2 + \cfrac{3^2 \cdot 4}{\cdots}}}}}}}$$

Identity 0.29.

$$\frac{9\pi - 28}{3\pi - 9} = \cfrac{1}{1 + \cfrac{1^2 \cdot 3}{2 \cdot 2^2 - \cfrac{1^2 \cdot 5}{1 + \cfrac{2^2 \cdot 5}{2 \cdot 4^2 - \cfrac{2^2 \cdot 7}{1 + \cfrac{3^2 \cdot 7}{2 \cdot 6^2 - \cfrac{3^2 \cdot 9}{\cdots}}}}}}}$$

Identity 0.30.

$$\pi = 4 - \cfrac{1}{1 + \cfrac{1^2 \cdot 1}{2 \cdot 2^2 - \cfrac{1^2 \cdot 3}{1 + \cfrac{2^2 \cdot 3}{2 \cdot 4^2 - \cfrac{2^2 \cdot 5}{1 + \cfrac{3^2 \cdot 5}{2 \cdot 6^2 - \cfrac{3^2 \cdot 7}{\cdots}}}}}}}$$

Identity 0.31.

$$\zeta(2) = \cfrac{2}{2 \cdot 1^2 - \cfrac{1^2 \cdot 1}{1 + \cfrac{1^2 \cdot 3}{2 \cdot 3^2 - \cfrac{2^2 \cdot 3}{1 + \cfrac{2^2 \cdot 5}{2 \cdot 5^2 - \cfrac{3^2 \cdot 5}{1 + \cfrac{3^2 \cdot 7}{\cdots}}}}}}}$$

Identity 0.32.

$$\frac{1276 - 1680\ln(2)}{105} = \cfrac{1}{1 - \cfrac{1^4}{2 \cdot 2^3 + \cfrac{1^4}{1 - \cfrac{2^4}{2 \cdot 4^3 + \cfrac{2^4}{1 - \cfrac{3^4}{2 \cdot 6^3 + \cfrac{3^4}{\dots}}}}}}}$$

Identity 0.33.

$$\frac{68 - 96\ln(2)}{12\ln(2) - 7} = \cfrac{1}{1 - \cfrac{1^2 \cdot 1}{2 \cdot 2^2 + \cfrac{1^2 \cdot 2}{1 - \cfrac{2^2 \cdot 2}{2 \cdot 4^2 + \cfrac{2^2 \cdot 3}{1 - \cfrac{3^2 \cdot 3}{2 \cdot 6^2 + \cfrac{3^2 \cdot 4}{\dots}}}}}}}$$