ENERGY CONSERVATION LAW FOR THE GRAVITATIONAL FIELD IN A 3D-BRANE UNIVERSE.

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Abstract. The 3D-brane universe model is an alternative theory of gravity. It was introduced using the concept of temporal coexistence. In the present paper the energy conservation law for the gravitational field and matter in the 3D-brane universe model is derived.

1. Introduction.

The concept of temporal coexistence was suggested in [1] (see also [2] and [3]). It applies to the events of the four-dimensional spacetime and subdivides them into those that had happened in the past, those happening right now, and forthcoming events that did not yet happen. This subdivision is global and it is absolute, i. e. it is irrespective to the choice of a coordinate frame and to the instrumentation used for measuring time. The events happening right now are called currently coexisting. They constitute a 3D-brane representing the current state of a universe in question.

Within the paradigm of a 3D-brane universe the gravitational field is described by a time-dependent 3D metric with the components

$$g_{ij} = g_{ij}(x^0, x^1, x^2, x^3), \quad 1 \leqslant i, j \leqslant 3, \tag{1.1}$$

where $x^0 = c\,t$ and c is the speed of light. We relate this 3D metric (1.1) to the four-dimensional metric of the standard 4D paradigm through the formulas

$$G_{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -g_{11} & -g_{12} & -g_{13} \\ 0 & -g_{21} & -g_{22} & -g_{23} \\ 0 & -g_{31} & -g_{32} & -g_{33} \end{Vmatrix}, \qquad G^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -g^{11} & -g^{12} & -g^{13} \\ 0 & -g^{21} & -g^{22} & -g^{23} \\ 0 & -g^{31} & -g^{32} & -g^{33} \end{Vmatrix}. \tag{1.2}$$

If the formulas (1.2) are fulfilled, then in the standard paradigm the time variable t in $x^0 = c\,t$ is interpreted as the cosmological time (see [4]), while x^1, x^2, x^3 are interpreted as comoving spatial coordinates (see [5]).

In the standard four-dimensional paradigm of general relativity and cosmology the four-dimensional metric should obey the standard Einstein's equation

$$r_{ij} - \frac{r}{2}\,G_{ij} - \Lambda\,G_{ij} = \frac{8\,\pi\,\gamma}{c^4}\,T_{ij}, \tag{1.3}$$

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see § 2 in Chapter V of [6]. Here c is the speed of light, γ is Newton's gravitational constant (see [7]), and Λ is the cosmological constant (see [8]). The quantities T_{ij} in the right hand side of (1.3) are the components of the energy-momentum tensor (see [9]). The term r_{ij} in (1.3) corresponds to the components of the four-dimensional Ricci tensor and r is the four-dimensional scalar curvature (see § 8 in Chapter IV of [10]). By substituting (1.2) into (1.3) in [1] the following equations were derived:

$$\begin{gathered}\frac{\partial b_{ij}}{\partial x^0}-\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}\,g_{ij}-\sum_{k=1}^{3}(b_{ki}\,b^k_j+b_{kj}\,b^k_i)-\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\\ -\frac{g_{ij}}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\sum_{k=1}^{3}b^k_k\,b_{ij}+R_{ij}-\frac{R}{2}\,g_{ij}+\Lambda\,g_{ij}=\frac{8\,\pi\,\gamma}{c^4}\,T_{ij},\end{gathered}\tag{1.4}$$

$$\sum_{k=1}^{3}\nabla_k\,b^k_j-\sum_{k=1}^{3}\nabla_j\,b^k_k=\frac{8\,\pi\,\gamma}{c^4}\,T_{0j},\tag{1.5}$$

$$-\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k+\frac{1}{2}\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q+\frac{R}{2}-\Lambda=\frac{8\,\pi\,\gamma}{c^4}\,T_{00}.\tag{1.6}$$

Here R_{ij} are the components of the three-dimensional Ricci tensor, R is the three-dimensional scalar curvature, and b_{ij} are given by the formula

$$b_{ij}=\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^0}=\frac{\dot g_{ij}}{2\,c}.\tag{1.7}$$

In [11] the Lagrangian approach to the 3D-brane universe model was applied. The action integral was taken as a sum of two integrals:

$$S=\iint\mathcal{L}_{\text{gr}}\,\sqrt{\det g}\;d^3x\,dx^0+\iint\mathcal{L}_{\text{mat}}\,\sqrt{\det g}\;d^3x\,dx^0.\tag{1.8}$$

The first integral in (1.8) corresponds to the gravitational field, while the second one is responsible for matter. The Lagrangian $\mathcal{L}_{\text{gr}}$ was taken in the standard four-dimensional form (see § 2 in Chapter V of [6]):

$$\mathcal{L}_{\text{gr}}=-\frac{c^3}{16\pi\gamma}\,(r+2\,\Lambda).\tag{1.9}$$

The four-dimensional scalar curvature r in (1.9) is associated with the four-dimensional metric (1.2). As is was shown in [1], it is expressed through the three-dimensional scalar curvature R in the following way:

$$r=-2\sum_{k=1}^{3}\frac{\partial b^k_k}{\partial x^0}-R-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_q\,b^q_k-\sum_{k=1}^{3}\sum_{q=1}^{3}b^k_k\,b^q_q\,.\tag{1.10}$$

Applying the stationary-action principle to (1.8), in [11] the equation (1.4) was re-derived along with the following purely three-dimensional expression for the spatial components of the four-dimensional energy-momentum tensor in it:

$$T_{ij}=-2\,c\,\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}},\quad 1\leqslant i,j\leqslant 3.\tag{1.11}$$

In [12] the Lagrangian approach to the 3D-brane universe model was considered again and the Hamiltonian approach to this model was developed. As a result the equation (1.4) was rederived twice more. As for the equations (1.5) and (1.6), they did not arise neither in [11] nor in [12]. The main goal of the present paper is to formulate and prove the energy conservation law for the gravitational field and matter within the 3D-brane universe paradigm.

2. Reducing the order of the Lagrangian.

Due to (1.7) the Lagrangian (1.9) with r given by (1.10) is of the second order with respect to time derivatives of the metric (1.1). In [12] the order of this Lagrangian was reduced by one. For this purpose in [12] the function (1.10) was replaced by the following function:

$$\rho = \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_k\, b^q_q - R - \sum_{k=1}^{3}\sum_{q=1}^{3} b^k_q\, b^q_k. \tag{2.1}$$

Then the Lagrangian (1.9) was rewritten as

$$\mathcal{L}_{\text{gr}} = -\frac{c^3}{16\pi\gamma}\,(\rho + 2\,\Lambda). \tag{2.2}$$

The action integral (1.8) with the Lagrangian (2.2), where ρ is given by the formula (2.1), leads to the same differential equation (1.4) as the action integral (1.8) with the Lagrangian (1.9). The Lagrangian (2.2) is of the first order with respect to time derivatives of the metric (1.1).

3. Euler-Lagrange equations.

The Lagrangian of matter $\mathcal{L}_{\text{mat}}$ was not written in full details in [11]. This was done later in [12]. Following [12], we write

$$\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{mat}}(Q_1, \ldots, Q_n, W_1, \ldots, W_n, \mathbf{g}, \mathbf{b}). \tag{3.1}$$

Here $Q_1, \ldots, Q_n$ are dynamic variables of matter and $W_1, \ldots, W_n$ are their time derivatives. They are related to each other through the formula

$$W_i = \frac{\partial Q_i}{\partial x^0} = \frac{\dot{Q}_i}{c}, \qquad i = 1, \ldots, n. \tag{3.2}$$

The quantities $Q_1, \ldots, Q_n$ and $W_1, \ldots, W_n$ are fields, i.e. they depend on the spatial variables x^1, x^2, x^3 and on the temporal variable $x^0 = c\,t$:

$$Q_i = Q_i(x^0, x^1, x^2, x^3), \qquad\qquad W_i = W_i(x^0, x^1, x^2, x^3). \tag{3.3}$$

The same is true for the components g_{ij} and b_{ij} of the fields $\mathbf{g}$ and $\mathbf{b}$ in (3.1). The formula (3.2) here is analogous to the formula (1.7).

The Lagrangian $\mathcal{L}_{\text{gr}}$ in (2.2) does not depend on the dynamic variables of matter $Q_1, \ldots, Q_n$ and $W_1, \ldots, W_n$. Therefore we write

$$\mathcal{L}_{\text{gr}} = \mathcal{L}_{\text{gr}}(\mathbf{g}, \mathbf{b}). \tag{3.4}$$

The total Lagrangian $\mathcal{L}$ in the action integral (1.8) is the sum

$$\mathcal{L} = \mathcal{L}_{\text{gr}} + \mathcal{L}_{\text{mat}} \tag{3.5}$$

of the Lagrangians (3.1) and (3.4). Therefore in this case we write

$$\mathcal{L} = \mathcal{L}(Q_1, \ldots, Q_n, W_1, \ldots, W_n, \mathbf{g}, \mathbf{b}). \tag{3.6}$$

Remark. Each entry Q_i in the argument list in (3.1) or in (3.6) means that the corresponding Lagrangian depends not only on Q_i, but on some finite number of partial derivatives of Q_i with respect to spatial variables x^1, x^2, x^3. The same is true for the arguments W_i and for the components g_{ij} and b_{ij} of the tensor fields $\mathbf{g}$ and $\mathbf{b}$ in (3.1), (3.4), and (3.6).

Relying on (3.3) and on the above remark, we introduce the following notations:

$$Q_i[i_1 \ldots i_s] = \frac{\partial Q_i}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad W_i[i_1 \ldots i_s] = \frac{\partial W_i}{\partial x^{i_1} \ldots \partial x^{i_s}}. \tag{3.7}$$

Then we introduce the notations similar to (3.7) for the components of $\mathbf{g}$ and $\mathbf{b}$:

$$g_{ij}[i_1 \ldots i_s] = \frac{\partial g_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}, \qquad b_{ij}[i_1 \ldots i_s] = \frac{\partial b_{ij}}{\partial x^{i_1} \ldots \partial x^{i_s}}. \tag{3.8}$$

In terms of the Lagrangian (3.5) the action integral (1.8) is written as

$$S = \iint \mathcal{L}\,\sqrt{\det g}\; d^3x\, dx^0. \tag{3.9}$$

Applying the stationary action principle to the action integral (3.9), in [12] the following Euler-Lagrange equations were derived:

$$-\frac{1}{2}\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} - \frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} \sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}} = 0, \tag{3.10}$$

$$-\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} - \Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}} \sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}} = 0. \tag{3.11}$$

In [12] it was shown that the equation (3.10) is equivalent to the equation (1.4) provided we admit the equality (1.11). For the variational derivative in the right hand side of (1.11) in [12] the following formulas were derived:

$$\begin{aligned}
&\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g^{ij}} = -\sum_{k=1}^{3}\sum_{q=1}^{3} \frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{kq}}\, g_{ik}\, g_{qj},\\
&\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}} = -\frac{1}{2}\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} -\\
&\qquad -\frac{1}{2}\,\Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}} \sum_{q=1}^{3} b^q_q + \Bigl(\frac{\delta\mathcal{L}_{\text{mat}}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}.
\end{aligned}$$

As for the equation (3.11), it describes the dynamics of matter. Since $\mathcal{L}_{\mathrm{gr}}$ does not depend on Q_i and W_i (see (3.4), this equation can be rewritten as

$$-\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}-\Bigl(\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\sum_{q=1}^{3}b^q_q+\Bigl(\frac{\delta\mathcal{L}_{\mathrm{mat}}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}=0.$$

4. Legendre transformation and the energy density.

The Legendre transformation determined by the Lagrangian (3.5) of the gravitational field and matter is given by the following formulas:

$$\beta^{ij}=\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}},\qquad\qquad P^i=\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}.\tag{4.1}$$

Here β^{ij} and P^i are generalized momenta associated with the generalized velocities b_{ij} and W_i. The transformation (4.1) was introduced in [12].

The energy density is introduced by the following formula from [12]:

$$\mathcal{H}=\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{n}P^i\,W_i-\mathcal{L}.\tag{4.2}$$

The quantities β^{ij} and P^i in (4.2) are given by the formulas (4.1). Let Ω be a three-dimensional domain in a 3D-brane universe. The energy of the gravitational field and matter within this domain is given by the following integral:

$$E(\Omega)=\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x.\tag{4.3}$$

The main goal of the next section is to derive a formula for the time derivative of the energy integral (4.3).

5. The energy conservation law.

Let's consider a small increment of the variable x^0. We write it as follows:

$$\hat{x}^0=x^0+\varepsilon.\tag{5.1}$$

Then we apply (5.1) to the dynamic variables of the gravitational field and matter:

$$\hat{g}_{ij}=g_{ij}(\hat{x}^0,x^1,x^2,x^3),\qquad\qquad \hat{Q}_i=Q_i(\hat{x}^0,x^1,x^2,x^3),\tag{5.2}$$

$$\hat{b}_{ij}=b_{ij}(\hat{x}^0,x^1,x^2,x^3),\qquad\qquad \hat{W}_i=W_i(\hat{x}^0,x^1,x^2,x^3),\tag{5.3}$$

$$\hat{\beta}^{ij}=\beta^{ij}(\hat{x}^0,x^1,x^2,x^3),\qquad\qquad \hat{P}^i=P^i(\hat{x}^0,x^1,x^2,x^3).\tag{5.4}$$

Applying the relationships (1.7) and (3.2) to (5.2), we get

$$\hat{g}_{ij}=g_{ij}+2\,\varepsilon\,b_{ij}+\ldots\,,\qquad\qquad \hat{Q}_i=Q_i+\varepsilon\,W_i+\ldots\,.\tag{5.5}$$

In the case of (5.3) we use partial derivatives:

$$\hat{b}_{ij}=b_{ij}+\varepsilon\,\frac{\partial b_{ij}}{\partial x^0}+\ldots\,,\qquad\qquad \hat{W}_i=W_i+\varepsilon\,\frac{\partial W_i}{\partial x^0}+\ldots\,. \tag{5.6}$$

And in the case of (5.4) we we apply the relationships (4.1):

$$\begin{aligned}
&\hat{\beta}^{ij}=\beta^{ij}+\varepsilon\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}+\ldots\,,\\
&\hat{P}^i=P^i+\varepsilon\,\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}+\ldots\,.
\end{aligned} \tag{5.7}$$

Through dots in (5.5), (5.6), (5.7), and in what follows below we denote higher order terms with respect to the small parameter $\varepsilon\to 0$.

The next step is to apply (5.1) to the integral (4.3) taking into account (4.2):

$$\begin{aligned}
\hat{E}(\Omega)=&\int\limits_{\Omega}\Bigl(\sum_{i=1}^{3}\sum_{j=1}^{3}\beta^{ij}\,b_{ij}+\sum_{i=1}^{n}P^i\,W_i\Bigr)\sqrt{\det\hat{g}}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}b_{ij}\,\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\frac{\partial b_{ij}}{\partial x^0}\,\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{n}\frac{\partial}{\partial x^0}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}W_i\,\sqrt{\det g}\;d^3x\,+\\
&+\varepsilon\int\limits_{\Omega}\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\frac{\partial W_i}{\partial x^0}\,\sqrt{\det g}\;d^3x-\hat{L}(Q)+\ldots\,.
\end{aligned} \tag{5.8}$$

The last term $\hat{L}(Q)$ in (5.8) is determined by the Lagrangian $\mathcal{L}$ in (4.2):

$$\hat{L}(Q)=\int\limits_{\Omega}\hat{\mathcal{L}}\sqrt{\det\hat{g}}\;d^3x. \tag{5.9}$$

In order to transform (5.9) we should note that the formulas (5.5) and (5.6) are similar to small variations of the tensor fields $\mathbf{g}$ and $\mathbf{b}$ and to small variations of the dynamic variables of matter $Q_1,\,\ldots\,,\,Q_n$ and $W_1,\,\ldots\,,\,W_n$:

$$\begin{aligned}
&\hat{g}_{ij}=g_{ij}+\varepsilon\,h_{ij}+\ldots\,, &&\hat{Q}_i=Q_i+\varepsilon\,h_i+\ldots\,,\\
&\hat{b}_{ij}=b_{ij}+\varepsilon\,\eta_{ij}+\ldots\,, &&\hat{W}_i=W_i+\varepsilon\,\eta_i+\ldots\,.
\end{aligned} \tag{5.10}$$

The functions h_{ij}, h_i, η_{ij}, and η_i in (5.10) are functions with compact support (see [13]). They are applied to the integral over the whole 3D-brane universe:

$$L=\int\mathcal{L}\sqrt{\det g}\;d^3x. \tag{5.11}$$

Applying (5.10) to (5.11), we would write

$$\hat{L} = L + \varepsilon \int \biggl(\sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\,\eta_{ij} + \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}\,h_{ij} + \\ + \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\,\eta_i + \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}\,h_i\biggr)\sqrt{\det g}\,d^3x + \ldots . \tag{5.12}$$

The difference of (5.5) and (5.6) from (5.10) is that small variations in (5.5) and (5.6) are not functions with compact support. For this reason the analog of the formula (5.12) has an extra term with boundary integral:

$$\hat{L}(\Omega) = L(\Omega) + \varepsilon \int\limits_{\Omega} \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta b_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{g}}\,\frac{\partial b_{ij}}{\partial x^0}\,\sqrt{\det g}\,d^3x + \\ + \varepsilon \int\limits_{\Omega} \biggl(\sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta W_i}\Bigr)_{\mathbf{Q},\mathbf{g},\mathbf{b}}\,\frac{\partial W_i}{\partial x^0} + \sum_{i=1}^{n}\Bigl(\frac{\delta\mathcal{L}}{\delta Q_i}\Bigr)_{\mathbf{W},\mathbf{g},\mathbf{b}}\,W_i\biggr)\sqrt{\det g}\,d^3x + \\ + \varepsilon \int\limits_{\Omega} \sum_{i=1}^{3}\sum_{j=1}^{3}\Bigl(\frac{\delta\mathcal{L}}{\delta g_{ij}}\Bigr)_{\mathbf{W},\mathbf{Q},\mathbf{b}}\,2\,b_{ij}\,\sqrt{\det g}\,d^3x + \varepsilon \int\limits_{\partial\Omega} \bigl(\mathcal{J}^1\,dx^2\wedge dx^3 + \\ + \mathcal{J}^2\,dx^3\wedge dx^1 + \mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\frac{\sqrt{\det g}}{c} + \ldots . \tag{5.13}$$

The extra term with boundary integral in (5.13) is associated with the energy flow. We shall study this term in the next section.

Let's return to the formula (5.8). The square root in the first integral of the formula (5.8) is transformed in the following way:

$$\sqrt{\det\hat{g}} = \sqrt{\det g} + \varepsilon\,\frac{\partial(\sqrt{\det g}\,)}{\partial x^0} + \ldots . \tag{5.14}$$

This formula can be transformed further. It yields

$$\sqrt{\det\hat{g}} = \sqrt{\det g} + \varepsilon \sum_{q=1}^{3} b^q_q\,\sqrt{\det g} + \ldots . \tag{5.15}$$

In deriving (5.15) from (5.14) we use the well-known Jacobi's formula for differentiating determinants (see [14]).

Now we can apply (5.13) and (5.15) to (5.8). In doing it we take into account the formulas (4.1) and the Euler-Lagrange equations (3.10) and (3.11). As a result the formula (5.8) reduces to the following one:

$$\hat{E}(\Omega) = E(\Omega) - \varepsilon \int\limits_{\partial\Omega} \bigl(\mathcal{J}^1\,dx^2\wedge dx^3 + \\ + \mathcal{J}^2\,dx^3\wedge dx^1 + \mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\frac{\sqrt{\det g}}{c} + \ldots . \tag{5.16}$$

On the other hand, applying the transformation (5.1) to the integral (4.3) directly, we obtain the following relationship:

$$\hat{E}(\Omega)=E(\Omega)+\varepsilon\,\frac{\partial E(\Omega)}{\partial x^0}+\ldots\,. \tag{5.17}$$

Comparing (5.16) and (5.17), we derive

$$\begin{gathered}\frac{\partial}{\partial x^0}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\\
+\,\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\frac{\sqrt{\det g}}{c}=0.\end{gathered} \tag{5.18}$$

The surface integral of the second kind in (5.18) can be transformed to a surface integral of the first kind. Indeed, we can write

$$\frac{\partial}{\partial x^0}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\frac{1}{c}\int\limits_{\partial\Omega}\bigl(\mathcal{J}^1\,n_1+\mathcal{J}^2\,n_2+\mathcal{J}^3\,n_3\bigr)\,dS. \tag{5.19}$$

Here n_1, n_2, n_3 are covariant components of the unit normal vector $\mathbf{n}$ perpendicular to the boundary $\partial\Omega$ and dS is the infinitesimal area element of the boundary. The quantities $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ in (5.19) are interpreted as the components a vector field. This vector field $\mathbf{J}$ is interpreted as the density of the total energy flow. Since $x^0=c\,t$, we can write the formula (5.19) as

$$\frac{\partial}{\partial t}\int\limits_{\Omega}\mathcal{H}\,\sqrt{\det g}\;d^3x+\int\limits_{\partial\Omega}\sum_{i=1}^{3}\mathcal{J}^i\,n_i\,dS. \tag{5.20}$$

The equality (5.20) can be formulated as the following theorem.

Theorem 5.1. *The increment of the total energy of the gravitational field and matter per unit time in a closed 3D-domain Ω is equal to the energy supplied to the domain per unit time through its boundary $\partial\Omega$.*

In order to transform the integral equality (5.20) to a differential form we apply the Ostrogradsky-Gauss formula (see [15]) along with the formulas (5.15) and (5.14). This yields the following differential relationship:

$$\frac{\partial\mathcal{H}}{\partial t}+\sum_{q=1}^{3}\mathcal{H}\,b^q_q+\sum_{i=1}^{3}\nabla_{\!i}\mathcal{J}^i=0. \tag{5.21}$$

The first term in (5.21) is the time derivative of the total energy density of the gravitational field and matter. The third term is the divergence of the density vector for the total energy flow. These two terms are standard. The second term in (5.21) is the Hubble term. It is associated with the Hubble expansion (see [16]) of a 3D-universe in our 3D-brane universe model. Note that some formulas including a Hubble term for the electromagnetic energy were derived in [17] within the standard four-dimensional paradigm.

6. Density of the total energy flow.

The vector $\mathbf{J}$ with the components $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ arises in (5.13) when deriving an analog of the formula (5.12) where the small variations of the dynamic variables are not functions with compact support. We know that the Lagrangian (3.6) depends not only on the the functions in its argument list, but on their spatial derivatives of the form (3.7) and (3.8) as well. Let's choose $b_{ij}[i_1 \ldots i_s]$ in (3.8) and consider its entry to the Lagrangian (3.6). The variation of b_{ij} in (5.10) contributes to the variational expansion of the integral (5.9) through the term

$$I(\mathbf{b}) = \varepsilon \int\limits_{\Omega} \Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1 \ldots i_s]\,d^3x. \tag{6.1}$$

Let's denote through ι_q a linear mapping acting upon differential 3-forms and producing differential 2-forms such that

$$\iota_q(dx^1 \wedge dx^2 \wedge dx^3) = \begin{cases} dx^2 \wedge dx^3 & \text{if } q = 1, \\ dx^3 \wedge dx^1 & \text{if } q = 2, \\ dx^1 \wedge dx^2 & \text{if } q = 3. \end{cases} \tag{6.2}$$

Now we can integrate (6.1) by parts. The result is written using (6.2):

$$\begin{aligned} &\varepsilon \int\limits_{\Omega} \Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1 \ldots i_s]\,d^3x = \\ &= \varepsilon \int\limits_{\partial\Omega} \Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1 \ldots i_{s-1}]\,\iota_{i_s}(dx^1 \wedge dx^2 \wedge dx^3) - \\ &- \varepsilon \int\limits_{\Omega} \frac{\partial}{\partial x^{i_s}}\Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}[i_1 \ldots i_{s-1}]\,d^3x. \end{aligned} \tag{6.3}$$

The last term in (6.3) is similar to the first term in it. Therefore we can iterate the integration by parts in (6.3). The result is written as follows:

$$\begin{aligned} I(\mathbf{b}) = \sum_{r=1}^{s} \varepsilon \int\limits_{\partial\Omega} (-1)^{r-1}\,&\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}} \ldots \partial x^{i_s}}\Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\times \\ &\times\,\eta_{ij}[i_1 \ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1 \wedge dx^2 \wedge dx^3) + \\ + \varepsilon \int\limits_{\Omega} (-1)^s\,&\frac{\partial^s}{\partial x^{i_1} \ldots \partial x^{i_s}}\Bigl(\frac{\partial \mathcal{L}}{\partial b_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_{ij}\,d^3x. \end{aligned} \tag{6.4}$$

The last term in (6.4) contributes to the bulk integrals in (5.13). The previous terms contribute to the boundary integral in (5.13).

The variation of g_{ij} in (5.10) contributes to the variational expansion of the integral (5.9) through the following term:

$$I(\mathbf{g}) = \varepsilon \int\limits_{\Omega} \Bigl(\frac{\partial \mathcal{L}}{\partial g_{ij}[i_1 \ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}[i_1 \ldots i_s]\,d^3x. \tag{6.5}$$

Integrating by parts iteratively in (6.5), we derive a formula similar to (6.4):

$$
\begin{gathered}
I(\mathbf{g})=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\times\\
\times\,h_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\\
+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_{ij}\,d^3x.
\end{gathered}
\tag{6.6}
$$

Further two steps are similar to the previous two. The analogs of the formulas (6.1) and (6.5) for the dynamic variables of matter in (5.10) are

$$
\begin{aligned}
&I(\mathbf{W})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_i[i_1\ldots i_s]\,d^3x,\\
&I(\mathbf{Q})=\varepsilon\int\limits_{\Omega}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_i[i_1\ldots i_s]\,d^3x.
\end{aligned}
\tag{6.7}
$$

Integrating by parts iteratively in (6.7), we get formulas similar to (6.4) and (6.6):

$$
\begin{gathered}
I(\mathbf{W})=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\times\\
\times\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\\
+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,\eta_i\,d^3x,
\end{gathered}
\tag{6.8}
$$

$$
\begin{gathered}
I(\mathbf{Q})=\sum_{r=1}^{s}\varepsilon\int\limits_{\partial\Omega}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\times\\
\times\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)+\\
+\,\varepsilon\int\limits_{\Omega}(-1)^s\,\frac{\partial^s}{\partial x^{i_1}\ldots\partial x^{i_s}}\Bigl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\,\sqrt{\det g}\,\Bigr)\,h_i\,d^3x.
\end{gathered}
\tag{6.9}
$$

The quantities η_{ij}, h_{ij}, η_i, and h_i in the formulas (6.4), (6.6), (6.8), and (6.9) should be replaced with the following ones:

$$
\eta_{ij}=\frac{\partial b_{ij}}{\partial x^0},\qquad\qquad \eta_i=\frac{\partial W_i}{\partial x^0},
\tag{6.10}
$$

$$
h_{ij}=2\,b_{ij},\qquad\qquad h_i=W_i,
\tag{6.11}
$$

The formulas (6.10) and (6.11) are derived by comparing (5.5) and (5.6) with (5.10).

The last step in calculating the components of the vector $\mathbf{J}$ consists in collecting boundary terms from (6.4), (6.6), (6.8), and (6.9) into one formula. Assume N to be the maximal order of the partial derivatives of the form (3.7) and (3.8) in $\mathcal{L}$. Then from (6.4), (6.6), (6.8), (6.9) and (5.13) we derive

$$
\begin{aligned}
&\bigl(\mathcal{J}^1\,dx^2\wedge dx^3+\mathcal{J}^2\,dx^3\wedge dx^1+\mathcal{J}^3\,dx^1\wedge dx^2\bigr)\,\frac{\sqrt{\det g}}{c}=\\
&=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial b_{ij}[i_1\ldots i_s]}\times\\
&\qquad\times\sqrt{\det g}\,\biggr)\,\eta_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial g_{ij}[i_1\ldots i_s]}\times\\
&\qquad\times\sqrt{\det g}\,\biggr)\,h_{ij}[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial W_i[i_1\ldots i_s]}\times\\
&\qquad\times\sqrt{\det g}\,\biggr)\,\eta_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3)\,+\\
&+\sum_{i=1}^{n}\sum_{s=1}^{N}\sum_{r=1}^{s}(-1)^{r-1}\,\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}\biggl(\frac{\partial\mathcal{L}}{\partial Q_i[i_1\ldots i_s]}\times\\
&\qquad\times\sqrt{\det g}\,\biggr)\,h_i[i_1\ldots i_{s-r}]\,\iota_{i_{s-r+1}}(dx^1\wedge dx^2\wedge dx^3).
\end{aligned}
\tag{6.12}
$$

Note that the formulas (6.10) and (6.11) apply to (6.12) as well as to the previous formulas (6.1), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), and (6.9). Note also that the partial differentiation operators of the form

$$\frac{\partial^{r-1}}{\partial x^{i_{s-r+2}}\ldots\partial x^{i_s}}$$

are omitted in those terms of (6.12) where $r=1$. The same is true for all previous formulas where these operators are used.

7. Concluding remarks.

The main result of the present paper is Theorem 5.1 presenting the total energy conservation law for the gravitational field and matter in a 3D-brane universe. This law is expressed by the formula (5.19) or equivalently by the formula (5.20). The components $\mathcal{J}^1$, $\mathcal{J}^2$, $\mathcal{J}^3$ of the vector $\mathbf{J}$ expressing the density of the total energy flow are given by the formula (6.12), where $\mathcal{L}$ is the total Lagrangian (3.5) of the gravitational field and matter.

In the standard four-dimensional paradigm the energy conservation law is united with the momentum conservation law and is formulated as the energy-momentum conservation law which is expressed by the formula

$$\sum_{q=0}^{3}\nabla_q T^{kq}=0, \tag{7.1}$$

see (3.6) in § 3 of Chapter V of [6]. Our formula (5.21) expressing the total energy conservation law in differential form replaces in part the formula (7.1). As for the momentum conservation law, in the 3D-brane universe paradigm it is a separate law that should be studied separately. This will be done in a separate paper.

8. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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