An Inconsistent Hierarchy of Sets in [0, 1]

By Jim Rock

Abstract: Two contradictory arguments are developed from a hierarchy of sets in [0, 1]. One argument is a proof by contradiction and its conclusion is true. The other argument is an existence argument and while its conclusion is not true, it follows logically from the a valid assumption followed by three true statements that precede the conclusion.

Introduction. For all rational numbers \(a\) in the closed interval [0, 1] and \{0\} define the collection of all \(R_a\) sets equal \{y is a rational number \(0 \leq y < a\) \} and \{0\}

The following four true statements characterize the collection of \(R_a\) sets and \{0\}.

a) The collection forms a hierarchy of sets with \(R_1\) at the top and \{0\} on the bottom.
b) Each \(R_a\) set contains all the elements in sets below it in the set hierarchy.
c) Each set is a proper subset of all the \(R_a\) sets above it in the set hierarchy.
d) Used in Arg #1 step 4. Each individual \(R_a\) set contains at least one element that is not in any of the sets below it in the hierarchy. Otherwise, the entire hierarchy would collapse.

Argument #1: \(R_1\) contains a largest element.
1) Let \(c\) and \(d\) be two elements of \(R_1\) with \(c > d\).
2) \(d\) is an element of \(R_c\), which is a proper subset of \(R_1\).
3) For any two elements in \(R_1\) the smaller element is contained in a proper subset of \(R_1\).
4) d) \(R_1\) contains a largest element not contained in any set below it in the set hierarchy.

Argument #2: \(R_1\) contains no largest element.
1) Suppose there is a largest element \(a'\) in \(R_1\).
2) \(a' < \frac{(a + a')}{2} < a\).
3) Let \(b = \frac{(a + a')}{2}\).
4) Then \(b\) is in \(R_1\) and \(a' < b\).

When a largest element is assumed in Argument #2 it leads to a contradiction so there is no largest element in \(R_1\). A valid proof by contradiction.

The difference between the two arguments is no attempt is made to specify a largest element in argument #1. It is an existence argument only.

But in argument #1, step 1 is a valid assumption and statements 2, 3, and d) from the Introduction are true statements. Step 4 follows logically from steps 1, 2, 3, and d).

Thus, we have two contradictory arguments that can be developed in any formal system containing sets, arithmetic, and relations between the rational numbers.

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