MOND from FLRW

Alireza Jamali

Independent Researcher

alireza.jamali.mp@gmail.com

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Abstract

After proving that the universal acceleration scale of MOND is the acceleration of light in an expanding Universe, it is shown that accelerating null rays require a modification of the metric of velocity space, hence the differential of velocity. Consequently the canonical momentum, and from there the law of motion are changed. After some approximations, MOND's essence is vindicated and it is seen as a necessary consequence of the acceleration of the Universe.

The fundamental assumption of MOND is the existence of a universal acceleration scale a_0 [1, 2, 3]. The situation is similar to that of Relativity in which the existence of a universal *velocity* scale c required a thorough revision of foundations of physics. If we want to take MOND seriously we must first understand it, and to understand it we need to know what precisely a_0 is. There already exists a hint, which was noticed by Milgrom himself,

$$a_0 \sim cH_0,$$

although he (and others) never took this serious enough to promote it to the level of an *exact equality* and this curiosity was never thought of as more than a 'numerical coincidence'.
I believe, however, that there are no 'numerical coincidences' in fundamental physics, so as the first step, I promote this curiosity to an identity

$$\boxed{a_0 = cH_0} \tag{1}$$

Then we would naturally need to know *whose entity is this acceleration?*
The appearance of H_0 points us to consider the FLRW metric[1]

$$ds^2 = -c^2dt^2 + R^2(t)d\vec{x}^2. \tag{2}$$

[1]For simplicity, without lack of generality and in accord with empirical evidence, I assume a spatially-flat Universe.

Also the appearance of c points us to consider the path of light in this spacetime, i.e. null rays $ds = 0$, therefore

$$cdt = \pm R(t) d\vec{x},$$

implying

$$\frac{d\vec{x}}{dt} = \vec{v} = \pm \frac{c}{R(t)}.$$

Differentiating with respect to t we would have

$$\boxed{\frac{d\vec{v}}{dt} = \mp \frac{cH(t)}{R(t)}} \tag{3}$$

where

$$H(t) := \frac{\dot{R}}{R},$$

is the Hubble parameter.

This equation, in the current epoch of the Universe $t = t_0$ is

$$\frac{d\vec{v}}{dt} = \mp cH_0$$

which is nothing but a_0.
We therefore arrive at the key observation that **the universal acceleration scale of MOND a_0 is the *acceleration of light* in an expanding Universe**.

How could then kinematics modify the laws of dynamics? From the experience of Special Relativity[2] we know that this is achieved via the metric, viz.

$$\mathbf{P} = m\frac{d\mathbf{X}}{d\tau} = m\frac{d\mathbf{X}}{dt}\frac{dt}{d\tau} = m\gamma\frac{d\mathbf{X}}{dt}. \tag{4}$$

To turn (3) into a metric, recall the method from SR, by which we turned the null rays

$$d\vec{x} = \pm cdt$$

into the Minkowski metric by *measuring how much a hypothetical path would differ from being a null ray*, viz.

$$d\vec{x}^2 - c^2dt^2 =: ds^2.$$

We also know from SR that by modifying the metric of spacetime we will always get a *function of velocity* (e.g. the γ factor), not one of acceleration, therefore *to get a function of acceleration we must investigate the metric of velocity space.*

Altogether then, by combining (2) and (3) we are led to

$$du^2 = (cH)^2dt^2 + R^2(t)d\vec{v}^2, \tag{5}$$

as the metric of velocity space, whereas since according to SR (in an expanding Universe)

$$\mathbf{U} = \begin{pmatrix} c\gamma \\ R(t)\gamma\vec{v} + \gamma\dot{R}\vec{x} \end{pmatrix} = \begin{pmatrix} c\gamma \\ \gamma\big(H\vec{x} + \vec{v}\big) \end{pmatrix},$$

[2]Hereafter *SR*.

meaning that *time is not a dimension of the velocity space*, we have

$$du^2 \stackrel{?}{=} -c^2 d\gamma^2 + \Big(d\big(\gamma(H\vec{x}+\vec{v})\big)\Big)^2. \tag{6}$$

We are thus led to another key observation: Relativity does not take the acceleration of null rays (light) into account which translates to *neglecting time as a dimension of the velocity space*. The situation is similar to what happened with Relativity itself: Galilean relativity does not take into account the velocity of null rays which translates to neglecting time as a dimension of the position space.

Therefore, adding time as a dimension to the velocity space, we must perfect (6) to

$$\boxed{du^2 = (cH)^2 dt^2 - c^2 d\gamma^2 + \Big(d\big(\gamma(H\vec{x}+\vec{v})\big)\Big)^2} \tag{7}$$

as *the* metric of velocity space.

It is now easy to show the *necessity of MOND in an accelerating Universe*. The key here is that *as our differential of velocity has changed we must find an expression of dynamical laws that employs differential of velocity*. This is not possible using the spacetime expression of Newton's Second law

$$\vec{F} = m\frac{d\vec{p}}{dt},$$

but is readily done using the configuration space expression (Euler-Lagrange equation)[3]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v}\right) = \frac{\partial \mathcal{L}}{\partial x},$$

according to which momentum is defined as

$$p = \frac{\partial \mathcal{L}}{\partial v} = \frac{d}{dv}(\frac{1}{2}mv^2). \tag{8}$$

But now dv is changed to du from (7), like in SR where dt is changed to $d\tau$ from the Minkowski metric. Therefore

$$p = \frac{d}{du}(\frac{1}{2}mv^2) = mv\frac{dv}{du} = mv\frac{dv}{dt}\frac{dt}{du} = mva\frac{dt}{du} =: mva\bar{\mu} \tag{9}$$

hence

$$F = \frac{dp}{dt} = m(\frac{dv}{dt})^2\bar{\mu} + mv\frac{da}{dt}\bar{\mu} + mva\frac{da}{dt}\frac{d\bar{\mu}}{da}, \tag{10}$$

if we neglect terms containing jerk da/dt we arrive at

$$F \simeq ma^2\bar{\mu}. \tag{11}$$

To find $\bar{\mu} = dt/du$, we make two approximations on (7),

1. $\gamma \approx 1$, so $d\gamma \approx 0$,

[3] For simplicity and without lack of generality we work in 1+1 dimensions hereafter.

2. We consider the metric in the current epoch of the Universe, so $H \to H_0$; then we would have

$$\boxed{du_0^2 = (cH_0)^2 dt^2 + (H_0 d\vec{x} + d\vec{v})^2} \tag{12}$$

thus

$$(\frac{du_0}{dt})^2 = (cH_0)^2 + (H_0\vec{v} + \vec{a})^2,$$

which is finally

$$\boxed{\frac{dt}{du_0} = \bar{\mu} = \frac{1}{\sqrt{(cH_0)^2 + (H_0\vec{v} + \vec{a})^2}}} \tag{13}$$

If we compare (11) and (13) with MOND's assumption

$$F = ma\mu, \tag{14}$$

we can conclude

$$\mu = \bar{\mu}a. \tag{15}$$

In the deep-MOND regime $a \to 0$ thus, my modification of $F = ma$ is

$$\tilde{F} = \frac{ma^2}{\sqrt{(cH_0)^2 + (H_0\vec{v})^2}}, \tag{16}$$

instead of

$$F_{\mathrm{M}} = m\frac{a^2}{cH_0}.$$

Equating (16) with the centrifugal force yields

$$\boxed{v^4 = GM\sqrt{(cH_0)^2 + (H_0 v)^2}} \tag{17}$$

It remains to be calculated whether this can explain discrepancies like the Bullet Cluster.

References

[1] J. Bekenstein and M. Milgorm. Does the missing mass problem signal the breakdown of Newtonian gravity? *Astrophysical Journal*, 286, 1984.

[2] M. Milgorm. A modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophysical Journal*, 270:365–370, 1983.

[3] F. Benoît and Stacy S. McGaugh. Modified Newtonian Dynamics (MOND): Observational Phenomenology and Relativistic Extensions. *Living Reviews in Relativity*, 15(1), 2012.