The mysterious links of 137 to the fine structure constant,,

electrons' Coulomb to gravitational force ratio, and mass of elementary particles

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Abstract

In this work, we unfold the mysteries surrounding the fine structure constant and explain the physical origin of the fine structure constant as a result of gauge invariance, Einstein's mass-energy relation, and spacetime quantization. We obtain an estimate of an electron's radius $R_e = 1.40934 \times 10^{-15}\ m$, and also link the magic 137 beyond electromagnetism, to electrons' Coulomb-gravitational force ratio $F_C/F_G = 3 \times (137\pi)^{16}$, and the mass ratios of an electron to other particles such as a proton, Higgs boson, W/Z bosons, and quarks. With the proposed quantized spacetime, singularity divergence, and vacuum catastrophe problems in continuum quantum field theory can be avoided.

One of the most baffling mysteries in physics is the fine structure constant α, [1]. This universal constant α quantifies the strength of the electromagnetic interactions between charged particles. The value of about 1/137 has remained a mystery for over a century. Feynman once said, “It’s one of the greatest damn mysteries of physics, you might say the hand of God wrote that number, but we don’t know how he pushed his pencil” [2]. In this work, we hope to shed some light on this mystery and also point out its possible relation to other types of forces in nature.

In this work, we present a model for the physical origin of the fine structure constant, based on spacetime quantization, gauge symmetry, and Einstein’s mass-energy relation. In quantized spacetime, space and time are not continuous and cannot be divided indefinitely. The concept of spacetime quantization is also assumed in the loop quantum gravity theory [3] to treat the quantization of gravity [4]. As will be shown later, by applying Einstein’s mass-energy relation to the quantized gauge function we could obtain the following equations:

$$\begin{aligned} n_0{}^2\,\hbar c/e^2 &= n_1{}^2+n_2{}^2+n_3{}^2+n_4{}^2 \\ n_5{}^2 &= n_1{}^2+n_2{}^2+n_3{}^2 \\ n_0{}^2\,\hbar c/e^2 &= n_4{}^2+n_5{}^2, \end{aligned} \tag{1}$$

where $\hbar c/e^2$ and n_5 must be prime numbers, and $n_0=1$ for the primary set of integer solutions. These constraints are necessary for the solution to represent the fundamental mode instead of higher harmonic modes. Before deriving Eq. (1) here we present the number theory [5] of this magic prime 137. The constraints for the primary set of solution for Eq. (1) lead naturally to $\hbar c/e^2 = 137$. $\{n_0, n_1, n_2, n_3, n_4, n_5\} = \{1, 2, 6, 9, 4, 11\}$ so that $137 = 2^2+6^2+9^2+4^2$ with 137 as a Pythagorean prime quintuple, and $137 = 4^2+11^2$ with 137 as a Pythagorean prime triple. In addition, one has $11^2 = 2^2+6^2+9^2$ with 11 as a prime and a Pythagorean quadruple. We have derived the ideal fine structure constant α_0 to be 1/137. The small deviation from its experimental value of $1/\alpha_{\exp} = 137.035999206\ (11)$ [6] is likely due to gauge symmetry breaking by weak interaction or the interaction of an electron with its own field. Such a situation arises in the deviation of the gyromagnetic ratio from 2 for an ideal Dirac’s electron [7]. A slight increase in the effective fine structure constant observed at ~90 GeV is not surprising, because at such a regime hadron contributions due to interaction increase. The ideal 1/137 is like Dirac’s theory of a “bare” electron before QED renormalization. We have obtained an empirical fit to the experimental value by

$1/\alpha_1 = 137.03597454 = 137 + 3(\zeta + \zeta^2)/4$, with ~$10^{-7}$ in error for two expansion terms of $\zeta \equiv 2\pi/137$. With five correction terms, one has $1/\alpha_2 = 137 + (3\zeta + 3\zeta^2 + \zeta^3)/4 + {}^4\zeta^4/9 + 5\zeta^5/17 = 137.035999207$, showing an error of 10^{-11}.

We now explain how to derive Eq. (1), based on the quantized gauge function in lattice spacetime. In continuous spacetime, the gauge transform of the wave function is invariant if $\lambda(t, \mathbf{r})$ satisfies [8]

$$\begin{gathered} \frac{e}{c}\mathbf{A} \;\rightarrow\; \frac{e}{c}\mathbf{A} - \frac{e}{c}\nabla\lambda(t,\mathbf{r}), \quad e\phi \;\rightarrow\; e\phi + \frac{\partial}{c\,\partial t}e\lambda(t,\mathbf{r}) \\ \frac{\partial^2}{c^2\partial t^2}\lambda(t,\mathbf{r}) - \nabla\cdot\nabla\lambda(t,\mathbf{r}) = 0, \end{gathered} \tag{2A}$$

and with discrete time and space, coordinates one has

$$\begin{gathered} \lambda(t,\mathbf{r}) \rightarrow \sqrt{\frac{c\hbar}{e^2}}\Lambda\left(t_n, x_i, y_j, z_k\right) \\ \Psi(t, \mathbf{r}) \rightarrow \exp\left(i\sqrt{c\hbar/e^2}\Lambda(t_n, x_i, y_j, z_k)\right)\Psi(t_n, x_i, y_j, z_k), \end{gathered} \tag{2B}$$

where $\sqrt{c\hbar/e^2}$ and $\Lambda(t_n, x_i, y_j, z_k)$ are dimensionless, and the latter is related to the electric potential and vector potential. In quantized spacetime, $\Lambda(t_n, x_i, y_j, z_k)$ is no longer a continuous scalar function of time and space, so it needs to be replaced by operators, and the spacetime coordinates need to be expressed in terms of a fundamental length unit L and time unit $T = L/c$. Because the standing wave of the fundamental mode has a wavelength λ equals to twice the lattice length L, the fundamental unit for the wave vector and frequency are given by $K = \pi/L$ and $\Omega = c\pi/L$, respectively.

Using an operator approach and a discrete Fourier transform, the gauge transformation $\Lambda(t_n, x_i, y_j, z_k)$, which has a unit like a momentum, can be replaced by a dimensionless operator $\mathbf{\Lambda}$ in 4D spacetime. One can define fundamental units for the wave vector $K = \pi/L$ and frequency $\Omega = c\pi/L$. The gauge function $\sqrt{\hbar c/e^2}n_0 K\mathbf{F}_0$ can be expressed in terms of four anti-commutative operators as

$$\left(-\sqrt{\hbar c/e^2}n_0 K\mathbf{F}_0 + n_1 K\mathbf{F}_1 + n_2 K\mathbf{F}_2 + n_3 K\mathbf{F}_3 + n_4 K\mathbf{F}_4\right)|\Psi\rangle = 0, \tag{3A}$$

and

$$\sqrt{\hbar c/e^2}\, n_0 K\mathbf{F}_0 \,|\Psi\rangle = \left(n_5 K\ \mathbf{F}_5 + n_4 K\mathbf{F}_4\right)|\Psi\rangle\ , \tag{3B}$$

where $\{\mathbf{F}_\mu, \mathbf{F}_\nu\} = 2\delta_{\mu\nu}\mathbf{I},\ \ \mu, \nu = 0, 1, 2, 3, 4,$ and $\{\mathbf{F}_5, \mathbf{F}_4\} = 0.$ One can use anti-commutative and orthonormal matrices $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}$ as in Dirac's original paper [9] to represent $\mathbf{F}_\mu$, as

$$\begin{aligned}
&\mathbf{F}_k = \begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, k = 1, 2, 3,\ \ \mathbf{F}_0 = \begin{pmatrix} 0 & -\mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}, \mathbf{F}_5 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix}, \mathbf{F}_4 = \begin{pmatrix} 0 & \mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}, \\
&\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{4}$$

where $\mathbf{F}_k = \boldsymbol{\alpha}_k$ and $\mathbf{F}_4 = \boldsymbol{\beta}$ are idempotent, i.e., $i\hbar\,\partial\Psi/\partial t = \left(c\boldsymbol{\alpha}\cdot\mathbf{P} + \boldsymbol{\beta}\, m_0 c^2\right)\Psi$. By taking the square of these operators in Eq. (3A), one has

$$\begin{aligned}
&\left(-\sqrt{\hbar c/e^2}\, n_0\mathbf{F}_0 + n_1\mathbf{F}_1 + n_2\,\mathbf{F}_2 + n_3\,\mathbf{F}_3 + n_4\,\mathbf{F}_4\right)^2|\Psi\rangle \\
&= \left(\left(\hbar c/e^2\right) n_0{}^2\mathbf{F}_0{}^2 + n_1{}^2\mathbf{F}_1{}^2 + n_2{}^2\,\mathbf{F}_2{}^2 + n_3{}^2\,\mathbf{F}_3{}^2 + n_4{}^2\,\mathbf{F}_4{}^2\right)|\Psi\rangle,
\end{aligned} \tag{5A}$$

where all the cross terms vanish because of the anti-commutative relations $\{\mathbf{F}_\mu, \mathbf{F}_\nu\} = 2\delta_{\mu\nu}\mathbf{I}$. One has

$$n_0{}^2\left(\hbar c/e^2\right) = n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2\ . \tag{5B}$$

Finally, from Eqs. (3A) and (3B) we obtain the following equation involving a set of integers as

$$\begin{aligned}
&n_0{}^2\,\hbar c/e^2 = n_1{}^2 + n_2{}^2 + n_3{}^2 + n_4{}^2 \\
&n_0{}^2\,\hbar c/e^2 = n_4{}^2 + n_5{}^2 \\
&n_5{}^2 = n_1{}^2 + n_2{}^2 + n_3{}^2,
\end{aligned} \tag{6}$$

where $\hbar c/e^2$ and n_5 must be prime numbers for the solution to represent a fundamental mode, instead of higher harmonic modes. For the fundamental mode's solution, we obtained $n_0 = 1$ and $\hbar c/e^2 = 137$. With the value of $\hbar c/e^2 = 137$ determined, we found $\{n_0, n_1, n_2, n_3, n_4, n_5\} = \{1, 2, 6, 9, 4, 11\}$. It indicates that 137 is not only a prime but also a Pythagorean prime quintuple with $137 = 2^2 + 6^2 + 9^2 + 4^2$, and a Pythagorean prime triple with $137 = 11^2 + 4^2$. One also has $11^2 = 2^2 + 6^2 + 9^2$, showing 11 a Pythagorean quadruple. From Eq. (6) one can notice that the index 11 is related to the magnitude of the total momentum vector, whereas the indices 2, 6, 9 are related to the momentum components along three spatial axes, and 4 is related to the internal energy due to the electromagnetic interaction. It is interesting to point out that $2\times 6\times 9\times 4 = 432$

, the product of the four integers for the Pythagorean prime quintuple of 137, equals approximately to 137π. The lowest set of integers is called the primary set, and all other solutions correspond to higher harmonic modes with the index n_0 greater than 1. Using this primary set of integers, one can restore Eq. (6) to the original fundamental units $K = \pi/L$ and frequency $\Omega = c\pi/L$ to obtain

$$\alpha_0 = \frac{e^2}{\hbar c} = \frac{1}{137}, \tag{7A}$$

$$137\left(\frac{n_0\pi\hbar c}{L}\right)^2 = \left(\frac{n_0\pi\hbar c}{L/4}\right)^2 + \left(\frac{n_0\pi\hbar c}{L/2}\right)^2 + \left(\frac{n_0\pi\hbar c}{L/6}\right)^2 + \left(\frac{n_0\pi\hbar c}{L/9}\right)^2, \tag{7B}$$

and

$$\left(\frac{n_0\pi\hbar c}{L/11}\right)^2 = \left(\frac{n_0\pi\hbar c}{L/2}\right)^2 + \left(\frac{n_0\pi\hbar c}{L/6}\right)^2 + \left(\frac{n_0\pi\hbar c}{L/9}\right)^2, \tag{7C}$$

For the primary mode with $n_0 = 1$, Eq. (7B) is analogous to Einstein's mass-energy-momentum relation $E^2 = m_0^2c^4 + c^2P_1^2 + c^2P_2^2 + c^2P_3^2$ for a relativistic particle at the center of mass system in discrete spacetime. The first term $\hbar c/4L$ on the right-hand side of Eq. (7B) represents the internal energy for an electron from the internal structure due to electromagnetic interactions. The last three terms $\hbar c/2L$, $\hbar c/6L$, $\hbar c/9L$ are related to the particle's kinetic energy along three spatial axes, and the term $\sqrt{137}\,\hbar c/L$ on the left-hand side of the equation represents the total mass energy. Eq. (7) contains different axial lengths in 4D spacetime and represents a hyper-cell structure. An electron can be regarded as in entangled coherent superposition of these degenerate eigenstates, and because of the couplings of the 4D coordinates to those anti-commutative operators, an electron possesses a ½ spin and can be regarded as a hyper-dimensional Möbius-type structure. In Eq. (7) L is a scalable length factor, one might consider to assign it to the Planck length, but the corresponding energy of ~10^{18} GeV, a scale for grand unification [10], is unsuitable for the known elementary particles. It is reasonable to assign its value to cover all elementary particles in the Standard Model [11]. Because of the ubiquitous presence of the scaling factor 137π in those empirical m formulae in Table 1, in addition to the constraints in Eq. (6), we also include a constraint for the search of the prime value for $\hbar c/e^2$ with the ratio of the hyper-cell volume to the suitable prime be sufficiently close to an integer multiple of π. Because $432/137 \approx 1.0037\pi \sim \pi$ in our screening procedure for the prime value for $\hbar c/e^2$, we impose that the remain of the quotient to be < 0.5%. After a screening algorithm to search all combinations of integers below10000, we have only found one primary solution with $\{n_1, n_2, n_3, n_4, n_5\}$ $= \{2, 6, 9, 4, 11\}$ that meets the constraints. The adjustable parameter K in Eq. (7) with $K = \pi/L$ is related to L, the cube lattice constant, half wavelength of the fundamental standing wave. Before we discuss its link to the mass of elementary particles, we present in Table 1 a list of mass ratios between some elementary particles and that of an electron which is the lightest and most accurately determined value among fermions. We found links between 137 and the ratio between the Planck length [12] and R_e, and the mass ratios of the Higgs boson, W/Z bosons, top quark, and proton. All these findings are summarized in Tables 1. These simple relations provide hints about a possible role of 137 in all these particles.

Table 1. Links between 137 and the empirical mass ratios formulae of fundamental particles

Electron $m_e c^2 = 0.510998910(13)\ MeV$	Higgs boson $m_{Higgs}/m_e = (137\pi)^2 \times (5/3)^2 \sqrt{2/3} \times 1.0003$ $m_{Higgs} c^2 = 125.25\ (17)\ GeV$
W boson $m_W/m_e = (137\pi)^2 \times (5/3)^2 \sqrt{3/2}/4 \times 0.9993$ $m_W c^2 = 80.4555\ (64)\ GeV$	Z boson $m_Z/m_e = (137\pi)^2 \times (5/3)^2 \sqrt{3}/5 \times 1.0011$ $m_Z c^2 = 91.1876\ (21)\ GeV$
Proton $m_p/m_e = (137\pi) \times 3\sqrt{2} \times 1.0055.$ $m_p c^2 = 0.93827208816(29)\,GeV$	Top quark $m_t/m_e = (137\pi)^2 \times (3/2)^{3/2} \times 0.9960$ $m_t c^2 = 173.210\ (710)\,GeV$
Bottom quark $m_b/m_e = (137\pi) \times 11\sqrt{3} \times 0.9975$ $m_b c^2 = 4.180\ (30)\,GeV$	Charm quark $m_c/m_e = (137\pi) \times 10/\sqrt{3} \times 1.004$ $m_c c^2 = 1.275\ (25)\,GeV$

To determine a suitable unit length, we use the electron as a reference. We obtain $L_e \equiv 2 \times R_e = 2.87907 \times 10^{-15}\ m$ with $m_e c^2 = 0.51100\ MeV$ [13], and the radius of an electron is $R_e \equiv \pi \hbar c / \left(m_{es} c^2 \times \sqrt{137} \times 36\right) = 1.4395 \times 10^{-15}\ m$. The factor 36 is the least common multiple of four axial length

4, 2, 6 and 9 for constructing a 4D perfect hyper-cube from hyper-cuboids. An electron can be regarded as in entangled coherence of three degenerate eigenstates, therefore, an electron has a symmetric shape.

By equating the mass energy of an electron to the electrostatic energy of two point-like particle with the same electric charge as an electron, the distance is found to be $2.8179\times10^{-15}\,m$, which is very close to the we obtained. In comparison, the theoretical classical radius of an electron is $2.818\times10^{-15}\,m$ [14], the experimental proton's radius is about $0.842\times10^{-15}\,m$ [15], and the radius of a quark is about $10^{-18}\,m$ [16]. We have shown in Table 1, there are three tiers for the mass distribution of these elementary particles according to the power of their dependence on 137π. Thus, it is reasonable for us to define the value of the minimum unit length $L_0 \equiv L_e/(137\pi)^2$ so that the effective mass energy could cover the particles belonging to the 2nd tier, such as the Higgs boson, W/Z bosons and top quark.

Aside from the role that 137 plays in the electromagnetic force, the ratio between the Planck length and L_0, we have also discovered a link between 137 and the gravitational force. More specifically, by closely analyzing the ratio of the Coulomb to gravitational forces for a pair of electrons $(F_C/F_G)_{\exp} = 4.165185\times10^{42}$, we found a very simple formula $3\times(137\pi)^{16}\times1.00135$ with an error of only one part per thousand. Therefore, the presence of 137 in the simple dimensionless constant ratio $K_c e^2/G\,m_e^2 \equiv 3\times(137\pi)^{16}$ strongly suggests a link of the fine structure constant in both the Coulomb force and the gravitation force. This formula hints that the prime 137 also plays an intricate role in gravity, and the power of 16 is likely related to sixteen pairwise operators in 4D spacetime according to the geometry algebra formalism. Our conjecture could provide a guideline toward theoretical development of quantum gravity. We obtained an estimate of $R_e = 1.4395\times10^{-15}m$, and found a relation for the ratio between the Planck length $L_{Planck} = 1.61625\times10^{-35}\,m$ and the electron's radius R_e as $L_{Pkanck}/R_e = (21\pi/5)\times(137\pi)^{-8}\times1.002$. The comparisons between the experimental values and our empirical formulae that link these values to 137 are summarized in Tables 2.

Table 2. The links between 137 and the Planck length and the ratio between Coulomb and gravitational forces for two electrons

speed of light	Electron's charge
$c = 2.99792458\times10^{8}m/s$	$e = 1.602176634\times10^{-19}coulomb$

Electron mass and radius R_e $m_e = 9.10938291(40)\times 10^{-31}\ kg$ $unit\ length\ \ L_e \equiv 2R_e = \pi\hbar c/\left(m_{es}c^2\sqrt{137}\times 36\right)$ $radius\ \ R_e \equiv L_e/2 = 1.43957\times 10^{-15}\ \ m$	Planck constant $h = 6.62607015\times 10^{-34}\ J\,s$ $\hbar = 1.054571817\times 10^{-34}\ J\,s$
Gravitational constant & Planck length $G = 6.67498(30)\times 10^{-11}\,kg^{-1}m^3s^{-2}$ $L_{Planck} \equiv \sqrt{\hbar G/c^3} = .616255\times 10^{-35}\ m$	Coulomb constant $K_C = 8.9875517923(14)\times 10^9\ kg\cdot m^3\cdot s^{-4}A^{-2}$
Ratio between electrons' Coulomb and gravitational forces $F_C/F_G = 3\times(137\pi)^{16}\times 1.00135$	Ratio between Planck length and electron's radius $L_{Pkanck}/R_e = (137\pi)^{-8}\times\left(3^3/2\right)\times 1.003$

In summary, we presented a model to explain the origin of the fine structure constant, and via Eq. (1) we can relate its inverse value to the number theory behind 137 as a Pythagorean prime of a triple, i.e., $137 = 4^2+11^2$, and also a Pythagorean prime of a quintuple, $137 = 4^2 +2^2 +6^2 +9^2$. We show that Eq. (1) is a natural consequence of spacetime quantization, gauge symmetry of the Lorentz group, and Einstein's mass-energy relation. Due to spacetime quantization, the gauge function is shown to be quantized and can be expressed as a sum of 4x4 anti-commutative matrix operators $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}$, which are used by Dirac in his theory of electrons. The derivation of the prime 137 as an ideal value of the fine structure constant compels us to postulate spacetime quantization. In this work, we unravel the mysterious role of the prime 137 in the fine structure constant. We also found some very simple empirical formulae, such as $m_p/m_e = (137\pi)\times 3\sqrt{2}\times 1.0055$ for the mass ratio between a proton and an electron, $F_C/F_G = 3\times(137\pi)^{16}\times 1.0015$ for the ratio of Coulomb and gravitation forces between two electrons. In addition, we also obtained a formula $L_{Pkanck}/R_e = \left(3^3/2\right)\times(137\pi)^{-8}\times 1.003$ for the ratio between the Planck length and electron's radius. All these surprisingly simple relationships seem to imply that the magic 137 plays an important role not only in electromagnetism but also has an intricately link to the other fundamental forces in nature. The hypothesis of the quantized spacetime is essential in our model, without it this 137 value could not have arisen. We have found that with a quantized spacetime lattice the energy is quantized as an integer multiple of $\hbar\pi/L$, which is the lowest quantized energy with one quantum, and the vacuum corresponds to a state with no quanta that has no energy. This leads to the so-called vacuum catastrophe for the universe, where the predicted total energy for each kind of quantized field becomes a value about 120 orders of magnitude greater than the experimental value.

The core hypothesis in our model that links 137 to the fine structure constant is the quantized spacetime. Also shared by loop quantum gravity community. Could alleviate some problems raised by singularity and black hole singularity. We have provided possible links between 137 to the masses of the quarks, proton, W and Z bosons, Higgs boson, the Planck length, and the Coulomb-to-gravitational force ratio, as shown in Tables 1 and 2, and these simple formulae appear to imply deep relationships among all four forces in nature. In this work, we present a model to explain the origin of the mysterious fine structure constant, and also shed light on the links between 137 and other types of forces. The prescribed formulae in this work could potentially point a viable path toward development of quantum gravity and grand unification theories. The perturbation refinement of the fine structure constant due to symmetry breaking of the Lorentz group or interactions of an electron with its own field awaits further studies. Further theoretical developments are needed to quantitatively explain the origins behind the simple formulae that we have found and described in this report.

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