Mapping Relativistic Space to Quantum Space via the Hopf Fibration: 
A Final Theory of Quantum Space-Time

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Abstract
To allow for a general theory of travel in spacetime, the metric must be mappable. An overview of the methodology of mapping space-time is presented. The core concept of a navigable spacetime is explored in a number of ways. The Riemann sphere of quantum mechanics is shown to be the Riemann sphere of general relativity. A preferred metric is chosen for mapping between the quantum states and the states in general relativity, which is generalizable to all of physics.

To the memory of my mother, Nancy Ellen Entwistle Nielsen, who first taught me how to think about time.

The Minkowski View
In a relativistic (i.e. Minkowski) space-time, the “relativity” of simultaneity tells us that there is no preferred reference frame or unique unstacking of reference frames. The major application of Minkowski space dominates in the Newtonian limit of systems without significant gravitation and is used to describe physical systems in special relativistic space over finite distances. To apply the Minkowski metric to situations under GR, the Minkowski metric (“Minkowski tensor”) was derived, whose elements are defined by the matrix

\[
(\eta)_{\text{off}} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

which provides, as Minkowski space (or “pseudo-Riemann”) space a decent description of an infinitesimal region in most cases where the surrounding region prohibits gravitational

1 The relativity of simultaneity is the concept that distant simultaneity – whether two spatially separated events occur at the same time – is not absolute, but depends on the observer’s reference frame.
singularities.\(^2\) As we know, over large distances, the Minkowski spacetime becomes curved and then ignorable in favor of general relativity. The solution is to acknowledge Riemann manifold tensors, as Einstein did in 1912\(^3\).

**The Riemann View (OR: Introducing Time-Space)**
*i.e., The Correct Way to Think About the Riemann Metric*

The Riemann tensor (e.g., Schutz 1985), known also as the Riemann curvature tensor (e.g., Weinberg 1972, p. 133; Arfken 1985, p. 123, Misner et al., 1973, p. 218), or the Riemann metric tensor, is a four-index tensor that is paramount to general relativity. The Riemann metric tensor, which can be extended to an arbitrary metric bundle, fiber metric, or vector bundle\(^4\) in differential geometry, can be represented a number of ways. As a matrix, it is represented as

**The Riemann Metric**

Not to be confused with the Minkowski metric, a separate enterprise.

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

This could also be written in *spherical coordinates*:

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}
\]

\[
x^\mu = (ct, r, \theta, \varphi)
\]

The significance of the Riemann sphere, over which the metric manifold or metric tensor is mapped, is to introduce the real beauty of general relativity, the gluing of time to space, or rather, *of space to time*. There is a way to work with the theory—the time-space approach—in which complex time dominates, and space is modeled usefully as a *relatum* of time. In this approach,

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\(^2\)To be more exact, in the presence of gravity spacetime, the Minkowski space is defined by a four-dimensional manifold, usually curved, for which the tangent space to any point is a four-dimensional Minkowski space

\(^3\)Riemann’s tensors were central to the formation of GR. Nevertheless, Einstein later observed, “Physicists were still far removed from such a way of thinking.”

https://www.aps.org/publications/apsnews/201306/physicshistory.cfm

\(^4\) See, for an example of this, *Lie Group Cosmology*, by Garret Lisi; he’s pulled all sorts of cool but somewhat arbitrary stuff out the fiber bundles
complex time is the primary object of concern, and spatial coordinates are examined in relation to the complex time variables.

Complex time-space (also conceptualizable as “Hilbert block time”, Augustinian time\(^5\), or “eternity space”) extends the notion of time as a map of events which exist “simultaneously” in a preferred frame. Depending on the subspace of the Hilbert complex spacetime you’re looking at, you can utilize it as the space of all possible events in all possible orderings, as in the Many Worlds interpretation but mapped isometrically on relativity, or you can utilize a subspace for the existence of an entire timeline at once\(^6\). But what does this mean? And how does this connect to quantum mechanics and QFT?

**The Solution**

An element of a Hilbert space may be uniquely specified by its coordinates w/ respect to an orthonormal basis in a countably infinite basis. (The countably is critical here.) The space in a basis arrives as an analogy to classically Cartesian geometric coordinates. Since this basis is countably infinite, it allows an isomorphism (an isomorphism being a map that preserves sets and relations among elements of a subset) of the Hilbert space with the space of the infinite sequences that are square-summable (i.e., the space occupied by the state vector in quantum mechanics). This latter space is generally, in the older literature, referred to as THE Hilbert space, and it is trivial to construct this space along a Riemann (complex) sphere. There is an existing map of the Riemann sphere upon the Hilbert space.

To say this again another way, for the case of \(\mathbb{C}^2\), the complex sphere, the existing isomorphic scheme of the Hilbert scheme of points is smooth and symplectic (i.e., alternating and non-degenerate). The complex sphere is merely an analogue of the Riemann sphere.

The three-sphere \(S^3\) becomes an arena for torsion states of the particles, eliminating abstract Hilbert space vectors (e.g., Soiguine, 2022).

The Schrodinger equation governs the evolution of the points on \(S^3\), which act on observables as operators in measurements. This arena may also be visualized as a quantum schematic of the Hopf bundle on the Riemann 3-sphere (i.e., a four dimensional complex N-sphere; this is the whole thing, the fibers through the sphere represents the paths of particles through all of the manifolds in all of general relativity. The Hopf fibration has been mapped, and is the generalizable bundle of all bundles through the N-sphere! The Lie groups are just a derivative of the Hopf metric; the Hopf fibration is everything!

\(^6\) If this seems strange, consider that it is well accepted that the collection of all possible states of a given system in quantum mechanics corresponds to a separable Hilbert space over the complex number field. This is a restatement of the same thing.
The [Author's Proposed] Metric (2008, UArk Fellowship)

The Hopf fibration is a map from the four-dimensional sphere $S^3$ onto the three-dimensional complex projective space $\mathbb{C}P^2$. It is described by the following equation: $f: S^3 \rightarrow \mathbb{C}P^2, (z_1, z_2, z_3) \rightarrow [z_1 : z_2 : z_3]$. The Hopf fibration is a topological mapping used to visualize properties of all of the four-dimensional sphere in a three-dimensional space. It is used to represent the bundle structure of the sphere, which is composed of circles that are drawn on the sphere, but this is just an approximation for a spacetime metric.

The Hopf fibration is the useful tool in the study of quantum field theory and quantum mechanics, as it provides a way to visualize the world lines of all the particles. The three-sphere $S^3$ becomes an arena for the torsion-type states, cancelling the Hilbert space vectors. The 4-sphere $S^4$ points evolve in time, governed by the Schrodinger equation, and act in measurements as observables on operators.

Thus, I have demonstrated that it is trivial to show that the "Hilbert Space" (coordinate mapping) of a quantum system in time-space picture (i.e., complex spacetime / space and time with "imaginary time") can be visualized as a subset of a fiber bundle on the Riemann sphere. The Hopf fibration mapped over the Riemann sphere can easily be used to identify a metric of the spacetime continuum as a whole, as it is isometric on manifolds.

The Hopf manifold is simply the Hopf bundle on the Riemann sphere. There are a number of ways to write this, but it all means the same thing, the worldlines of the quantum are drawn on this sphere and the gauge is adjustable by their motions. This is it!

[Conclusions]

The Hopf metric through spacetime through the Riemann sphere is the final theory of spacetime evolution sought by Einstein, Heinz Hopf, Godel, and the other leading thinkers at the turn of the 20th century. There is nothing else. This is the last thing. We must work with it.
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