Riemann Hypothesis True ? or False ?

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Abstract

In this working paper I will add a complement about the Riemann hypothesis

let ζ the zeta function and η the diriklet function $\forall s \in \mathbb{C}\ with\ Re(s) > 0\ \ \eta(s) = \sum_{k=1}^{k=+\infty} \frac{(-1)^{k+1}}{k^s}$

We know that $\forall s \in \mathbb{C}\ with\ \ Re(s) > 0 \quad \left(1 - 2^{(1-s)}\right)\zeta(s) = \eta(s)$

Let $s = a + ib$ a complex number with $a, b \in \mathbb{R}\,; 0 < a < 1\,, b \neq 0$ such that $\zeta(s) = 0$

We have also $\zeta(1 - s) = 0$

So $\eta(s) = 0$ (because $s \neq 1 + \frac{2k\pi i}{\ln 2}$ $, k \in \mathbb{Z}$) and also $\eta(1 - s) = 0, \eta(\bar{s}) = 0\ and\ \eta(1 - \bar{s}) = 0$

Since $\eta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{x^{(s-1)}}{e^x+1} = 0$ we have $\int_0^{+\infty} \frac{x^{(s-1)}}{e^x+1} = 0$ and also $\int_0^{+\infty} \frac{x^{(-s)}}{e^x+1} = 0$

an integration by substitution$(x = e^t)$ gives $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ and also $\int_{-\infty}^{+\infty} \frac{e^{(1-s)t}}{e^{e^t}+1} = 0$

Let the complex function f $\forall z \in \mathbb{C}\ \ f(z) = \frac{e^{sz}}{e^{e^z}+1}$ f is meromorphic and poles of f are :

$z_{k,k'} = \ln(|2k + 1|\pi) + sgn(2k + 1)i\frac{\pi}{2} + i2k'\pi \qquad k, k' \in \mathbb{Z}$ where $sgn(2k + 1)$ is the sign of $(2k + 1)$

$z_{k,k'} = \ln((2k + 1)\pi) \pm i\frac{\pi}{2} + i2k'\pi \qquad k \in \mathbb{N}, k' \in \mathbb{Z}$

See that $Re\left(z_{k,k'}\right)$ is strictly positive

Let $n, m \in \mathbb{N}^*\,and\ \varepsilon \in \mathbb{R}\ with\ 0 < \varepsilon < \frac{1}{2}\ and\ A \in \mathbb{R}, A = A_n = \ln\left((2n + \varepsilon)\pi\right)$

Let $K_{(n,m)}$ the compact set in $\mathbb{C}$ (the rectangle)

$K_{(n,m)} = \{x + iy, x, y \in \mathbb{R}\ \ -m \leq x \leq A_n\ and\ 0 \leq y \leq 2\pi\}$

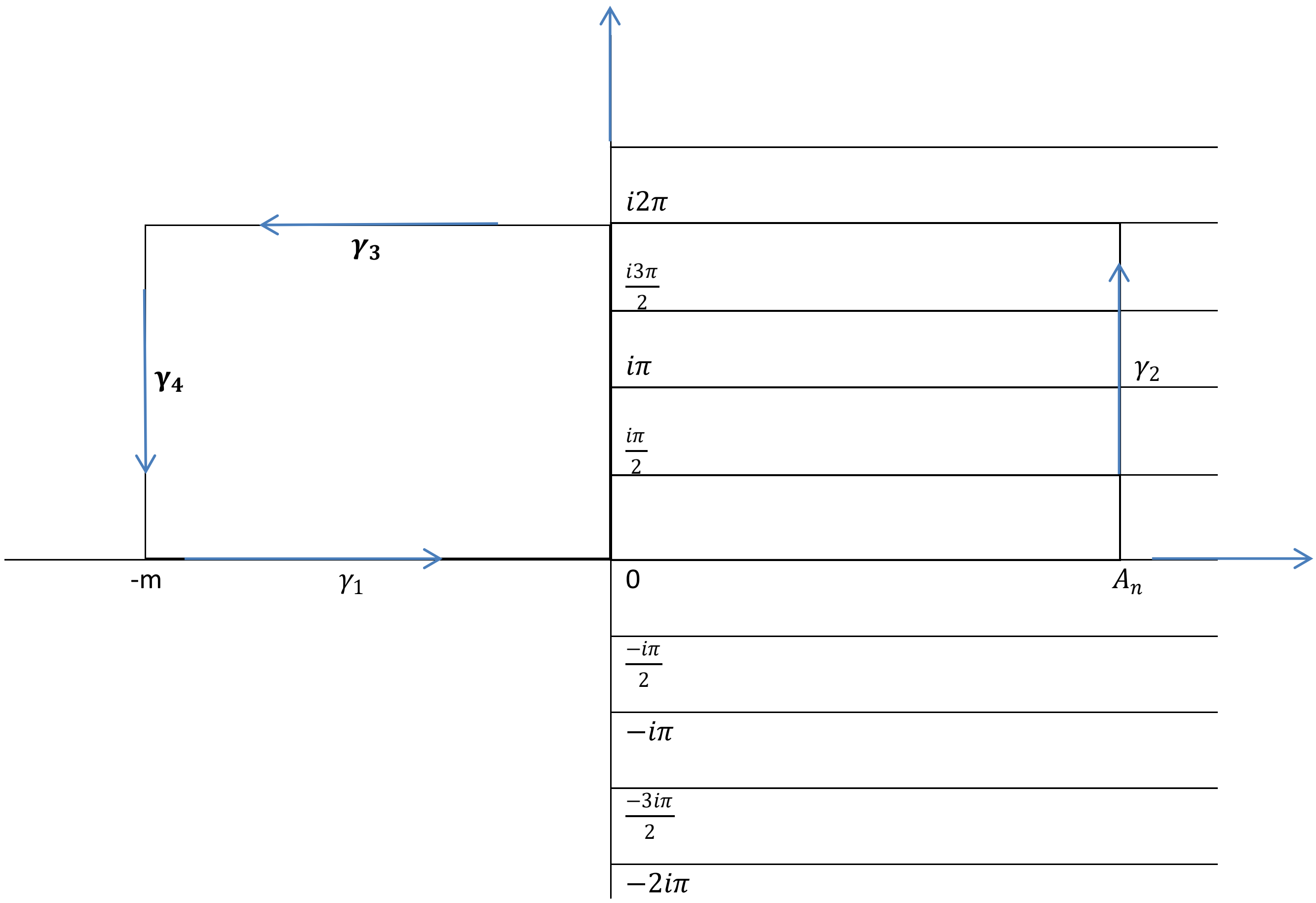
Poles of f in $K_{(n,m)}$ are

$z_k = ln((2k + 1)\pi) + i\frac{\pi}{2}$ and $z'_k = ln((2k + 1)\pi) + i\frac{3\pi}{2} \quad 0 \leq k \leq (n - 1)$

(see the graph below)

The residu formula gives

$$\oint_{\partial K_{(n,m)}} f(z)dz = 2\pi i\left(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k)\right)$$



$i2\pi$
$\frac{i3\pi}{2}$
$i\pi$
$\frac{i\pi}{2}$
0
$\frac{-i\pi}{2}$
$-i\pi$
$\frac{-3i\pi}{2}$
$-2i\pi$
-m
A_n
γ_1
γ_2
γ_3
γ_4

$$\oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz + \oint_{\gamma_4} f(z)dz = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$$

$$\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt - \int_{-m}^{A} \frac{e^{s(t+2\pi i)}}{e^{e^{(t+2\pi i)}}+1} dt - i\int_0^{2\pi} \frac{e^{s(it-m)}}{e^{e^{(it-m)}}+1} dt$$

$$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$$

$$\left(1 - e^{s2\pi i}\right)\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt - ie^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt \qquad (1)$$

$$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$$

Let's calculate $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt$

$\forall\, z \in \mathbb{C}\; with\; |z| \le 1\; |e^z + 1| \ne 0$ so the function$z \to |e^z + 1|$ has aminima $p > 0$ On the cmopact

$\{z \in \mathbb{C}\; with\; |z| \le 1\}$

So $\forall\, z \in \mathbb{C}\; with\; |z| \le 1\; |e^z + 1| \ge p$

$\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|e^{(it-m)}\right| = e^{(-m)} \le 1$ so $\left|e^{e^{(it-m)}} + 1\right| \ge p$

So $\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|\frac{e^{sit}}{e^{e^{(it-m)}}+1}\right| \le \frac{e^{-bt}}{p}$

Since$\int_0^{2\pi} e^{-bt} du < \infty$ So $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt = 0$

When m tends to$+\infty$ the equation (1) becomes

$$\left(1 - e^{s2\pi i}\right)\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$$

Since $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ we have $\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1} dt = -\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt$

$$-\left(1 - e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$$

Let's calculate $\sum_{k=0}^{(n-1)} Res(f,z_k)$

$$Res(f,z_k) = \frac{e^{sz_k}}{e^{e^{z_k}} \times e^{z_k}} = \frac{e^{sz_k}}{(-1)\times e^{z_k}} = -e^{(s-1)z_k} = -e^{(s-1)\left(ln((2k+1)\pi) + i\frac{\pi}{2}\right)} = -\pi^{(s-1)} e^{(s-1)i\frac{\pi}{2}} \times \frac{1}{(2k+1)^{(1-s)}}$$

$$\sum_{k=0}^{(n-1)} Res(f,z_k) = -\pi^{(s-1)} e^{(s-1)i\frac{\pi}{2}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

By the same we have

$$\sum_{k=0}^{(n-1)} Res(f,z'_k) = -\pi^{(s-1)} e^{(s-1)i\frac{3\pi}{2}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

So

$$-\left(1 - e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = -2i\pi^{s}(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}) \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1 - e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = -2i\pi^{s}(-ie^{si\frac{\pi}{2}} + ie^{si\frac{3\pi}{2}}) \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\int_0^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=-2\pi^s(e^{si\frac{\pi}{2}}-e^{si\frac{3\pi}{2}})\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} \quad (2)$$

$$\int_0^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_{\pi}^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(i(t+\pi)+A)}}{e^{e^{(i(t+\pi)+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{-e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(it+i\pi+A)}e^{e^{(it+A)}}}{e^{e^{(it+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}e^{s(it+i\pi+A)}dt-\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{e^{(it+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt-\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{e^{(it+A)}}+1}dt+\frac{1}{si}\left[e^{s(it+i\pi+A)}\right]_0^{\pi}=\left(1-e^{si\pi}\right)\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\frac{1}{si}\left(e^{s(i2\pi+A)}-e^{s(i\pi+A)}\right)$$

$$=\left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt+\frac{1}{si}e^{sA}\left(e^{si2\pi}-e^{si\pi}\right)$$

So equality (2) becomes

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt+\frac{1}{s}e^{sA}\left(e^{si2\pi}-e^{si\pi}\right)$$

$$=-2\pi^s(e^{si\frac{\pi}{2}}-e^{si\frac{3\pi}{2}})\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt-\frac{1}{s}e^{sA}e^{\frac{si\pi}{2}}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} \quad (3)$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{sit}}{e^{ie^{(it+A)}}+1}dt=\int_{\frac{-\pi}{2}}^{0}\frac{e^{sit}}{e^{(e^A(-\sin t+i\cos t))}+1}dt+\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{(e^A(-\sin t+i\cos t))}+1}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{(e^A(\sin t+i\cos t))}+1}dt+\int_0^{\frac{\pi}{2}}\frac{e^{sit}e^{\left(e^A(\sin t-i\cos t)\right)}}{e^{(e^A(\sin t-i\cos t))}+1}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{(e^A(\sin t+i\cos t))}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt+\int_0^{\frac{\pi}{2}}e^{sit}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{(e^A(\sin t+i\cos t))}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt+\frac{1}{si}\left(e^{\frac{si\pi}{2}}-1\right)$$

So equality (3) becomes

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{(e^A(\sin t+i\cos t))}+1}dt-ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt+\frac{ie^{sA}}{si}\left(e^{\frac{si\pi}{2}}-1\right)-\frac{1}{s}e^{sA}e^{\frac{si\pi}{2}}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{(e^A(\sin t+i\cos t))}+1}dt-ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt-\frac{1}{s}e^{sA}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

We multiply by $e^{(3-s)A}$ we get

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{3A}\left(\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)-\frac{1}{s}e^{3A}$$

$$=-2\pi^s e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

Let's calculate $\lim_{n\to+\infty}e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt$

$$\left|\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt\right|\le\int_A^{+\infty}\left|\frac{e^{st}}{e^{e^t}+1}\right|dt=\int_A^{+\infty}\frac{e^{at}}{e^{e^t}+1}dt\le\int_A^{+\infty}\frac{e^{at}}{e^{e^t}}dt\le\frac{1}{e^{\left(\frac{1}{2}e^A\right)}}\int_A^{+\infty}\frac{e^{at}}{e^{\left(\frac{1}{2}e^t\right)}}dt \qquad \frac{1}{e^{e^t}}=\frac{1}{e^{\left(\frac{1}{2}e^t\right)}}\times\frac{1}{e^{\left(\frac{1}{2}e^t\right)}}$$

So $\left|e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt\right|\le\frac{e^{(3-a)A}}{e^{\left(\frac{1}{2}e^A\right)}}\int_A^{+\infty}\frac{e^{at}}{e^{\left(\frac{1}{2}e^t\right)}}dt$

Clearly $\lim_{n\to+\infty}e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt=0$

so

$$-2\pi^s e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)-\frac{1}{s}e^{3A}+o(1) \qquad (4)$$

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{\left(\frac{1}{2}(-s^2+3s-2)\right)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}-e^{it}-(s-1)\left(e^{2it}-e^{it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

lemma :

$\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt=0$ and $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}\right|dt=0$

proof

$$e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt=e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt+e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{e^A(\sin t+i\cos t)}+1}\right|dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{\left|e^{\left(e^A(\sin t+i\cos t)\right)}+1\right|}dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{\left|e^{\left(e^A(\sin t+i\cos t)\right)}\right|-1}dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{e^{\left(e^A\sin t\right)}-1}dt$$

$$\le\frac{1}{e^{\left(\frac{\sqrt{2}e^A}{2}\right)}-1}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}t^3dt$$

$$e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt\le\frac{e^{3A}}{e^{\left(\frac{\sqrt{2}e^A}{2}\right)}-1}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}t^3dt$$

So $\lim_{n\to+\infty}e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt=0$

$$\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt \leq \int_0^{\frac{\pi}{4}}\frac{t^3}{\left|e^{(e^A(\sin t+i\cos t))}\right|-1}dt \leq \int_0^{\frac{\pi}{4}}\frac{t^3}{e^{e^A\sin t}-1}dt = \int_0^{\frac{\sqrt{2}}{2}}\frac{(\arcsin(u))^3}{e^{(e^A u)}-1}\times\frac{1}{\sqrt{1-u^2}}du$$

By substitution ($t = \arcsin(u)$)

We have $\forall u \in \left[0,\frac{\sqrt{2}}{2}\right]$ $\frac{1}{\sqrt{1-u^2}} \leq \sqrt{2}$ so

$$\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt \leq \sqrt{2}\int_0^{\frac{\sqrt{2}}{2}}\frac{(\arcsin(u))^3}{e^{(e^A u)}-1}du$$

The taylor formula with gives

$\forall u \in \left[0,\frac{\sqrt{2}}{2}\right]$ $\arcsin(u) = \arcsin(0) + u \times \frac{1}{\sqrt{1-(\xi_u)^2}}$ where $\xi_u \in]0,u[$

So $\forall u \in \left[0,\frac{\sqrt{2}}{2}\right]$ $\arcsin(u) \leq u\sqrt{2}$ (because $\frac{1}{\sqrt{1-(\xi_u)^2}} \leq \sqrt{2}$)

So $\forall u \in \left[0,\frac{\sqrt{2}}{2}\right]$ $(arcsin\,(u))^3 \leq 2\sqrt{2}u^3$

So $\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt \leq 4\int_0^{\frac{\sqrt{2}}{2}}\frac{u^3}{e^{(e^A u)}-1}du = 4e^{-4A}\int_0^{\frac{e^A\sqrt{2}}{2}}\frac{v^3}{e^v-1}dv \leq 4e^{-4A}\int_0^{+\infty}\frac{v^3}{e^v-1}dv$

(By substitution $e^A u = v$)

So $e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt \leq 4e^{-A}\int_0^{+\infty}\frac{v^3}{e^v-1}dv$

Since $\int_0^{+\infty}\frac{v^3}{e^v-1}dv < \infty$ we get $\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt = 0$

So $\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt = 0$

We deduce that $\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t-i\cos t))}+1}\right|dt = 0$

Let's prove that $\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)(e^{-2it}-e^{-it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt = 0$ and
$\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{sit}-e^{it}-(s-1)(e^{2it}-e^{it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt = 0$

The taylor formula with integral gives

$$\exists M \in \mathbb{R}^{*+}\ \forall\, t \in \left[0,\frac{\pi}{2}\right]\quad \left|e^{-sit} - e^{-it} - (s-1)\left(e^{-2it} - e^{-it}\right) - \frac{1}{2}(-s^2+3s-2)t^2\right| \leq Mt^3$$

So $\left|\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)(e^{-2it}-e^{-it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt\right| \leq M\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt$

So $\left|e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)(e^{-2it}-e^{-it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt\right| \leq Me^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt$

Using the lemma we get $\lim\limits_{n\to+\infty}\left|e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)(e^{-2it}-e^{-it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt\right| = 0$

So $\lim\limits_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)(e^{-2it}-e^{-it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt = 0$

By the same we get $\lim_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}} \frac{e^{sit}-e^{it}-(s-1)(e^{2it}-e^{it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt = 0$

We deduce that $(as\ n\to+\infty)$

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right) = e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right) + e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(\frac{1}{2}(-s^2+3s-2))t^2}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+o(1) \qquad (5)$$

We deduce from equality (4)

$$-2\pi^{s}e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} = e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right) + e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(\frac{1}{2}(-s^2+3s-2))t^2}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+o(1)$$

Let the complex function h $\quad\forall z\in\mathbb{C}\quad h(z)=\frac{e^{qz}}{e^{e^{z}}+1}\quad q\in\mathbb{N}^*$

The residu formula on $K_{(n,m)}$ gives

$$\left(1-e^{q2\pi i}\right)\int_{-m}^{A}\frac{e^{qt}}{e^{e^{t}}+1}dt + i\int_0^{2\pi}\frac{e^{q(it+A)}}{e^{e^{(it+A)}}+1}dt - ie^{-sm}\int_0^{2\pi}\frac{e^{qit}}{e^{e^{(it-m)}}+1}dt$$

$$= 2\pi i\left(\sum_{k=0}^{(n-1)}Res(h,z_k)+\sum_{k=0}^{(n-1)}Res(h,z'_k)\right)$$

When m tends to$+\infty$ we get

$$i\int_0^{2\pi}\frac{e^{q(it+A)}}{e^{e^{(it+A)}}+1}dt == -2\pi^{q}\left(e^{qi\frac{\pi}{2}}-e^{qi\frac{3\pi}{2}}\right)\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

By the same as above we have

$$i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{q(it+A)}}{e^{ie^{(it+A)}}+1}dt - \frac{1}{q}e^{qA}e^{\frac{qi\pi}{2}} = -2\pi^{q}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

$$i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{q(it+A)}}{e^{ie^{(it+A)}}+1}dt = -2\pi^{q}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}} + \frac{1}{q}e^{qA}e^{\frac{qi\pi}{2}}$$

$$ie^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-qit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - ie^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{qit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt - \frac{1}{q}e^{3A}$$

$$= -2\pi^{q}e^{(3-q)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-qit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt - i\int_0^{\frac{\pi}{2}}\frac{e^{qit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$= -2\pi^{q}e^{(3-q)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}} + \frac{1}{q}e^{3A} \qquad (6)$$

Let's calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$

Using equality (6) for $q=1$ we get

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=-2\pi e^{2A}n+e^{3A}=-\pi e^{2A}(2n+\varepsilon-\varepsilon)+e^{3A}$$

$$=-\pi e^{2A}(2n+\varepsilon)+\pi\varepsilon e^{2A}+e^{3A}=-e^{3A}+\pi\varepsilon e^{2A}+e^{3A}=\pi\varepsilon e^{2A}$$

So $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=\pi\varepsilon e^{2A}$

Let's calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(e^{-2it}-e^{-it})}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$

Using equality (6) for $q=2$ we get

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-2it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{2it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$=-2\pi^2e^A\sum_{k=0}^{(n-1)}(2k+1)+\frac{1}{2}e^{3A}$$

$$=-2\pi^2e^An^2+\frac{1}{2}e^{3A}$$

So

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(e^{-2it}-e^{-it})}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=-2\pi^2e^An^2+\frac{1}{2}e^{3A}-\pi\varepsilon e^{2A}$$

$$=\frac{1}{2}e^A(-4\pi^2n^2+e^{2A}-2\pi\varepsilon e^A\ )=-\frac{1}{2}e^A\left(-4\pi^2n^2+\left((2n+\varepsilon)\pi\right)^2-2\pi\varepsilon(2n+\varepsilon)\pi\right)$$

$$=\frac{1}{2}e^A\pi^2\left(-4n^2+(2n+\varepsilon)^2-2\varepsilon(2n+\varepsilon)\right)=\frac{1}{2}e^A\pi^2(-4n^2+4n^2+4\varepsilon n+\varepsilon^2-4\varepsilon n-2\varepsilon^2)=-\frac{1}{2}\pi^2\varepsilon^2e^A$$

By the same we can calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$ we find

$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=C+o(1)$ where C is constante depending only on ε

(By using the equation (5) wich is also true for $q\in\mathbb{N}^*$ we can take for example $q=3$ there is a lot of calculus)

So equality (5) becomes

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=\pi\varepsilon e^{2A}-\frac{1}{2}(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}(-s^2+3s-2)C+o(1)$$

Thus equality (4) gives

$$-2\pi^se^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$=\pi\varepsilon e^{2A}-\frac{1}{2}(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}(-s^2+3s-2)C-\frac{1}{s}e^{3A}+o(1)$$

$$2\pi^sse^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}=e^{3A}-\pi\varepsilon se^{2A}+\frac{1}{2}s(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}s(s^2-3s+2)C+o(1)$$

$2\pi^s s\big((2n+\varepsilon)\pi\big)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$

$=\big((2n+\varepsilon)\pi\big)^3-\pi\varepsilon s\big((2n+\varepsilon)\pi\big)^2+\frac{1}{2}s(s-1)\pi^2\varepsilon^2(2n+\varepsilon)\pi+\frac{1}{2}s(s^2-3s+2)C+o(1)$

$2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$

$=(2n+\varepsilon)^3-\varepsilon s(2n+\varepsilon)^2+\frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)+\frac{1}{2\pi^3}s(s^2-3s+2)C+o(1)$

Let $C'(s)=\frac{1}{2\pi^3}s(s^2-3s+2)C$ so

$$2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}=(2n+\varepsilon)^3-\varepsilon s(2n+\varepsilon)^2+\frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)+C'(s)+o(1) \qquad (7)$$

We have also

$2s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$

$$=(2n+\varepsilon)^s-\varepsilon s(2n+\varepsilon)^{(s-1)}+\frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)^{(s-2)}+C'(s)(2n+\varepsilon)^{(s-3)}+o\big((2n+\varepsilon)^{(s-3)}\big) \qquad (8)$$

Let the sequence U such that $\forall\, n\in\ \mathbb{N}^*\ \ U(n,s)=2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}-\frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)$

So $\forall\, n\in\ \mathbb{N}^*\ \ U(n,s)=(2n+\varepsilon)^3-\varepsilon s(2n+\varepsilon)^2+C'(s)+o(1)$

Since $\eta(1-s)=0$ we have

$\forall\, n\in\ \mathbb{N}^*\ \ U\big(n,(1-s)\big)=(2n+\varepsilon)^3-\varepsilon(1-s)(2n+\varepsilon)^2+C'(1-s)+o(1)$

So $\forall\, n\in\ \mathbb{N}^*\ \ U(n,s)+U\big(n,(1-s)\big)=2(2n+\varepsilon)^3-\varepsilon(2n+\varepsilon)^2+C'(s)+C'(1-s)+o(1)$

Since $\eta(\bar{s})=0\ and\ \eta(1-\bar{s})=0$ we have

$\forall\, n\in\ \mathbb{N}^*\ \ U(n,\bar{s})+U\big(n,(1-\bar{s})\big)=2(2n+\varepsilon)^3-\varepsilon(2n+\varepsilon)^2+C'(\bar{s})+C'(1-\bar{s})+o(1)$

So $\forall\, n\in\ \mathbb{N}^*\ \ \Big(U(n,s)+U\big(n,(1-s)\big)\Big)-\Big(U(n,\bar{s})+U\big(n,(1-\bar{s})\big)\Big)$

$=(C'(s)+C'(1-s))-\big(C'(\bar{s})+C'(1-\bar{s})\big)+o(1)$

Let the sequence V such that

$\forall\, n\in\ \mathbb{N}^*\ \ V(n,s)=\Big(U(n,s)+U\big(n,(1-s)\big)\Big)-\Big(U(n,\bar{s})+U\big(n,(1-\bar{s})\big)\Big)$

We have $\forall\, n\in\ \mathbb{N}^*\ \ V(n,s)=(C'(s)+C'(1-s))-\big(C'(\bar{s})+C'(1-\bar{s})\big)+o(1)$

So $\lim\limits_{n\to+\infty}V(n,s)=(C'(s)+C'(1-s))-\big(C'(\bar{s})+C'(1-\bar{s})\big)$

So $\lim\limits_{n\to+\infty}\big(V((n+1),s)-V(n,s)\big)=0$

$\forall\, n\in\ \mathbb{N}^*\ V\big((n+1),s\big)-V(n,s)=\Big[\big(U((n+1),s)-U(n,s)\big)+\Big(U\big((n+1),(1-s)\big)-U(n,(1-s))\Big)\Big]$

$-\Big[\big(U((n+1),\bar{s})-U(n,\bar{s})\big)+\Big(U\big((n+1),(1-\bar{s})\big)-U\big(n,(1-\bar{s})\big)\Big)\Big]$

For each $n \in \mathbb{N}^*$ let's calculate $\left(U((n+1),s)-U(n,s)\right)$

$\forall\, n \in \mathbb{N}^*\ U\big((n+1),s\big)-U(n,s)$

$= 2s(2(n+1)+\varepsilon)^{(3-s)}\sum_{k=0}^{n}\frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$

$-\frac{1}{2}s(s-1)\varepsilon^2\big((2(n+1)+\varepsilon)-(2n+\varepsilon)\big)$

$= 2s(2n+2+\varepsilon)^{(3-s)}\sum_{k=0}^{n}\frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} - s(s-1)\varepsilon^2$

$= 2s(2n+2+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} - 2s(2n+\varepsilon)^{(3-s)}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} + 2s(2n+2+\varepsilon)^{(3-s)} \times \frac{1}{(2n+1)^{(1-s)}}$

$-s(s-1)\varepsilon^2$

$= \big((2n+2+\varepsilon)^{(3-s)}-(2n+\varepsilon)^{(3-s)}\big)2s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} + 2s(2n+2+\varepsilon)^{(3-s)} \times (2n+1)^{(s-1)} - s(s-1)\varepsilon^2$

We have $(as\ n \to +\infty)$

$(2n+2+\varepsilon)^{(3-s)} = (2n+\varepsilon)^{(3-s)}\left(1+\frac{2}{2n+\varepsilon}\right)^{(3-s)}$

$= (2n+\varepsilon)^{(3-s)}\left(1+\frac{2(3-s)}{2n+\varepsilon}+\frac{4(3-s)(2-s)}{2(2n+\varepsilon)^2}+\frac{8(3-s)(2-s)(1-s)}{6(2n+\varepsilon)^3}+o(\frac{1}{(2n+\varepsilon)^3})\right)$

$= (2n+\varepsilon)^{(3-s)} + 2(3-s)(2n+\varepsilon)^{(2-s)} + 2(3-s)(2-s)(2n+\varepsilon)^{(1-s)}$

$+\frac{4}{3}(3-s)(2-s)(1-s)(2n+\varepsilon)^{(-s)} + o((2n+\varepsilon)^{(-s)})$

So $\big((2n+2+\varepsilon)^{(2-s)}-(2n+\varepsilon)^{(2-s)}\big)2s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} =$

$\Big[\ 2(3-s)(2n+\varepsilon)^{(2-s)} + 2(3-s)(2-s)(2n+\varepsilon)^{(1-s)} + \frac{4}{3}(3-s)(2-s)(1-s)(2n+\varepsilon)^{(-s)} + o((2n+\varepsilon)^{(-s)})\ \Big] \times$
$\Big[\ (2n+\varepsilon)^{s} - \varepsilon s(2n+\varepsilon)^{(s-1)} + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)^{(s-2)} + C'(s)(2n+\varepsilon)^{(s-3)} + o((2n+\varepsilon)^{(s-3)})\Big]$

$= 2(3-s)(2n+\varepsilon)^2 - 2\varepsilon s(3-s)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 + 2(3-s)(2-s)(2n+\varepsilon) - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1)$

$= 2(3-s)(2n+\varepsilon)^2 + \big(-2\varepsilon s(3-s)+2(3-s)(2-s)\big)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1)$

So $\big((2n+2+\varepsilon)^{(2-s)}-(2n+\varepsilon)^{(2-s)}\big)2s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$

$= 2(3-s)(2n+\varepsilon)^2 + \big(-2\varepsilon s(3-s)+2(3-s)(2-s)\big)(2n+\varepsilon) + s(s-1)(3-s)\varepsilon^2 - 2\varepsilon s(3-s)(2-s)$

$+\frac{4}{3}(3-s)(2-s)(1-s) + o(1)$ (9)

We have $(as\ n \to +\infty)$

$(2n+2+\varepsilon)^{(3-s)} \times (2n+1)^{(s-1)} = (2n+1+(1+\varepsilon))^{(3-s)} \times (2n+1)^{(s-1)}$

$= (2n+1))^{(3-s)}\left(1+\frac{(1+\varepsilon)}{2n+1}\right)^{(3-s)} \times (2n+1)^{(s-1)}$

$= (2n+1)^2 \left(1 + \frac{(3-s)(1+\varepsilon)}{2n+1} + \frac{(3-s)(2-s)(1+\varepsilon)^2}{2(2n+1)^2} + o(\frac{1}{(2n+1)^2})\right)$

$= (2n+1)^2 + (3-s)(1+\varepsilon)(2n+1) + \frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2 + o(1)$

$= (2n+\varepsilon+(1-\varepsilon))^2 + (3-s)(1+\varepsilon)(2n+\varepsilon+(1-\varepsilon)) + \frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2 + o(1)$

$= (2n+\varepsilon)^2 + 2(1-\varepsilon)(2n+\varepsilon) + (1-\varepsilon)^2 + (3-s)(1+\varepsilon)(2n+\varepsilon) + (3-s)(1+\varepsilon)(1-\varepsilon)$

$+\frac{1}{2}(3-s)(2-s)(1+\varepsilon)^2 + o(1)$

$= (2n+\varepsilon)^2 + \big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)(2n+\varepsilon) + (1-\varepsilon)^2 + (3-s)(1-\varepsilon^2)$

$+\frac{1}{2}(3-s)(2-s)(1+2\varepsilon+\varepsilon^2) + o(1)$

$= (2n+\varepsilon)^2 + \big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)(2n+\varepsilon) + (1-\varepsilon)^2 + (3-s) - (3-s)\varepsilon^2$

$+\frac{1}{2}(3-s)(2-s) + (3-s)(2-s)\varepsilon + \frac{1}{2}(3-s)(2-s)\varepsilon^2 + o(1)$

$= (2n+\varepsilon)^2 + \big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)(2n+\varepsilon) + \frac{1}{2}(3-s)(-s)\varepsilon^2 + (3-s)(2-s)\varepsilon$

$+\frac{1}{2}(3-s)(4-s) + (1-\varepsilon)^2 + o(1)$

$= (2n+\varepsilon)^2 + \big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)(2n+\varepsilon) - \frac{1}{2}s(3-s)\varepsilon^2 + (3-s)(2-s)\varepsilon$

$+\frac{1}{2}(3-s)(4-s) + (1-\varepsilon)^2 + o(1)$

so

$2s(2n+2+\varepsilon)^{(2-s)} \times (2n+1)^{(s-1)}$

$= 2s(2n+\varepsilon)^2 + 2s\big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)(2n+\varepsilon) - s^2(3-s)\varepsilon^2 + 2s(3-s)(2-s)\varepsilon$

$+s(3-s)(4-s) + 2s(1-\varepsilon)^2 + o(1)$ (10)

From equalties (9) and (10) we deduce that

$U\big((n+1),s\big) - U(n,s)$

$= 6(2n+\varepsilon)^2 + \Big(-2\varepsilon s(3-s) + 2(3-s)(2-s) + 2s\big((3-s)(1+\varepsilon) + 2(1-\varepsilon)\big)\Big)(2n+\varepsilon)$

$+s(s-1)(3-s)\varepsilon^2 - 2\varepsilon s(3-s)(2-s) + \frac{4}{3}(3-s)(2-s)(1-s) - s^2(3-s)\varepsilon^2 + 2s(3-s)(2-s)\varepsilon$

$+s(3-s)(4-s) + 2s(1-\varepsilon)^2 - s(s-1)\varepsilon^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(-\varepsilon s(3-s) + (3-s)(2-s) + s(3-s)(1+\varepsilon) + 2s(1-\varepsilon)\big)(2n+\varepsilon)$

$+\big(s(s-1)(2-s) - s^2(3-s)\big)\varepsilon^2 + \frac{4}{3}(3-s)(2-s)(1-s) + s(3-s)(4-s) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(-\varepsilon s(3-s) + (3-s)(2-s) + s(3-s) + \varepsilon s(3-s) + 2s(1-\varepsilon)\big)(2n+\varepsilon)$

$-s(s^2-3s+2+3s-s^2)\varepsilon^2 + \frac{1}{3}[4(3-s)(2-s)(1-s) + 3s(3-s)(4-s)] + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big((3-s)(2-s) + s(3-s) + 2s(1-\varepsilon)\big)(2n+\varepsilon)$

$-2s\varepsilon^2 + \frac{1}{3}[4(3-s)(2-s)(1-s) + 3s(3-s)(4-s)] + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(s^2 - 5s + 6 + 3s - s^2 + s(1-\varepsilon)\big)(2n+\varepsilon)$

$-2s\varepsilon^2 + \frac{1}{3}(3-s)[4(s^2-3s+2) + 12s - 3s^2] + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2\big(-2s + 6 + 2s(1-\varepsilon)\big)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 + 2(-2s\varepsilon + 6)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 - 4(s\varepsilon - 3)(2n+\varepsilon) - 2s\varepsilon^2 + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + o(1)$

$= 6(2n+\varepsilon)^2 - 4(s\varepsilon - 3)(2n+\varepsilon) + \frac{1}{3}(3-s)(s^2+8) + 2s(1-\varepsilon)^2 + 2s(1-2\varepsilon) + o(1)$

Let $p(s) = \frac{1}{3}(3-s)(s^2+8) + 2s(1-2\varepsilon)$

So $\quad \forall\, n \in \mathbb{N}^*\ U\big((n+1), s\big) - U(n,s) = 6(2n+\varepsilon)^2 - 4(s\varepsilon - 3)(2n+\varepsilon) + p(s) + o(1)$

We have also

$\forall\, n \in \mathbb{N}^*\ U\big((n+1),(1-s)\big) - U(n,(1-s)) = 6(2n+\varepsilon)^2 - 4\big((1-s)\varepsilon - 3\big)(2n+\varepsilon) + p(1-s) + o(1)$

So $\big(U((n+1),s) - U(n,s)\big) + \Big(U\big((n+1),(1-s)\big) - U(n,(1-s))\Big)$

$= 12(2n+\varepsilon)^2 - 4(\varepsilon - 6)(2n+\varepsilon) + p(s) + p(1-s) + o(1)$

We have also

$\forall\, n \in \mathbb{N}^*\ \big(U((n+1),\bar{s}) - U(n,\bar{s})\big) + \Big(U\big((n+1),(1-\bar{s})\big) - U\big(n,(1-\bar{s})\big)\Big)$

$= 12(2n+\varepsilon)^2 - 4(\varepsilon - 6)(2n+\varepsilon) + p(\bar{s}) + p(1-\bar{s}) + o(1)$

So $\quad \forall\, n \in \mathbb{N}^*\ V\big((n+1),s\big) - V(n,s) = (p(s) + p(1-s)) - (p(\bar{s}) + p(1-\bar{s})) + o(1)$

Since $\quad \lim_{n\to+\infty}\big(V((n+1),s) - V(n,s)\big) = 0$

We deduce that $\quad (p(s) + p(1-s)) - \big(p(\bar{s}) + p(1-\bar{s})\big) = 0$

Let's calculate $(p(s) + p(1-s))$

Let $p(s) = \frac{1}{3}(3-s)(s^2+8) + 2s(1-2\varepsilon) = -\frac{1}{3}(s^3 + 8s - 3s^2 - 24) + 2s(1-2\varepsilon)$

$= -\frac{1}{3}(s^3 - 3s^2) - \frac{8}{3}s + 8 + 2s(1-2\varepsilon) = -\frac{1}{3}(s^3 - 3s^2) - 2s\left(2\varepsilon + \frac{1}{3}\right) + 8$

So $\ p(s) = -\frac{1}{3}(s^3 - 3s^2) - 2s\left(2\varepsilon + \frac{1}{3}\right) + 8$

$p(1-s) = -\frac{1}{3}((1-s)^3 - 3(1-s)^2) - 2(1-s)(2\varepsilon + \frac{1}{3}) + 8$

$p(s) + p(1-s) = -\frac{1}{3}(s^3 - 3s^2 + (1-s)^3 - 3(1-s)^2) - 2(2\varepsilon + \frac{1}{3}) + 16$

$s^3 - 3s^2 + (1-s)^3 - 3(1-s)^2 = s^3 - 3s^2 + 1 - 3s + 3s^2 - s^3 - 3s^2 + 6s - 3 = -3s^2 + 3s - 2$

$p(s) + p(1-s) = -\frac{1}{3}(-3s^2 + 3s - 2) - 2\left(2\varepsilon + \frac{1}{3}\right) + 16 = (s^2 - s) - 2(2\varepsilon + \frac{1}{3}) + 16 + \frac{2}{3}$

$p(s) + p(1-s) = (s^2 - s) - 4\varepsilon + 16$

$p(\bar{s}) + p(1-\bar{s}) = (\bar{s}^2 - \bar{s}) - 4\varepsilon + 16$

So $(p(s) + p(1-s)) - (p(\bar{s}) + p(1-\bar{s})) = (s^2 - s) - (\bar{s}^2 - \bar{s})$

We deduce that $(s^2 - s) - (\bar{s}^2 - \bar{s}) = 0$

So $s^2 - s - \bar{s}^2 + \bar{s} = 0$

So $s^2 - \bar{s}^2 - (s - \bar{s}) = 0$

So $(s - \bar{s})(s + \bar{s}) - (s - \bar{s}) = 0$

So $(s - \bar{s})(s + \bar{s} - 1) = 0$

So $2ib(2a - 1) = 0$

Since $b \neq 0$ we have $2a - 1 = 0$

Thus $a = \frac{1}{2}$

So if there exist a complex number s with $0 < Re(s) < 1$ such that $\zeta(s) = 0$ then $Re(s) = \frac{1}{2}$

So Riemann hypothesis seems to be true but it is false (let's see the rest)

Let the sequence W such that

$$\forall\, n \in \mathbb{N}^* \;\; W(n,s) = 2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} + \varepsilon s(2n+\varepsilon)^2 - \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)$$

So $\forall\, n \in \mathbb{N}^* \;\; W(n,s) = (2n+\varepsilon)^3 + C'(s) + o(1)$

Since $\eta(1-s) = 0$ we have

$\forall\, n \in \mathbb{N}^* \;\; W\big(n,(1-s)\big) = (2n+\varepsilon)^3 + C'(1-s) + o(1)$

So $\forall\, n \in \mathbb{N}^* \;\; W(n,s) - W\big(n,(1-s)\big) = C'(s) - C'(1-s) + o(1)$

Let the sequence T such that

$$\forall\, n \in \mathbb{N}^* \;\; T(n,s) = W(n,s) - W\big(n,(1-s)\big)$$

We have $\forall\, n \in \mathbb{N}^* \;\; T(n,s) = C'(s) - C'(1-s) + o(1)$

So $\lim_{n\to+\infty} T(n,s) = C'(s) - C'(1-s)$

So $\lim_{n\to+\infty} \big(T((n+1),s) - T(n,s)\big) = 0$

$\forall\, n \in \mathbb{N}^* \; T\big((n+1),s\big) - T(n,s) = \Big(W\big((n+1),s\big) - W(n,s)\Big) - \Big(W\big((n+1),(1-s)\big) - W(n,(1-s))\Big)$

For each $n \in \mathbb{N}^*$ let's calculate $\left(W((n+1),s) - W(n,s)\right)$

$$\forall\, n \in \mathbb{N}^* \quad W(n,s) = 2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} + \varepsilon s(2n+\varepsilon)^2 - \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)$$

$$\forall\, n \in \mathbb{N}^* \quad W(n,s) = U(n,s) + \varepsilon s(2n+\varepsilon)^2$$

$$\forall\, n \in \mathbb{N}^* \quad \Big(W\big((n+1),s\big) - W(n,s)\Big) = \Big(U\big((n+1),s\big) - U(n,s)\Big) + (\varepsilon s(2(n+1)+\varepsilon)^2 - \varepsilon s(2n+\varepsilon)^2)$$

$$= \Big(U\big((n+1),s\big) - U(n,s)\Big) + \varepsilon s((2n+\varepsilon+2)^2 - (2n+\varepsilon)^2)$$

$$= \Big(U\big((n+1),s\big) - U(n,s)\Big) + \varepsilon s(4(2n+\varepsilon)+4)$$

$$= \Big(U\big((n+1),s\big) - U(n,s)\Big) + 4\varepsilon s\,(2n+\varepsilon) + 4\varepsilon s$$

We have $\quad \forall\, n \in \mathbb{N}^* \; U\big((n+1),s\big) - U(n,s) = 6(2n+\varepsilon)^2 - 4(s\varepsilon-3)(2n+\varepsilon) + p(s) + o(1)$

where $p(s) = \frac{1}{3}(3-s)(s^2+8) + 2s(1-2\varepsilon)$

so $\Big(W\big((n+1),s\big) - W(n,s)\Big) = 6(2n+\varepsilon)^2 - 4(s\varepsilon-3)(2n+\varepsilon) + p(s) + 4\varepsilon s\,(2n+\varepsilon) + 4\varepsilon s + o(1)$

so $\Big(W\big((n+1),s\big) - W(n,s)\Big) = 6(2n+\varepsilon)^2 + 12(2n+\varepsilon) + p(s) + 4\varepsilon s + o(1)$

$$p(s) + 4\varepsilon s = \frac{1}{3}(3-s)(s^2+8) + 2s(1-2\varepsilon) + 4\varepsilon s = \frac{1}{3}(3-s)(s^2+8) + 2s$$

So $\Big(W\big((n+1),s\big) - W(n,s)\Big) = 6(2n+\varepsilon)^2 + 12(2n+\varepsilon) + q(s) + o(1)$

where $q(s) = \frac{1}{3}(3-s)(s^2+8) + 2s$

Since $\eta(1-s) = 0$ we have also

$$\Big(W\big((n+1),(1-s)\big) - W(n,(1-s))\Big) = 6(2n+\varepsilon)^2 + 12(2n+\varepsilon) + q(1-s) + o(1)$$

$$\Big(W\big((n+1),s\big) - W(n,s)\Big) - \Big(W\big((n+1),(1-s)\big) - W\big(n,(1-s)\big)\Big) = q(s) - q(1-s) + o(1)$$

So $T\big((n+1),s\big) - T(n,s) = q(s) - q(1-s) + o(1)$

So $\lim\limits_{n\to+\infty}\big(T((n+1),s) - T(n,s)\big) = q(s) - q(1-s)$

Since $\lim\limits_{n\to+\infty}\big(T((n+1),s) - T(n,s)\big) = 0$ we have $q(s) - q(1-s) = 0$

Let's calculate $\big(q(s) - q(1-s)\big)$

$$q(s) = \frac{1}{3}(3-s)(s^2+8) + 2s = -\frac{1}{3}[(s-3)(s^2+8) - 6s] = -\frac{1}{3}[s^3 + 8s - 3s^2 - 24 - 6s]$$

$$= -\frac{1}{3}(s^3 - 3s^2 + 2s) + 8 = -\frac{1}{3}s(s-1)(s-2) + 8$$

So $q(s) = -\frac{1}{3}s(s-1)(s-2) + 8$

$$q(1-s) = -\frac{1}{3}(1-s)(1-s-1)(1-s-2) + 8 = -\frac{1}{3}(1-s)(-s)(-s-1) + 8 = \frac{1}{3}s(s-1)(s+1) + 8$$

So $q(1-s)=\frac{1}{3}s(s-1)(s+1)+8$

We have $q(s)-q(1-s)=0$

So $-\frac{1}{3}s(s-1)(s-2)-\frac{1}{3}s(s-1)(s+1)=0$

So $s(s-1)(s-2)+s(s-1)(s+1)=0$

So $-3s(s-1)(2s-1)=0$

So $s=0$ or $s=1$ or $s=\frac{1}{2}$ (this is a contradiction because $0<a<1$ and $b\neq 0$)

So there is no complex number s $with$ $0<Re(s)<1$ and $Im(s)\neq 0$ such that $\eta(s)=0$

So there is no complex number s $with$ $0<Re(s)<1$ and $Im(s)\neq 0$ such that $\zeta(s)=0$

So Riemann hypothesis is false .