Nilakantha’s formula for pi

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Abstract. The Nilakantha series was developed in the 15th century as a way to calculate the value of pi.

1. Introduction

The number pi is defined by

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right) \tag{1}$$

Nilakantha series is described as

$$\pi = 3+4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+3)^3-(2n+3)} = 3+4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)(2n+3)(2n+4)} \tag{2}$$

The series was published in the 15th century by the Indian mathematician Nilakantha Samayaji (1445-1545).

In this note we give some formulas related to (2).

2. Formulas

Entry 1.

$$\pi = 3+\frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\binom{2n+2}{2n-1}} \tag{3}$$

$$\pi = 3+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2(n+1)+n(n+1)^2} \tag{4}$$

$$\pi = 3+\frac{1}{6}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{1^2+2^2+3^2+\ldots+n^2} \tag{5}$$

$$\pi = 3+2\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k+1} \tag{6}$$

$$\pi = 3+\sum_{n=1}^{\infty}\frac{4n+5}{(n+1)(n+2)(2n+1)(2n+3)}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} \tag{7}$$

$$\pi = 3+2\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k(k+1)} \tag{8}$$

$$\pi = 3+4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n+1)}{(2n+1)(2n+3)}\sum_{k=1}^{n}\frac{1}{k(k+1)} \tag{9}$$

$$\pi = 3+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(k+1)(2k+1)} \tag{10}$$

$$\pi = 3+6\sum_{n=1}^{\infty}\frac{1}{(2n+1)(2n+3)(4n+3)(4n+5)} \tag{11}$$

$$\pi = 3 + 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)(2n+3)} - 12\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)(2n+3)(2n+4)} \tag{12}$$

$$\pi = 3 + 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)(2n+4)^2} + 8\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)(2n+3)(2n+4)^2} \tag{13}$$

$$\pi = 3 + 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)^2(2n+3)} - 8\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)^2(2n+3)(2n+4)} \tag{14}$$

$$\pi = 3 + 24\sum_{n=1}^{\infty} \frac{1}{(4n-2)(4n-1)(4n+1)(4n+2)} \tag{15}$$

$$\pi = 3 + 24\sum_{n=1}^{\infty} \frac{1}{(16n^2-4)(16n^2-1)} = 3 + 6\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)(16n^2-1)} \tag{16}$$

Entry 2.

$$\pi = 3 + \sum_{n=0}^{\infty} \left(1 - 2^{-2n-2}\right) 2^{-2n-3}\, \zeta(2n+4) \tag{17}$$

$$\pi = 2 + \sum_{n=0}^{\infty} \left(1 - 2^{-2n-1}\right) 2^{-2n}\, \zeta(2n+2) \tag{18}$$

$$\pi = 3 + \frac{1}{6} - \sum_{n=0}^{\infty} (-1)^n \left(1 - 2^{-n-1}\right)\left(1 - \left(1 - 2^{-n-2}\right)\zeta(n+3)\right) \tag{19}$$

$$\pi = 3 + \sum_{n=0}^{\infty} \left(1 - 2^{-n-1}\right)\left(1 - \left(1 - 2^{-n-2}\right)\zeta(n+3)\right) \tag{20}$$

$$\pi = 3 + 4\sum_{n=0}^{\infty} \left(1 - 2^{-4n-6}\left(\zeta\left(2n+3, \frac{1}{4}\right) - \zeta\left(2n+3, \frac{3}{4}\right)\right)\right) \tag{21}$$

$$\pi = 3 + \sum_{n=0}^{\infty} 2^{-2n-1}\, \Phi\left(-1, 2n+3, \frac{3}{2}\right) \tag{22}$$

$$\pi = 3 + 4\sum_{n=0}^{\infty} \left(1 - 2^{-2n-3}\, \Phi\left(-1, 2n+3, \frac{1}{2}\right)\right) \tag{23}$$

Remark 1: $\zeta(x)$ is the Riemann zeta function:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}\ ,\ \mathrm{Re}(x) > 1 \tag{24}$$

Remark 2: $\zeta(x,y)$ is the Hurwitz zeta function:

$$\zeta(x, u) = \sum_{n=0}^{\infty} \frac{1}{(u+n)^x}\ ,\ u > 0,\ \mathrm{Re}(x) > 1 \tag{25}$$

Remark 3: $\Phi(z,s,u)$ is the Lerch transcendent:

$$\Phi(z, x, u) = \sum_{n=0}^{\infty} \frac{z^n}{(u+n)^x}\ ,\ u > 0,\ |z| < 1, \mathrm{Re}(x) > 0;\ \ |z| = 1,\ \mathrm{Re}(x) > 1 \tag{26}$$

Entry 3.

$$\pi = 3 + 2 \cdot \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(2k+2)(2k+3)(2k+4)} \tag{27}$$

$$\pi = 3 + 4 \cdot \sum_{n=0}^{\infty} 3^{-n-1} \sum_{k=0}^{n} \binom{n}{k} (-2)^k \left(\frac{1}{2k+2} - \frac{2}{2k+3} + \frac{1}{2k+4}\right) \tag{28}$$

$$\pi = 3 + \sum_{n=0}^{\infty} \left(\frac{1}{(n+1)(n+2)} - \frac{4}{(n+1)(n+2)(n+3)} + \frac{6}{(n+1)(n+2)(n+3)(n+4)} \right) \sum_{k=0}^{[n/2]} (-2)^{-k} \binom{n-k}{k} \tag{29}$$

$$\pi = 3 + \sum_{n=0}^{\infty} \left(\frac{1}{(n+3)(n+4)} \right) \sum_{k=0}^{[n/2]} (-2)^{-k} \binom{n-k}{k} \tag{30}$$

Entry 4.

$$\pi = 3 + \sum_{n=0}^{\infty} (-1)^n 2^{-2n-3} \left(\frac{4}{2n+2} - \frac{4}{2n+3} + \frac{1}{2n+4} \right) + \sum_{n=0}^{\infty} 2^{-n-4} \left(\frac{2}{n+3} - \frac{1}{n+4} \right) \sum_{k=0}^{[n/2]} (-2)^{-k} \binom{n-k}{k} \tag{31}$$

$$\pi = 3 + 2 \sum_{n=0}^{\infty} (-1)^n \phi^{-2n-4} \left(\frac{\phi^2}{2n+2} - \frac{2\phi}{2n+3} + \frac{1}{2n+4} \right) + \sum_{n=0}^{\infty} \phi^{-2n-8} \left(\frac{\phi^2}{n+3} - \frac{1}{n+4} \right) \sum_{k=0}^{[n/2]} (-2)^{-k} \binom{n-k}{k} \tag{32}$$

where $\phi = \frac{1+\sqrt{5}}{2}$.

Entry 5.

$$2\sqrt{3}\,\pi - 6\ln\left(\frac{4}{3}\right) = 9 + 4 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-n}}{(2n+2)(2n+3)(2n+4)} \tag{33}$$

3. Future Research

Recall that

$$\binom{z}{n} = \frac{z(z-1)(z-2)\ldots(z-n+1)}{n!} \quad , z \in \mathbb{C},\ n = 0, 1, 2, 3, \ldots \tag{34}$$

Entry 6. for $z \in \mathbb{C}$, we have

$$\pi = 6\sqrt{3} \sum_{n=1}^{\infty} \frac{2^{-2n}}{n} \binom{n+z}{n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} (-3)^k + 6\sqrt{3} \sum_{n=1}^{\infty} \frac{(-3)^{-n}}{2n-1} \binom{z}{2n-1} \tag{35}$$

$$\pi = 8 \sum_{n=1}^{\infty} \frac{\left(2\sqrt{2}\right)^{-n}}{n} \binom{n+z}{n} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2k+1} (-1)^k \left(\sqrt{2}-1\right)^{n-2k-1} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n \left(\sqrt{2}-1\right)^{2n-1}}{2n-1} \binom{z}{2n-1} \tag{36}$$

4. References

[1] J. Arndt and C. Haenel, Pi-Unleashed, 2nd ed.,Spriger, Berlin, 2001.

[2] D. Brink, Nilakantha’s accelerated series for pi, Acta Arithmetica 171(4), 2015.

[3] R. Roy, The discovery of the series formula for π by Leibniz, Gregory and Nilakantha, Math. Mag. 63, 1990.