A proposal for mass variation under gravity

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Abstract – We review the special relativistic properties of a rotating system of coordinates, as considered by Einstein, in his initial considerations of time and length changes in gravity. Then, following Einstein's application of the principle of equivalence, we propose that an object's energy gain implies a gravitational potential dependent mass increase (GPDM). We then show how this mass-energy change is a function of frequency changes in light and hence time dilation. Applying this to gravitational lensing, we present predictions of GPDM in galaxies and large gravitational bodies. We explore various possible cosmological implications, including rapid clumping in the early universe, the formation of the cosmic web and dark matter-like phenomena. Experiments to test the theory in the terrestrial domain are also suggested.

Introduction. – In special relativity, the invariant mass is defined by $E^2 - \boldsymbol{p}^2c^2 = m^2c^4$, where this mass can be measured in its rest frame, with $m = E_0/c^2$. However, for composite particles the kinetic energy can contribute to the objects rest mass, measured in the centre of momentum frame (COM), see Appendix A. In terms of the nature and origin of inertial mass, Ernst Mach, famously gave Newton's rotating bucket experiment a new relativistic interpretation [1]. Whereas Newton saw the concave shape of the rotating water as evidence for inertial motion with respect to absolute space, Mach re-interpreted this as being relative motion with respect to other matter in the universe. Einstein also embraced Mach's ideas when developing General Relativity (GR) [2], as it was consistent with the principle of the general relativity of all motion. Indeed, it has now been shown using the equations of GR, that an external rotating mass shell modifies the geodesics on a static disk to match the centrifugal accelerations felt on a rotating disk [3,4], confirming Mach's interpretation.

Einstein also considered the idea of a variable inertial mass for objects in gravity [1], stating in 1950, that surrounding masses and their motions [5]: "...must determine completely and unequivocally the inertial behavior of our mass". In 1912, Einstein, through considering gravitational binding energy [6], determined the equation for a variable rest mass, dependent on its proximity to a neighbouring mass, of

$$m = m_0\left(1 + \frac{GM}{rc^2}\right). \tag{1}$$

[1]Einstein's conceptual approach explained in a letter to Gustav Mie in 1917, Einstein Archive, reel 17-221.

One of the simplest extensions to GR, which includes this idea, is the Brans-Dicke theory [7], which introduces an additional scalar field, allowing inertial mass to vary as a function of spacetime. This theory remains a viable alternative to GR, although constrained by current astronomical observations. A different theory, based more on Mach's principle, which includes a variation in inertial mass, is the Hoyle-Narlikar theory [8]. More recent theories, such as string theory can include additional scalar fields that vary the inertial mass, and in brane-world theories, mass can vary with cosmological time [9].

Historically, after publishing the Special theory of relativity (SR) in 1905, Einstein looked to generalise this framework to include gravity. As part of this effort, he presented an analysis of the spinning disk in 1912 [10], with further consideration in 1916 [11].

We begin by quoting Einstein's description of his disk thought experiment. We do this because our own thought experiment exactly mirrors Einstein's. The difference being we apply it to mass, not just time and length, as Einstein did.

Einstein considered a rotating system of coordinates K', centred on the axis of a non-rotating inertial frame K, as follows:

The principle of equivalence demands that in dealing with Galilean regions we may equally well make use of non-inertial systems, that is, such co-ordinate systems as, relatively to inertial systems, are not free from acceleration and rotation...

For let K' be a system of co-ordinates whose z'−axis coincides with the z−axis of K, and which rotates about the latter axis with constant angular velocity.

Imagine, further, that we have given a large number of rigid rods, all equal to each other. We suppose these laid in series along the periphery and the diameter of the circle, at rest relatively to K'. If U is the number of these rods along the periphery, D the number along the diameter... With respect to K all the rods upon the periphery experience the Lorentz contraction, but the rods upon the diameter do not experience this contraction (along their lengths!). It therefore follows that

$$\frac{U}{D} > \pi. \tag{2}$$

It therefore follows that the laws of configuration of rigid bodies with respect to K' do not agree with the laws of configuration of rigid bodies that are in accordance with Euclidean geometry. If, further, we place two similar clocks (rotating with K'), one upon the periphery, and the other at the centre of the circle, then, judged from K, the clock on the periphery will go slower than the clock at the centre. The same thing must take place, judged from K',...

Space and time, therefore, cannot be defined with respect to K' as they were in the special theory of relativity with respect to inertial systems. But, according to the principle of equivalence, K' may also be considered as a system at rest, with respect to which there is a gravitational field (field of centrifugal force, and force of Coriolis).

We therefore arrive at the result: the gravitational field influences and even determines the metrical laws of the space-time continuum [12, 13].

What is remarkable in this thought experiment, is that Einstein's deep conviction in the correctness of the equivalence principle for all kinds of non inertial forces, which allowed him to deduce that spacetime is warped in gravity.

That is, using the principle of equivalence [2] [14, 15], an observer on the disk's periphery, experiencing a radial acceleration, is deemed completely equivalent to one lying in a static gravitational field, and therefore that clocks would also run slow when in gravity.

Following Einstein's line of reasoning, that the increased relative velocity at the periphery of the disk causes time and length to be scaled by the relativistic gamma, then we also expect that the relativistic energy of an annulus of mass on the periphery will rise by the same gamma factor. Now, because we have a centre of momentum frame (COM) we can weigh the disk, expecting an increased reading for inertial mass. Refer to supplemental materials for Einstein's full quote regarding the spinning disk.

A thought experiment on the preponderance of mass . Let O be at the centre of a rotating coordinate system. Let P be another observer positioned in the frame at some distance r from O, and rotating with the frame.

For an angular velocity of ω, we have a circumferential velocity of $v = r\omega$ with a consequent Lorentz factor $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. A non-rotating observer at O will then observes a relative velocity of v for P, producing, the expected time dilation and length contraction effects, predicted by Einstein [3].

Specifically, the clocks at P will appear to run slow as viewed by O, who observes t_O/γ, where t_O is the time on O's clock, and any standard measuring sticks at P, will appear shortened to ℓ_O/γ, where ℓ_O is the reference length in O's frame. However, as the radius vector is directed orthogonal to the circumferential velocity, the length r will have no length contraction. Now, if O switches to the co-rotating frame for P, then P appears to become at rest with respect to O. Then, by the equivalence principle, the centrifugal acceleration previously observed for P, can be interpreted as P lying in an equivalent gravitation field. That is, the equivalence principle selects out the radial direction on the frame, which is transverse to the direction of motion, and is now the radial direction of an equivalent gravitational field.

Additionally, it is also known from SR, that an object at P will be seen by O to have a relativistic energy, given by γmc^2, increasing from its original rest energy mc^2, where m is the rest mass. For simplicity, we imagine a mass annulus is located in flat space also at the radius r.

Hence, if the observer O now undertakes a test to determine the new inertial mass of the disk, then O will now determine an inertial mass of $E/c^2 = \gamma m$.

This is indeed a general principle, first pointed out by Einstein, that the changing energy content of a body ΔE implies a change in inertial mass of $\Delta E/c^2$. Then, based on $E^2 - p^2c^2 = m^2c^4$, this inertial mass will be invariant for all frames. Hence, for the rotating annulus, O determines the mass to be

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4}$$

We can see that this relation is in inverse ratio to the time dilation.

Now, following Einstein's argument for time and length changes on the disk, we propose O can interpret P's inertial mass to vary radially in a gravitational well. Now via the weak equivalence principle, there is also a change in P's gravitational mass. [4]. That is we could envisage

[2]The weak equivalence principle refers to the equality between inertial and gravitational mass, which has been verified with a precision of 10^{-15} in the MICROSCOPE mission [14]

[3]While Einstein did use the SR γ factor on the disk, he used the rotating disk to motivate a new conception of gravity, based on the warping space and time. For as far as we know [16] he began with the Minkowski metric, and then with help from Grossman, acquired the more general form

$$ds^2 = g_{uv} dx^u dx^v. \tag{3}$$

This is the first mathematical description where the geometry itself, is now a function of the coordinates, which became a foundation for his theory. Likewise, also following Einstein, we are using the example of the disk to motivate the argument regarding the nature of gravity, and not to produce a special relativistic theory of gravity.

[4]We are not proposing that for objects moving in linear motion that,"gravitational mass" = "relativistic mass". However if one remains outside the disk, then its rotational motion cannot be transformed away. Hence, similar to the case of a spring weighing more

attaching between O and P an extremely accurate strain gauge, with a digital readout. We then expect, the meter will show a force reading, which all observers can agree, is greater than expected.

This result shows that in gravity P's mass is a function of radius, or gravitational potential. Hence, in order to avoid confusion with other mass definitions, we propose to refer to the mass gain under gravity as the, 'gravitational potential dependent mass' (GPDM).

The Schwarzschild metric and mass increase. – The Gullstrand-Painlevé coordinate system [17,18] can be used to write the Schwarzschild solution with an alternate set of coordinates [19] from the perspective of a 'rain frame' [20], where spacetime can be deemed as flowing into a black hole with a velocity given by the escape velocity of

$$v = \sqrt{\frac{2GM}{r}}. \tag{5}$$

The escape velocity shows the energy differential between a radius r and infinity, and so also showing the time dilation factor, and hence the mass increase factor. So substituting the escape velocity into Eq. (4), we arrive at a mass dependency for m_0 in a gravitational field, at radius r, to be

$$m = \frac{m_0}{\sqrt{1 - \frac{2MG}{rc^2}}}, \tag{6}$$

where M is the source mass, and m_0 is a test mass $m_0 \ll M$. We see therefore that a test mass m_0 will experience a mass gain in a gravitational field, based on its proximity to another mass. This then agrees with Einstein's original result in Eq. (1) to first order, although derived through a completely different approach.

The equivalence principle employed on a the spinning disk, translates to a static gravitational field. Hence, there is no need to consider mass in motion under gravity with consequent frame dragging effects, so we can simply employ the Schwarzschild metric.

We note that $\frac{2GM}{rc^2}$ is normally very small due the to large effect of the factor c^2, but becomes significant when approaching the event horizon. Furthermore, due to the mutuality of the effect, M will also undergo a slight increase in its internal energy from the influence of m_0.

Two-body mass increase to first order. There are about twenty exact solutions known to Einstein's field equations, however none of these are two-body solutions and the non-linear nature of gravity means that they cannot be superimposed.

As a first approximation, if we start with some initially large mass M, then from Newton's law, we have at some radius r, an acceleration

$$a = \frac{MG}{r^2}. \tag{7}$$

If we add an additional mass m to M, then the acceleration might be expected to increase to $\frac{(M+m)G}{r^2}$. However, we have claimed that in the presence of other masses, mass will increase according to Eq. (6). Hence, the Newtonian acceleration needs to be corrected to

$$\begin{aligned} a' &= \frac{(M' + m')G}{r^2} \\ &= \frac{\left(\frac{M}{\sqrt{1-\frac{2mG}{rc^2}}} + \frac{m}{\sqrt{1-\frac{2MG}{rc^2}}}\right)G}{r^2} \\ &\simeq \frac{\left(M\left(1 + \frac{mG}{rc^2}\right) + m\left(1 + \frac{MG}{rc^2}\right)\right)G}{r^2} \\ &\simeq \frac{(M+m)\,G}{r^2} + \frac{2mMG^2}{r^3c^2}, \end{aligned} \tag{8}$$

using the binomial expansion approximation. We are assuming in this approximation, that we can simply join two Schwarzschild solutions, which we can justify to first order providing we have small masses at distances creating small spatial distortion. This modified acceleration is not referring to the equivalence principle, but to additional mass appearing when combining masses. Therefore, to first order, we have gravitational mass increase, due to the proximity of the two masses, M and m of

$$(M+m)' = M + m + \frac{2mMG}{rc^2}. \tag{9}$$

We can see that this effect becomes more significant approaching the event horizon, as shown in Eq. (6). This result implies a strengthened gravitational field and therefore an increase in the force on a mass stationary in gravity.

The acceleration corrective term, in Eq. (8), is interesting, as it is identical to the Einstein correction to Newtonian gravity of $\frac{2M^2G^2}{r^3c^2}$, which describes an additional attractive acceleration. This is a small corrective term, but since it is radially dependent and Mercury has a fairly elliptical orbit, it may create a small variation of the predicted result in GR, which may be detectable in upcoming missions [21,22].

Reduction to classical physics and relativistic correction terms. We now show how the total energy of a test mass m_0 that we propose in Eq. (6), reduces to known quantities in the classical limit. Using $E = mc^2$, we find the total energy of a test mass

$$E_{\text{tot}} = \frac{m_0c^2}{\sqrt{1 - \frac{2MG}{rc^2}}}. \tag{10}$$

We can use the binomial expansion to write

$$\frac{1}{\sqrt{1 - \frac{2MG}{rc^2}}} = 1 + \frac{MG}{rc^2} + \frac{3M^2G^2}{2r^2c^4} + \frac{5M^3G^3}{2r^3c^6} \dots \tag{11}$$

We then have

$$E_{\text{tot}} = m_0c^2\left[1 + \frac{MG}{rc^2} + \frac{3M^2G^2}{2r^2c^4} + \dots\right]. \tag{12}$$

when compressed, all outside frames will agree that after kinetic energy is added to a spinning object then it weighs more in gravity.

Now, assuming $\frac{2MG}{rc^2} \ll 1$, we can take the first three terms as an approximation to give

$$E_{\text{tot}} \simeq m_0c^2 + \frac{m_0MG}{r} + \frac{3m_0M^2G^2}{2r^2c^2}. \tag{13}$$

We see that mc^2 is the rest energy, $\frac{mMG}{r}$ is the classical gravitational potential energy along with a 'relativistic' correction term

$$\frac{3mM^2G^2}{2r^2c^2}. \tag{14}$$

Hence, the energy contribution from the gravitational field to the particle is

$$E_{\text{grav}} \simeq \frac{mMG}{r} + \frac{3mM^2G^2}{2r^2c^2} = \frac{mMG}{r}\left(1 + \frac{3MG}{2rc^2}\right). \tag{15}$$

The corrective term of $\frac{3MG}{2rc^2}$, for the mass of the sun of $M = 1.989 \times 10^{30}$ kg where r is its radius of 6.96×10^5 km, increases the energy by a factor of 3.2×10^{-6}. We can see the interaction here between energy in the field and the mass increase, which maintains energy conservation.

Effects on emitted light frequency. Let us suppose we have two equal masses m_1 and m_2 at some equal distance from an observer O. If both masses are brought in exact proximity to each other, they can effectively be treated as one mass. Then, based on the Schwarzschild metric time coefficient, O will record the frequency of an emitted light source from the combined mass to be

$$\nu_r = \nu_s \sqrt{1 - \frac{2\left(m_1 + m_2 + \frac{2m_1m_2G}{r_0c^2}\right)G}{rc^2}}, \tag{16}$$

where ν_s is the source frequency and using Eq. (9) to find the additional mass increase. We can see that the mass increase hypothesis, as given to first order in Eq. (9), implies a drop in frequency of the detected light in addition to the standard gravitational redshift.

From the mass gain formula in Eq. (6) and the redshift formula from the Schwarzschild metric of

$$\nu_r = \nu_s \sqrt{1 - \frac{2MG}{rc^2}} \tag{17}$$

we can now relate several key variables, as follows

$$\frac{m}{m_0} = \frac{\nu_s}{\nu_r} = \frac{E}{E_0} = \frac{dt_0}{dt} = 1 + z_g, \tag{18}$$

so that the mass increase of m_0 changes in the same ratio as the frequency shift and the time dilation. We can also equate this to the gravitational redshift z_g, as $1 + z_g = \frac{\nu_s}{\nu_r}$. Hence, we have a general inverse relation between time dilation and mass increase under gravity. Therefore we can also write the apparent mass gain as

$$m = m_0\left(\frac{\nu_s}{\nu_r}\right) = m_0(1 + z_g), \tag{19}$$

which shows a general relation between mass, frequency and time.

In order to give an intuitive justification to this relation for mass increase. If we simply consider the case of a single photon entering a gravitational field with an initial energy at infinity of $E_0 = \hbar\omega_0$, then it will gain energy from the gravitational field, of [23]

$$E \approx E_0\left(1 + \frac{GM}{rc^2}\right). \tag{20}$$

This result also correlates directly with the observed redshift

$$f = f_0\left(1 + \frac{GM}{rc^2}\right). \tag{21}$$

Now, if we imagine a box of photons entering a gravitational field, then its inertial mass must also increase according to Eq. (20), consistent to first order with our result in Eq. (6).

Applications. –

Mass of galactic objects . Based on estimated amounts of galactic source mass, current gravitational lensing observations reveal greater than expected values. We wish to now derive a modified lensing formula [24] based on the mass gain relation above.[5] The estimated total mass that causes gravitational lensing from general relativity is $M'_T = \frac{\alpha rc^2}{4G}$, where α is the deflection angle, which includes the effect of space and time curvature.

Now, from Eq. (19), we have $m = m_0(\frac{\nu_s}{\nu_r})$ and so $M'_T = M_T(\frac{\nu_s}{\nu_r}) = M_T(1 + z_g)$, giving a modified lensing formula

$$M_T = \frac{\alpha rc^2}{4G(1 + z_g)}, \tag{22}$$

where z_g is the gravitational redshift of the intermediate lensing object. This indicates that for higher redshift structures additional gravitational lensing will be expected. This formula thus implies a hidden mass given by

$$M_{\text{add}} = \frac{\alpha rc^2}{4G}\left(1 - \frac{\nu_r}{\nu_s}\right) = \frac{\alpha rc^2}{4G}\left(\frac{z_g}{1 + z_g}\right). \tag{23}$$

We see that for values of $\nu_r \simeq \nu_s$ then $(1 - \frac{\nu_r}{\nu_s}) \to 0$ we have very little extra mass. If $\nu_r \ll \nu_s$ we have large amounts of extra mass, not easily visible due to the highly redshifted photons. In the extreme case, for $\nu_r \to 0$ then $M_{\text{add}} \to \frac{\alpha rc^2}{4G}$, and almost the entire lensing contribution is due to some form of unseen extra mass. Hence the matter is possibly undetectable except for its lensing effects.

The unseen matter, coupled to the mass increase is consistent, in principle, with current 'dark matter' proposals. Current estimates of dark matter are approximately

[5] That is, if in our proposal mass and spacetime all change by the factor $\sqrt{1 - \frac{2MG}{c^2r}}$, then any changes in spacetime correlating to gravitational lensing, also corresponds to a change in mass/energy.

a factor of 5.5 times the observed baryonic matter [25]. Gravitational lensing and redshift data, can thus be used to directly confirm the missing mass in a given galaxy or a galaxy cluster.

Estimating a stars mass gain under its formation. We can also use Eq. (19) to estimate how much a cloud of dust increases in mass upon collapse, as in star formation, for example. Now, before collapse we have an initial mass m_i, whereas after collapse, we have the mass

$$M_{\text{tot}} = m_i \left(\frac{\nu_s}{\nu_r} \right). \tag{24}$$

This implies that there is an effective additional mass of

$$M_{\text{add}} = M_{\text{tot}} - m_i = M_{\text{tot}} \left(1 - \frac{\nu_r}{\nu_s} \right). \tag{25}$$

For stars, M_{tot} is typically found using orbital mechanics of surrounding planets or stars. For the sun, we can also use the gravitational bending of light.

Hence, the degree of gravitational red shift, will yield the mass increase of any body formed in such a manner. As above when ν_r goes beyond radio frequencies, then most of the gravitational effect is due to unseen or invisible mass. Therefore at the more extreme case of a black hole, its mass gain dominates over its initial mass as given by Eq. (24).

A terrestrial application . If we measure the inertial mass of an object on the Earth's surface, and at the top of a building 100 m high, say, then from Eq. (6), using the known mass and radius of the Earth, we can calculate the expected change in inertial mass within the Earth's gravitational field. Substituting these values we find a mass change factor 1.1×10^{-14}. A Cavendish torsion balance can measure inertial mass to an accuracy of around 10^{-15} and so this small variation may feasibly be detectable with such an experiment. Other experiments include those undertaken to test deviations from an inverse square law [26].

Further implications. There is not yet direct evidence for discrepancies in mass change in pre and post star collapse, since the field is still relatively young, and many processes are not understood. However it has been suggested that progenitors for normal type Ib and Ic supernovae might be the result of very massive stars collapsing to a black hole with no supernova event [27]. These are referred to as supernova imposters. GPDM would expect a mass increase from such a collapse. Hence orbiting masses would undergo a change in trajectory, inclined toward the newly formed blackhole, along with an increase in trajectory speed.

Since the theory predicts that higher density masses emit more redshifted light, we expect to observe a direct correlation between total luminosity and higher mass. This correlation seems to exist with the Tully Fisher relation [28] of $L \propto W^{\alpha}$, where $\alpha \approx 3.5 - 4.0$, particularly since the observed luminosity L corresponds to the same regions where baryonic matter is found.

Related to this, are the current observations of certain galaxies in the infrared region, which show a greater number of active stars and the creation of new ones [29] A decrease in wavelength and increase in source mass/energy is what the theory predicts and hence, may shed light on this observation.

In line with the theory's faster then expected gravitational collapse rate, we could expect a larger number of well formed galaxies in the early universe as well as a more extensively formed cosmic web of 'dark matter,' [30] [31]. Following from this, the theory would expect faster than current predictions of ultra massive blackhole formation, at the centre of most galaxies. [32]

With respect to galaxy clusters, they seem to contain large amounts of dark matter. The theory would therefore expect them to emit significant radio emissions. Observations in 2019 showed much higher than expected radio emissions in a region connecting the two clusters of Abell 0399 and Abell 0401 [33]. Radio emissions of this kind are usually caused by so called synchrotron emission, by electrons moving at relativistic speeds. However the signal detected in this study is a factor of up to a hundred times brighter than some theoretical predictions for this type of synchrotron emission. This is an interesting find and seems to be another opportunity to test the validity of the theory.

With regard to the Bullet cluster, not only did the dark matter pass through, but so did the many galaxies in each cluster without obvious disturbance compared to the gas clouds [34]. This appears to indicate a close association of baryonic matter with the dark matter, as expected by this theory.

With respect to dark matter profiles, there are some areas of difficulty with the theory. In particular the core cusp problem [35]. However since GPDM associates dark matter with baryonic matter, one might expect, as per the Navarro-Frenk-White profile [36], the density of dark matter to increase towards the core of galaxies. However this is not always observed. Rather we observe more constant density cores in many galaxies. Several explanations have been offered for the problem [37, 38].

Recent observations of the super massive blackhole (SMBH), in galaxy M87, have revealed large amounts of dark matter, coming from the halo towards the accretion disk [39]. GPDM's association with baryonic matter appears consistent with this new find.

Finally, applying N-body simulations, to many of the above cases, would be useful in properly testing the theory.

Conclusion. – Using the principle of equivalence based on Einstein's rotating disk thought experiment, the paper presents the hypothesis that the inertial and gravitational mass/energy of a body will increase within a gravitational field according to Eq. (6) and hence depends on its distance to other masses, according to Eq. (9), to first order.

It also posits that a frequency decrease of radiation emitted from source masses in a gravitational field is accompanied by a mass/energy increase, according to Eq. (18), showing a general link between time dilation and inertial mass. Hence, by the weak equivalence principle this relation also applies to gravitational mass.

We also present a general relation for a mass increase for a body aggregating from dust in Eq. (23). Additionally we provide a modified gravitational lensing formula that also provides a potential candidate for dark matter, in Eq. (22).

Various other effects follow from these results, which may possibly be detected experimentally at very small distances or confirmed with cosmological measurements.

The theory's simple thesis of GPDM appears to have the potential to fit a diverse array of new cosmological observations.

***.** – Appendix A: The definition of mass in SR

Within SR, the invariant mass m is defined as the magnitude of the energy-momentum four-vector $[E, \boldsymbol{p}]$, with

$$m^2 = E^2 - \boldsymbol{p}^2, \tag{26}$$

using units with $c = 1$. In the rest frame of the object, with $\boldsymbol{p} = 0$, we therefore have $m = E_0$ [6]. The mass, being defined in this manner, is therefore invariant for all observers. However, if the mass absorbs or radiates photons of energy ΔE, there will be a change, in the invariant mass, of

$$\Delta m = \frac{\Delta E}{c^2}, \tag{27}$$

where for clarity we include a c^2 factor.

Now, for a group of relativistic particles, we can also define an invariant mass [40]. Since energy and momentum are separately conserved, we can sum over n particles by superposition, giving a generalised energy-momentum relation

$$M^2 = \left(\sum_{i=1}^{n} E_i\right)^2 - \left(\sum_{i=1}^{n} \boldsymbol{p}_i\right)^2. \tag{28}$$

Hence, in the centre of momentum (COM) frame, where $\sum_{i=1}^{n} \boldsymbol{p}_i = 0$, we have the invariant mass defined as

$$M = \sum_{i=1}^{n} E_i, \tag{29}$$

where $\sum_{i=1}^{n} E_i = \sum \gamma_i m_i c^2$, is the sum of the relativistic energies of each particle (in the COM frame). Since each particle has an invariant rest energy m_i, we can see that the additional mass is due to the energy of motion, or the relativistic kinetic energy

$$K = (\gamma - 1)mc^2 = \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots . \tag{30}$$

Hence, we can write for the invariant mass of a particle group

$$M = \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} K_i. \tag{31}$$

[6]For notational simplicity, we interpret the vector expression $\boldsymbol{p}^2 \equiv \boldsymbol{p} \cdot \boldsymbol{p} = p^2$ using the conventional Gibbs-Heaviside vector notation. The interested reader can find justification for this expression in the vector system of Clifford geometric algebra.

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