Formulae yielding $\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)$

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January 31, 2023

abstract

We give some integrals for $\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)$, where $\zeta(x)$ is the Riemann zeta function.

keywords: Integrals, zeta function, number Pi .

I. Introduction

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{1}$$

the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ , \ \ \mathrm{Re}(s) > 1 \tag{2}$$

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1}\, n^{-s} \ , \ \ \mathrm{Re}(s) > 0 \tag{3}$$

for $s = \frac{3}{2}$, we have

$$\zeta\left(\frac{3}{2}\right) = \sum_{n=1}^{\infty} n^{-3/2} = \left(2+\sqrt{2}\right)\sum_{n=1}^{\infty}(-1)^{n-1}\, n^{-3/2} = \frac{8+2\sqrt{2}}{7}\sum_{n=0}^{\infty}(2\,n+1)^{-3/2} \tag{4}$$

$$\zeta\left(\frac{3}{2}\right) = \left(\sqrt{2}+1\right)\sum_{n=1}^{\infty}\binom{2\,n}{n} 2^{-3\,n-1}\,(2\,n+1)\,\zeta\left(n+\frac{3}{2}\right) \tag{5}$$

$$\zeta\left(\frac{3}{2}\right) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{n}{(n+1)^2\left(\sqrt{n+1}+1\right)} \tag{6}$$

$$\zeta\left(\frac{3}{2}\right)\left(\frac{1}{2}-\frac{1}{4\sqrt{2}}\right) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\frac{n}{(2\,n+1)^2\left(\sqrt{2\,n+1}+1\right)} \tag{7}$$

$$\zeta\left(\frac{3}{2}\right)\left(\frac{4-\sqrt{2}}{4}\right) = \frac{\pi}{4} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+3}\sum_{k=0}^{n}(-1)^k\left(\frac{1}{\sqrt{2\,k+1}}+\frac{1}{\sqrt{2\,k+3}}\right) \tag{8}$$

In this note we give some integrals for $\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)$.

II. Integrals

Entry 1.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty \frac{x^2}{\sinh(x^2)}\,dx=\frac{1}{2}\int_0^\infty \frac{\sqrt{x}}{\sinh x}\,dx \tag{9}$$

Entry 2.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=-\frac{1}{4}\int_0^\infty \frac{1}{\sqrt{x}}\ln\left(\tanh\left(\frac{x}{2}\right)\right)dx=-\frac{1}{2\sqrt{2}}\int_0^\infty \frac{\ln(\tanh x)}{\sqrt{x}}\,dx \tag{10}$$

Entry 3.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty \tanh^{-1}\left(e^{-x^2}\right)dx=\int_0^\infty \sqrt{\ln(\coth x)}\;dx \tag{11}$$

Entry 4.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2}\int_1^\infty \frac{1}{x\sqrt{\ln x}}\tanh^{-1}\left(\frac{1}{x}\right)dx=\frac{1}{2}\int_0^1 \frac{\tanh^{-1} x}{x\sqrt{-\ln x}}\,dx \tag{12}$$

Entry 5.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2}\int_0^\infty \frac{\sqrt{\ln\left(x+\sqrt{1+x^2}\right)}}{x\sqrt{1+x^2}}\,dx=\frac{1}{2}\int_0^{\pi/2} \frac{\sqrt{\ln(\sec x+\tan x)}}{\sin x}\,dx \tag{13}$$

Entry 6.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=-\frac{1}{\sqrt{2}}\int_0^\infty \ln\left(\tanh(x^2)\right)dx=\frac{1}{\sqrt{2}}\int_0^\infty \sqrt{\tanh^{-1}(e^{-x})}\;dx \tag{14}$$

Entry 7.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2\sqrt{2}}\int_0^1 \frac{1}{x\sqrt{-\ln x}}\ln\left(\frac{1+x^2}{1-x^2}\right)dx=\frac{1}{4}\int_0^1 \frac{1}{x\sqrt{-\ln x}}\ln\left(\frac{1+x}{1-x}\right)dx \tag{15}$$

Entry 8.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_{-\infty}^\infty \frac{e^{-3x}}{\sinh\left(e^{-2x}\right)}\,dx=\int_{-\infty}^\infty \frac{e^{3x}}{\sinh\left(e^{2x}\right)}\,dx \tag{16}$$

Entry 9.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2}\int_1^\infty \frac{\sqrt{\ln\left(x+\sqrt{x^2-1}\right)}}{x^2-1}\,dx=\int_1^\infty \frac{\sqrt{\ln x}}{x^2-1}\,dx \tag{17}$$

Entry 10.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty \frac{\sqrt{\ln(\cosh x)}}{\sinh x}\,dx=\frac{1}{\sqrt{2}}\int_0^\infty \frac{\sqrt{\ln\left(1+x^2\right)}}{x\sqrt{1+x^2}}\,dx \tag{18}$$

Entry 11.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty \frac{x}{\sqrt{-\ln(\tanh x)}\,\sinh(2x)}\,dx=\int_0^\infty \frac{x}{\sqrt{\ln(\coth x)}\,\sinh(2x)}\,dx \tag{19}$$

Entry 12.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^{\pi/2}\frac{\sqrt{\ln(\sec x)}}{\sin x}\,dx \tag{20}$$

Entry 13.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{\sqrt{2}}\int_0^1\frac{1}{x}\sqrt{\tanh^{-1}x}\,dx=\frac{1}{2}\int_0^1\frac{1}{x}\sqrt{\ln\left(\frac{1+x}{1-x}\right)}\,dx \tag{21}$$

Entry 14.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\sqrt{2}\int_0^{\pi/4}\sqrt{-\ln(\tan x)}\,\tan(2x)\,dx \tag{22}$$

Entry 15.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2\sqrt{2}}\int_0^\infty\sqrt{\operatorname{sech}^{-1}\sqrt{1-e^{-x}}}\,dx=\frac{1}{2\sqrt{2}}\int_0^\infty\sqrt{\coth^{-1}\left(\frac{1}{\sqrt{1-e^{-x}}}\right)}\,dx \tag{23}$$

Entry 16.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2\sqrt{2}}\int_0^\infty\sqrt{\ln\left(\frac{1+e^{-x/2}}{\sqrt{1-e^{-x}}}\right)}\,dx \tag{24}$$

Entry 17.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_{-\infty}^\infty e^x\tanh^{-1}\left(e^{-e^{2x}}\right)dx=\int_{-\infty}^\infty e^{-x}\tanh^{-1}\left(e^{-e^{-2x}}\right)dx \tag{25}$$

Entry 18.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{4}\int_0^\infty\sqrt{\ln\left(\frac{\left(1+e^{x/2}\right)^2}{e^x-1}\right)}\,dx=\frac{1}{4}\int_0^\infty\sqrt{\coth^{-1}\left(\frac{e^x+1}{e^x-1}\right)}\,dx \tag{26}$$

Entry 19.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty\sqrt{\operatorname{sech}^{-1}\left(1-e^{-2\sqrt{2}\,x}\right)}\,dx-\int_0^{\frac{\ln(\sqrt{2})}{\sqrt{2}}}\sqrt{\operatorname{sech}^{-1}\left(e^{2\sqrt{2}\,x}-1\right)}\,dx \tag{27}$$

Entry 20.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^{\ln(\sqrt{2})}\sqrt{-\ln\left(e^{2x}-1\right)}\,dx+\int_0^\infty\sqrt{-\ln\left(1-e^{-2x}\right)}\,dx \tag{28}$$

Entry 21. for $u>0$, $v>0$, $v=-\ln\left(\tanh\left(u^2\right)\right)$, $u=\sqrt{\tanh^{-1}(e^{-v})}$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=u\,v+\frac{1}{\sqrt{2}}\int_u^\infty\ln\left(\coth\left(x^2\right)\right)dx+\frac{1}{\sqrt{2}}\int_v^\infty\sqrt{\tanh^{-1}(e^{-x})}\,dx \tag{29}$$

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=-u\,v+\frac{1}{\sqrt{2}}\int_0^u\ln\left(\coth\left(x^2\right)\right)dx+\frac{1}{\sqrt{2}}\int_0^v\sqrt{\tanh^{-1}(e^{-x})}\,dx \tag{30}$$

Entry 22. for $u>0$, $v>0$, $v=\tanh^{-1}\left(e^{-u^2}\right)$, $u=\sqrt{\ln(\coth v)}$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=u\,v+\int_u^\infty \tanh^{-1}\left(e^{-x^2}\right)dx+\int_v^\infty \sqrt{\ln(\coth x)}\;dx \tag{31}$$

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=-u\,v+\int_0^u \tanh^{-1}\left(e^{-x^2}\right)dx+\int_0^v \sqrt{\ln(\coth x)}\;dx \tag{32}$$

Entry 23.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{\sqrt{2}}\int_0^\infty \sqrt{x}\left(\frac{7-11\tanh x+7(\tanh x)^2-3(\tanh x)^3}{1+(\tanh x)^2}\right)dx \tag{33}$$

Entry 24.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{3}{2}\int_0^\infty \frac{\sqrt{x}}{e^x+1}\,dx+2\int_0^\infty \frac{\sqrt{x}}{e^{2x}+1}\,dx \tag{34}$$

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^\infty \frac{\sqrt{x}}{e^x-1}\,dx-\int_0^\infty \frac{\sqrt{x}}{e^{2x}-1}\,dx \tag{35}$$

Entry 25.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{4}\int_0^\infty \sqrt{x}\left(7-3\tanh\left(\frac{x}{2}\right)-4\tanh x\right)dx \tag{36}$$

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{\sqrt{2}}\int_0^\infty \sqrt{x}\,(7-3\tanh x-4\tanh(2x))\,dx \tag{37}$$

Entry 26.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2}\int_0^\infty \sqrt{x}\left(\coth\left(\frac{x}{2}\right)-\coth x\right)dx=\sqrt{2}\int_0^\infty \sqrt{x}\,(\coth x-\coth(2x))\,dx \tag{38}$$

Entry 27.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\sqrt{2}\int_0^\infty x^2\left(7-3\tanh\left(x^2\right)-4\tanh\left(2x^2\right)\right)dx \tag{39}$$

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=2\sqrt{2}\int_0^\infty x^2\left(\coth\left(x^2\right)-\coth\left(2x^2\right)\right)dx \tag{40}$$

Entry 28.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{\sqrt{2}}\int_0^1 \frac{\sqrt{\tanh^{-1}x}\,\left(7-4x+3x^2\right)}{(1+x)\left(1+x^2\right)}\,dx \tag{41}$$

Entry 29.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{1}{2}\int_0^\infty\int_0^\infty \frac{1}{\sinh\left(x+y^2\right)}\,dx\,dy=\int_0^\infty\int_0^\infty \frac{x}{\sinh\left(x^2+y^2\right)}\,dx\,dy \tag{42}$$

III. Endnote

Entry 30. for $m \in \mathbb{N}=\{1,\ 2,\ 3,\ \ldots\}$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\frac{\ln(\tanh 1)+\ln\left(\tanh m^2\right)}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\ln\left(\prod_{n=1}^{m}\tanh\left(n^2\right)\right)-$$
$$\frac{1}{\sqrt{2}}\int_0^1\ln\left(\tanh x^2\right)dx-\frac{1}{\sqrt{2}}\int_m^\infty\ln\left(\tanh x^2\right)dx+2\sqrt{2}\int_1^m\frac{x}{\sinh\left(2x^2\right)}\left(x-[x]-\frac{1}{2}\right)dx \tag{43}$$

Entry 31. for $u>0$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=u\tanh^{-1}\left(e^{-u^2}\right)+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{3/2}}\Gamma\left(\frac{1}{2},(2n+1)u^2\right)+\int_{\tanh^{-1}\left(e^{-u^2}\right)}^{\infty}\sqrt{\ln(\coth x)}\,dx \tag{44}$$

Remark: $\Gamma(x,y)$ is the incomplete gamma function.

Entry 32. for $0<u<\sqrt{\pi}$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=u-\frac{2u^5}{5\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}F\left(1,\frac{5}{4},\frac{9}{4},-\frac{u^4}{(n\pi)^2}\right)+\int_u^\infty\frac{x^2}{\sinh\left(x^2\right)}dx \tag{45}$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 33. for $0<u<\sqrt{\pi}$, we have

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=u-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2\left(2^{2n-1}-1\right)B_n u^{4n+1}}{(2n)!\,(4n+1)}+\int_u^\infty\frac{x^2}{\sinh\left(x^2\right)}dx \tag{46}$$

Remark: $B_n=\left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\ldots\right\}$ are the Bernoulli numbers.

Entry 34.

$$\frac{4-\sqrt{2}}{8}\sqrt{\pi}\,\zeta\left(\frac{3}{2}\right)=\int_0^1\tanh^{-1}\left(e^{-x^2}\right)dx+\sum_{n=1}^{\infty}\tanh^{-1}\left(e^{-n^2}\right)-\sum_{n=1}^{\infty}\int_0^1\tanh^{-1}\left(\frac{1-e^{-2nx-x^2}}{e^{n^2}-e^{-(n+x)^2}}\right)dx \tag{47}$$

Future Research

Entry 35.

$$\pi\left(\frac{1}{2}-\frac{1}{4}\sum_{n=0}^{\infty}\binom{4n+2}{2n+1}\frac{2^{-6n}}{4n+1}\left(\frac{1}{\sqrt{2}}-2^{-2n}\right)\zeta\left(2n+\frac{1}{2}\right)\right)=1-\frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}(-1)^n 2^{-3n}(2n+1)\left(\frac{1}{\sqrt{2}}-2^{-n-1}\right)\zeta\left(n+\frac{3}{2}\right) \tag{48}$$

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