On Born Reciprocal Relativity, Algebraic Extensions of the Yang and Quaplectic Algebra, and Noncommutative Curved Phase Spaces

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Abstract

After a brief introduction of Born's reciprocal relativity theory is presented, we review the construction of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* fiber coordinates and momenta $[X_a, X_b] \neq 0$; $[P_a, P_b] \neq 0$. This construction leads to more general algebras given by a two-parameter family of deformations of the Quaplectic algebra, and to further algebraic extensions involving antisymmetric tensor coordinates and momenta of higher ranks $[X_{a_1 a_2 \cdots a_n}, X_{b_1 b_2 \cdots b_n}] \neq 0$; $[P_{a_1 a_2 \cdots a_n}, P_{b_1 b_2 \cdots b_n}] \neq 0$. We continue by examining algebraic extensions of the Yang algebra in extended noncommutative phase spaces and compare them with the above extensions of the deformed Quaplectic algebra. A solution is found for the exact analytical mapping of the non-commuting x^μ, p^μ operator variables (associated to an $8D$ curved phase space) to the canonical Y^A, Π^A operator variables of a flat $12D$ phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, with the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an $8D$ curved phase space into a flat $12D$ phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the $8D$ cotangent space of the $4D$ spacetime. Consequently, one has found a direct link between noncommutative curved phase spaces in lower dimensions to commutative flat phase spaces in higher dimensions.

1 Introduction : Born Reciprocal Relativity Theory

Most of the work devoted to Quantum Gravity has been focused on the geometry of spacetime rather than phase space per se. The first indication that phase space should play a role in Quantum Gravity was raised by [3]. The principle behind Born's reciprocal relativity theory [5], [6] was based on the idea proposed long ago by [3] that coordinates and momenta should be unified on the same footing. Consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. The principle of maximal acceleration was advocated earlier on by [4]. A *maximal* speed limit (speed of light) must be accompanied with a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) [6].

We explored in [6] some novel consequences of Born's reciprocal Relativity theory in flat phase-space and generalized the theory to the curved spacetime scenario. We provided, in particular, some specific results resulting from Born's reciprocal Relativity and which are *not* present in Special Relativity. These are : momentum-dependent time delay in the emission and detection of photons; relativity of chronology; energy-dependent notion of locality; superluminal behavior; relative rotation of photon trajectories due to the aberration of light; invariance of areas-cells in phase-space and modified dispersion relations.

The generalized velocity and force (acceleration) boosts (rotations) transformations of the $flat$ $8D$ Phase space coordinates , where $X^i, T, E, P^i; i = 1,2,3$ are **c** -valued (classical) variables which are *all* boosted (rotated) into each-other, were given by [5] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. The $U(1,3) = SU(1,3) \times U(1)$ group transformations leave invariant the symplectic 2-form $\Omega = -\, dT \wedge dE + \delta_{ij} dX^i \wedge dP^j; i,j = 1,2,3$ and also the following Born-Green line interval in the $flat$ $8D$ phase-space

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;-\; (dY)^2 \;-\; (dZ)^2 \;+$$

$$\frac{1}{b^2}\;\left((dE)^2 \;-\; c^2 (dP_x)^2 \;-\; c^2 (dP_y)^2 \;-\; c^2 (dP_z)^2\right) \tag{1.1}$$

The maximal proper force is set to be given by b. The rotations, velocity and force (acceleration) boosts leaving invariant the symplectic 2-form and the line interval in the $8D$ phase-space are rather elaborate, see [5] for details.

These transformations can be simplified drastically when the velocity and force (acceleration) boosts are both parallel to the x-direction and leave the transverse directions Y, Z, P_y, P_z intact. There is now a subgroup $U(1,1) = SU(1,1) \times U(1) \subset U(1,3)$ which leaves invariant the following line interval

$$(d\omega)^2 \;=\; c^2 (dT)^2 \;-\; (dX)^2 \;+\; \frac{(dE)^2 \;-\; c^2 (dP)^2}{b^2} =$$

$$(d\tau)^2 \left(1 + \frac{(dE/d\tau)^2 - c^2(dP/d\tau)^2}{b^2} \right) = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right), \quad P = P_x \tag{1.2}$$

where one has factored out the proper time infinitesimal $(d\tau)^2 = c^2dT^2 - dX^2$ in (1.2). The proper force interval $(dE/d\tau)^2 - c^2(dP/d\tau)^2 = -F^2 < 0$ is "spacelike" when the proper velocity interval $c^2(dT/d\tau)^2 - (dX/d\tau)^2 > 0$ is timelike. The analog of the Lorentz relativistic factor in eq-(1.14) involves the ratios of two proper *forces*.

One may set the maximal proper-force acting on a fundamental particle of Planck mass to be given by $F_{max} = b \equiv m_P c^2/L_P$, where m_P is the Planck mass and L_P is the postulated minimal Planck length. Invoking a minimal/maximal length duality one can also set $b = M_U c^2/R_H$, where R_H is the Hubble scale and M_U is the observable mass of the universe. Equating both expressions for b leads to $M_U/m_P = R_H/L_P \sim 10^{60}$. The value of b may also be interpreted as the maximal string tension.

The $U(1,1)$ group transformation laws of the phase-space coordinates X, T, P, E which leave the interval (1.2) invariant are [5]

$$T' = T\ cosh\xi + (\frac{\xi_v\ X}{c^2} + \frac{\xi_a\ P}{b^2})\ \frac{sinh\xi}{\xi} \tag{1.3a}$$

$$E' = E\ cosh\xi + (-\xi_a\ X + \xi_v P)\ \frac{sinh\xi}{\xi} \tag{1.3b}$$

$$X' = X\ cosh\xi + (\xi_v\ T - \frac{\xi_a\ E}{b^2})\ \frac{sinh\xi}{\xi} \tag{1.4a}$$

$$P' = P\ cosh\xi + (\frac{\xi_v\ E}{c^2} + \xi_a\ T)\ \frac{sinh\xi}{\xi} \tag{1.4b}$$

ξ_v is the velocity-boost rapidity parameter; ξ_a is the force (acceleration) boost rapidity parameter, and ξ is the net effective rapidity parameter of the primed-reference frame. These parameters ξ_a, ξ_v, ξ are defined respectively in terms of the velocity $v = dX/dT$ and force $f = dP/dT$ (related to acceleration) as

$$tanh(\frac{\xi_v}{c}) = \frac{v}{c}; \quad tanh(\frac{\xi_a}{b}) = \frac{F}{F_{max}}, \quad \xi = \sqrt{(\frac{\xi_v}{c})^2 + (\frac{\xi_a}{b})^2} \tag{1.5}$$

The $U(3,1)$ generators $Z_{ab} = \frac{1}{2}(L_{[ab]} + M_{(ab)})$ are comprised of the 6 ordinary Lorentz generators $L_{[ab]}$, and 10 force (acceleration) boost/rotation generators $M_{(ab)}$ giving a total of 16 generators.

It is straight-forward to verify that the transformations (1.4) leave invariant the phase space interval $c^2(dT)^2 - (dX)^2 + ((dE)^2 - c^2(dP)^2)/b^2$ but *do not* leave separately invariant the proper time interval $(d\tau)^2 = c^2dT^2 - dX^2$, nor the interval in energy-momentum space $\frac{1}{b^2}[(dE)^2 - (dP)^2]$. Only the *combination*

$$(d\omega)^2 = (d\tau)^2 \left(1 - \frac{F^2}{F^2_{max}} \right) \tag{1.6}$$

is truly left invariant under force (acceleration) boosts (1.4). They also leave invariant the symplectic 2-form (phase space areas) $\Omega = - \, dT \wedge dE + dX \wedge dP$.

After this brief introduction of Born's reciprocal relativity theory is presented, in section **2** we review the construction of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* fiber coordinates and momenta $[X_a, X_b] \neq 0$; $[P_a, P_b] \neq 0$. This construction leads at the end of section **2** to more general algebras given by a two-parameter family of deformations of the Quaplectic algebra, and to local gauge theories of gravity based on the latter deformed Quaplectic algebras.

We continue in section **3** by examining the algebraic extensions of the Yang algebra in extended noncommutative phase spaces, and compare them with the extensions of the deformed Quaplectic algebra involving antisymmetric tensor coordinates and momenta of higher ranks $[X_{a_1 a_2 \cdots a_n}, X_{b_1 b_2 \cdots b_n}] \neq 0$; $[P_{a_1 a_2 \cdots a_n}, P_{b_1 b_2 \cdots b_n}] \neq 0$.

In section **4** a solution is found for the exact analytical mapping of the noncommuting x^μ, p^μ operator variables (associated to an $8D$ curved phase space) to the canonical Y^A, Π^A operator variables of a flat $12D$ phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, with the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an $8D$ curved phase space into a flat $12D$ phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the $8D$ cotangent space of the $4D$ spacetime. We finalize with some concluding remarks.

2 The Deformed Quaplectic Group and Complex Gravity

To begin this section we review the construction of the *deformed* Quaplectic group given by the semidirect product of $U(1,3)$ with the deformed (noncommutative) Weyl-Heisenberg group involving noncommutative coordinates and momenta [16]. And then we proceed to construct a two-parameter family of deformed Quaplectic algebras parametrized by two complex coefficients α, β. The deformed Weyl-Heisenberg algebra involves the generators

$$Z_a = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} - i \frac{P_a}{\lambda_p} \right); \quad Z_a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} + i \frac{P_a}{\lambda_p} \right); \quad a = 1,2,3,4. \quad (2.1)$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The local gauge theory based on the deformed Quaplectic algebra was constructed in the fiber bundle over the base manifold which is a $4D$ curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$ [16]. The (deformed) Quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore we must *not* confuse the *deformed* complex gravitational theory constructed in [16] with the noncommutative gravity work in the literature where the spacetime coordinates x^μ are not commuting.

The four fundamental length, momentum, temporal and energy scales are respectively

$$\lambda_l = \sqrt{\frac{\hbar c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar b}{c}}; \quad \lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_e = \sqrt{\hbar bc}. \tag{2.2}$$

where b is the *maximal* proper force associated with the Born's reciprocal relativity theory. In the natural units $\hbar = c = b = 1$ all four scales become *unity*. The gravitational coupling is given by

$$G = \frac{c^4}{\mathcal{F}_{max}} = \frac{c^4}{b}. \tag{2.3}$$

and the four scales coincide then with the Planck length, momentum, time and energy, respectively and we can verify that

$$\mathcal{F}_{\max} = m_P \frac{c^2}{L_P} \sim M_{Universe} \frac{c^2}{R_H} \tag{2.4}$$

The generators of the $U(1,3)$ algebra given by Z_{ab} are Hermitian $(Z_{ab})^\dagger = Z_{ab}$, with $a, b = 1,2,3,4$; while the generators of the *deformed* Weyl-Heisenberg algebra $Z_a, Z_a^\dagger$ are pairs of Hermitan-conjugates like $L_+ = L_x + iL_y, L_- = L_x - iL_y$ in the $SO(3)$ algebra. The standard Quaplectic group [5] is given by the semi-direct product of the $U(1,3)$ group and the unmodified Weyl-Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, Z_a^\dagger, \mathcal{I}$ described below with $a, b = 1,2,3,4$.

A careful analysis reveals that the generators $Z_a, Z_a^\dagger$ (comprised of Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl-Heisenberg algebra can be defined in terms of judicious linear combinations of the Hermitian $U(1,4)$ algebra generators Z_{AB} , where $A, B = 1,2,3,4,5$; $a, b = 1,2,3,4$; $\eta_{AB} = diag\ (+,-,-,-,-)$. The linear combination is defined after introducing the following *complex*-valued coefficients as follows

$$Z_a = (-i)^{1/2} (Z_{a5} - i Z_{5a}); \; Z_a^\dagger = (i)^{1/2} (Z_{a5} + i Z_{5a}); \; Z_{55} = \frac{\mathcal{I}}{2} \tag{2.5}$$

The reason behind this particular choice of the complex coefficients appearing in eq-(2.5) will be explained below in eq-(2.14). The Hermitian generators of the $U(1,4)$ algebra are given by $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}^b_a,\ \mathcal{E}^d_c] \ = \ -i\ \delta^b_c\ \mathcal{E}^d_a\ + i\ \delta^d_a\ \mathcal{E}^b_c; \quad [\mathcal{E}^d_c,\ \mathcal{E}^5_a] \ = \ -i\ \delta^d_a\ \mathcal{E}^5_c; \quad [\mathcal{E}^d_c,\ \mathcal{E}^a_5] \ = \ i\ \delta^a_c\ \mathcal{E}^d_5. \tag{2.6}$$

and $[\mathcal{E}^5_5,\ \mathcal{E}^a_5] = -i\ \delta^5_5\ \mathcal{E}^a_5$ .... such that now $\mathcal{I}(= 2Z_{55})$ *no* longer commutes with $Z_a, Z^\dagger_a$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz sub-algebra generators $L_{[ab]}$ and the "shear"-like generators $M_{(ab)}$ as

$$Z_{ab} \equiv \frac{1}{2}\ (M_{(ab)} + L_{[ab]}) \ \Rightarrow \ L_{ab} \equiv L_{[ab]} \ = \ (Z_{ab} - Z_{ba});\ \ M_{ab} \equiv M_{(ab)} = \ (Z_{ab} + Z_{ba}), \tag{2.7}$$

the "shear"-like generators $M_{(ab)}$ and the Lorentz generators $L_{[ab]}$ are Hermitian. The explicit commutation relations of the M_{ab}, L_{ab} generators is given by

$$[L_{ab},\ L_{cd}] \ = \ i\ (\eta_{bc}L_{ad} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \tag{2.8a}$$

$$[M_{ab},\ M_{cd}] \ = \ -\ i\ (\eta_{bc}L_{ad} + \eta_{ac}L_{bd} + \eta_{bd}L_{ac} + \eta_{ad}L_{bc}). \tag{2.8b}$$

$$[L_{ab},\ M_{cd}] \ = \ i\ (\eta_{bc}M_{ad} - \eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc}). \tag{2.8c}$$

Therefore, given $Z_{ab} = \frac{1}{2}(M_{ab} + L_{ab}),\ Z_{cd} = \frac{1}{2}(M_{cd} + L_{cd})$ after straightforward algebra it leads to the $U(1,3)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -\ i\ (\ \eta_{bc}\ Z_{ad}\ -\ \eta_{ad}\ Z_{cb}\ ). \tag{2.8d}$$

as expected. By extension, the $U(1,4)$ commutators are[1]

$$[\ Z_{AB},\ Z_{CD}\ ] \ = \ -\ i\ (\ \eta_{BC}\ Z_{AD}\ -\ \eta_{AD}\ Z_{CB}\ ). \tag{2.8e}$$

The commutators of the Lorentz boosts generators L_{ab} with the X_c, P_c generators are

$$[L_{ab},\ X_c] \ = \ i\ (\ \eta_{bc}\ X_a\ -\ \eta_{ac}\ X_b\ ); \quad [L_{ab},\ P_c] \ = \ i\ (\ \eta_{bc}\ P_a\ -\ \eta_{ac}\ P_b\ ) \tag{2.9}$$

The Hermitian M_{ab} generators are the "reciprocal" boosts/rotation transformations which *exchange* X for P, in addition to boosting (rotating) those variables, and one ends up with the commutators of M_{ab} with the X_c, P_c generators given by

$$[M_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \quad [M_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \tag{2.10}$$

[1]Strictly speaking, $U(1,4)$ is a pseudo-unitary group. After performing the Weyl unitary "trick" via an analytical continuation $U(1,4) \rightarrow U(5)$ one obtains the unitary group $U(5)$ comprised of 5×5 unitary matrices obeying $U^\dagger = U^{-1}$. A unitary matrix can be written as $U = e^A$ where A is an anti-Hermitian matrix $A^\dagger = -A$. And any anti-Hermitian matrix A can be written as $A = \pm iH$, where H is Hermitian, therefore all group elements can be written in the form $U = e^{\pm i\theta^{AB} Z_{AB}}$ where θ^{AB} are the corresponding parameters associated to every generator

The commutators in eq-(1.8d) and the definitions in eq-(2.5) lead to

$$[\ Z_{ab},\ Z_c\ ] \ = \ (-i)^{3/2}\ (\eta_{bc}\ Z_{a5}\ + i\ \eta_{ac}\ Z_{5b})$$
$$[\ Z_{ab},\ Z_c^\dagger\ ] \ = \ -\ (i)^{1/2}\ (i\ \eta_{bc}\ Z_{a5}\ +\ \eta_{ac}\ Z_{5b}) \tag{2.11}$$

which are consistent with the commutators in eqs-(2.8a-2.8c) and the definitions in eqs-(2.5,2.7). The right-hand side of eq-(2.11) can be rewritten in terms of $Z_a, Z_a^\dagger, Z_b, Z_b^\dagger$ after the following replacements

$$Z_{a5}\ =\ \frac{1}{2}\,[\ (-i)^{1/2}\ Z_a^\dagger + (i)^{1/2}\ Z_a\ ],\ \ Z_{b5}\ =\ \frac{1}{2i}\,[\ (-i)^{1/2}\ Z_a^\dagger - (i)^{1/2}\ Z_a\ ] \tag{2.12}$$

After some algebra one finds

$$[\ Z_{ab},\ Z_c\ ]\ =\ -\frac{i}{2}\ \eta_{bc}\ Z_a\ +\ \frac{i}{2}\ \eta_{ac}\ Z_b\ -\frac{1}{2}\ \eta_{bc}\ Z_a^\dagger\ -\ \frac{1}{2}\ \eta_{ac}\ Z_b^\dagger$$

$$[\ Z_{ab},\ Z_c^\dagger\ ]\ =\ -\frac{i}{2}\ \eta_{bc}\ Z_a^\dagger\ +\ \frac{i}{2}\ \eta_{ac}\ Z_b^\dagger\ +\ \frac{1}{2}\ \eta_{bc}\ Z_a\ +\ \frac{1}{2}\ \eta_{ac}\ Z_b\ . \tag{2.13}$$

The particular *choice* of the complex coefficients appearing in eq-(2.5) leads to the following *deformed* Weyl-Heisenberg algebra

$$[Z_a,\ Z_b^\dagger]\ =\ -\ (\ \eta_{ab}\ \mathcal{I}\ +\ M_{ab}\ );\ \ \ [Z_a,\ Z_b]\ =\ [Z_a^\dagger,\ Z_b^\dagger]\ =\ -i\ L_{ab} \tag{2.14a}$$

$$[Z_a,\ \mathcal{I}]\ =\ 2\ Z_a^\dagger;\ \ \ [Z_a^\dagger,\ \mathcal{I}]\ =\ -2\ Z_a;\ \ \ [Z_{ab},\ \mathcal{I}] = 0.\ \ \ \mathcal{I}\ =\ 2\ Z_{55}. \tag{2.14b}$$

where $[\frac{X_a}{\lambda_l}, \mathcal{I}] = 2i\frac{P_a}{\lambda_p}$; $[\frac{P_a}{\lambda_p}, \mathcal{I}] = 2i\frac{X_a}{\lambda_l}$ and the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices . The Planck constant is given in terms of the length and momentum scales of eq-(2.2) as $\hbar = \lambda_l \lambda_p$. In $\hbar = 1$ units, $\lambda_l \lambda_p \rightarrow 1$.

The deformed Quaplectic algebra is given explicitly by eqs-(2.8d, 2.11, 2.13, 2.14) and obeys the Jacobi identities by virtue of the definitions in eqs-(2.5,2.7). After recurring directly to definitions in eq-(2.1), one finds that eq-(2.14a) explicitly reflects the *deformation* of the Weyl-Heisenberg algebra resulting from the noncommutative algebra of coordinates and momenta given by

$$[\frac{X_a}{\lambda_l}, \frac{P_b}{\lambda_p}]\ =\ i\ (\eta_{ab}\ \mathcal{I} + M_{ab});\ [X_a, X_b]\ =\ -\ i(\lambda_l)^2\ L_{ab};\ [P_a, P_b]\ =\ i(\lambda_p)^2\ L_{ab}; \tag{2.15}$$

One could interpret the term $\eta_{ab}\ \mathcal{I}\ +\ M_{ab}$ as a matrix-valued Planck constant $\hbar_{ab}$ (in units of $\hbar = 1$). One may also note that the generator $\mathcal{I}$ no longer commutes with $Z_a, Z_a^\dagger$, but it *exchanges* them, as one can see from eq-(2.14b) resulting from the definition of $\mathcal{I}$ given by $\mathcal{I} \equiv 2Z_{55} = M_{55}$.

One of the salient features of the construction of the deformed Quaplectic (Weyl-Heisenberg) algebra is that by varying the values of the following complex coefficients α, β appearing in the linear combinations

$$Z_a\ =\ \alpha\ Z_{a5}\ + \beta\ Z_{5a};\ \ Z_a^\dagger\ =\ \alpha^*\ Z_{a5}\ + \beta^*\ Z_{5a};\ \ Z_{55} = \frac{\mathcal{I}}{2} \tag{2.16}$$

it furnishes different commutation relations than the ones described by eqs-(2.14,2.15). The latter commutators are found in the special case when $\alpha = (-i)^{1/2}$, $\beta = (-i)^{3/2}$ as chosen in eq-(2.5). For instance, if either $\alpha = 0$ or $\beta = 0$ it will lead instead to vanishing commutators $[Z_a, Z_b^\dagger] = [Z_a, Z_b] = [Z_a^\dagger, Z_b^\dagger] = 0$ as a result of eq-(2.8e). And, in turn, one would have had $[X_a, X_b] = [P_a, P_b] = [X_a, P_b] = 0$ instead of eqs-(2.15). Therefore, the introduction of non-vanishing complex coefficients α, β, via eq-(2.16), yield a two-parameter family of deformed fiber coordinates and momenta algebras parametrized by α, β. In particular, one may explicitly introduce these parameters by writing $Z_a(\alpha, \beta), Z_a^\dagger(\alpha^*, \beta^*)$.

After introducing the complex-valued vierbein $E_\mu^a = e_\mu^a + i f_\mu^a$, it leads to the complex metric

$$g_{\mu\nu} \equiv E_\mu^a \ (E_\nu^b)^* \eta_{ab} \ = \ g_{(\mu\nu)} \ + i \ g_{[\mu\nu]} \tag{2.18a}$$

with

$$g_{(\mu\nu)} \ = \ (e_\mu^a \ e_\nu^b \ + \ f_\mu^a \ f_\nu^b) \ \eta_{ab}, \quad i \ g_{[\mu\nu]} \ = \ - i \ (e_\mu^a \ f_\nu^b \ - \ e_\nu^b \ f_\mu^a) \ \eta_{ab} \tag{2.18b}$$

The 4×4 complex metric $g_{\mu\nu}$ is Hermitian $g_{\mu\nu}^\dagger = g_{\mu\nu}$ as a result of $g_{\nu\mu} = (g_{\mu\nu})^*$. To verify that $g_{[\mu\nu]} = -g_{[\nu\mu]}$ one just needs to relabel the indices $a \leftrightarrow b$ in (eq-2.18b) and recur to $\eta_{ba} = \eta_{ab}$.

The two-parameter family of $U(1,4)$-valued Hermitian gauge fields is given by

$$\mathbf{A}_\mu \ = \ \Omega_\mu^{ab} \ Z_{ab} \ + \frac{1}{L} \ [\ E_\mu^a \ Z_a(\alpha, \beta) \ + \ (E_\mu^a)^* \ Z_a^\dagger(\alpha^*, \beta^* \ ) \ ] \ + \ \Omega_\mu \ \mathcal{I} \ . \tag{2.19}$$

where L is a length scale that is introduced for dimensional reasons since the physical units of $\mathbf{A}_\mu$ are $(length)^{-1}$. $\Omega_\mu^{ab} Z_{ab}$ is given by $\frac{1}{2}(\Omega_\mu^{(ab)} M_{ab} + \Omega_\mu^{[ab]} L_{ab})$, and $Z_a(\alpha, \beta), Z_a^\dagger(\alpha^*, \beta^*)$ are displayed in eq-(2.16).

One can rewrite the two-parameter family of $U(1,4)$-valued Hermitian gauge fields (2.19) as

$$\mathbf{A}_\mu \ = \ \Omega_\mu^{ab} \ Z_{ab} \ + \ \Omega_\mu^{(a5)} \ M_{a5} \ + \ \Omega_\mu^{[a5]} \ L_{a5} \ + \ \Omega_\mu \ \mathcal{I}, \quad \Omega_\mu \equiv \Omega_\mu^{55}. \tag{2.20}$$

After some straightforward algebra one finds that the real-valued connection components $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ are given by suitable linear combinations of the e_μ^a, f_μ^a components of the complex-valued vierbein as follows

$$\Omega_\mu^{a5} \ = \ e_\mu^a \ (\frac{\alpha + \alpha^*}{L}) \ - \ f_\mu^a \ (\frac{\alpha - \alpha^*}{iL}); \quad \Omega_\mu^{5a} \ = \ e_\mu^a \ (\frac{\beta + \beta^*}{L}) \ - \ f_\mu^a \ (\frac{\beta - \beta^*}{iL}) \tag{2.21a}$$

and such that

$$\Omega_\mu^{(a5)} \ \equiv \ \frac{1}{2} \ (\Omega_\mu^{a5} \ + \ \Omega_\mu^{5a}), \quad \Omega_\mu^{[a5]} \ \equiv \ \frac{1}{2} \ (\Omega_\mu^{a5} \ - \ \Omega_\mu^{5a}) \tag{2.21b}$$

Because $\alpha \neq \beta$, one finds that $\Omega_\mu^{a5} \neq \Omega_\mu^{5a}$, consequently, $\Omega_\mu^{(a5)} \neq 0; \Omega_\mu^{[a5]} \neq 0$. Therefore, the introduction of the two distinct complex coefficients α, β is tantamount of choosing an infinite family of real-valued connection components $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ given by the many different linear combinations of e_μ^a and f_μ^a. The real valued coefficients of these linear combinations are given by the real and imaginary parts of α and β as displayed in eq-(2.21a). One should also emphasize that *no* zero torsion conditions were imposed in reaching the relations in eqs-(2.21) between $\Omega_\mu^{a5}, \Omega_\mu^{5a}$ and e_μ^a, f_μ^a.

The Hermitian $U(1,4)$-valued field strength is defined by

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i\, [\mathbf{A}_\mu, \mathbf{A}_\nu] \tag{2.22}$$

from which one can read-off the curvature components $R_{\mu\nu}^{(ab)}; R_{\mu\nu}^{[ab]}$, and the other components of the field strength (like torsion), in terms of the connection components (and their derivatives) of eq-(2.19) from the following decomposition of the field strength

$$\mathbf{F}_{\mu\nu} = R_{\mu\nu}^{(ab)}\, M_{ab} + R_{\mu\nu}^{[ab]}\, L_{ab} + \frac{1}{L}\, [\, F_{\mu\nu}^a\, Z_a(\alpha, \beta) + (F_{\mu\nu}^a)^*\, Z_a^\dagger(\alpha^*, \beta^*\,)\,] + F_{\mu\nu}\, \mathcal{I} \tag{2.23}$$

By proceeding as one did in [16] one may then construct the generalized actions for complex gravity after using the complex metric (vierbein) and its inverse to raise and lower indices. The most simple actions can have terms linear and quadratic in the curvature, and also quadratic terms in the torsion. For further details we refer to [16].

Alternatively, one could instead start with the $U(1,4)$-valued Hermitian gauge field in eq-(2.20) leading to the field strength

$$\mathbf{F}_{\mu\nu} = R_{\mu\nu}^{(ab)}\, M_{ab} + R_{\mu\nu}^{[ab]}\, L_{ab} + R_\mu^{(a5)}\, M_{a5} + R_\mu^{[a5]}\, L_{a5} + F_{\mu\nu}\, \mathcal{I} \tag{2.24}$$

and expressed in terms of $\Omega_\mu^{(ab)}, \Omega_\mu^{[ab]}, e_\mu^a, f_\mu^a, \Omega_\mu^{55} = \Omega_\mu$ and their derivatives. Note that $U(1,4)$ has 25 generators, whereas the metric affine group in $4D$, given by the semi-direct product of $GL(4,R)$ with the translation group T_4, has 20 generators. Therefore, the complex gravitational theory based on $U(1,4)$, and inspired from Born reciprocal relativity theory, has more degrees of freedom than the metric affine theory of gravity in $4D$. This is not surprising since one is dealing with gravity in curved phase spaces. There is also torsion in our construction.

A curved phase-space action associated with the geometry of the cotangent bundle of spacetime and based on Lagrange-Finsler and Hamilton-Cartan Geometry [17], [18] can be found in [15]. To conclude this section, there are two different approaches to construct generalized gravitational theories in curved phase spaces : (i) via the $U(1,4)$ local gauge theory construction presented here, or (ii) via Finsler geometric methods.

3 The Yang Algebra versus the Deformed Quaplectic Algebra

This section is devoted to an extensive analysis of the Yang and the deformed Quaplectic algebras associated with noncommutative phase spaces. Secondly, we present extensions of such algebras involving antisymmetric tensor coordinates and momenta of different ranks.

3.1 The Yang Algebra and its Extension via Generalized Angular Momentum Operators in Higher Dimensions

Given a flat $6D$ spacetime with coordinates $Y^M = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$, and a metric $\eta_{MN} = diag(-1, +1, +1, \ldots, +1)^2$, the Yang algebra [2], which is an extension of the Snyder algebra [1], can be derived in terms of the $SO(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators[3]

$$J^{MN} \; = \; - \; (Y^M \; \Pi^N \; - \; Y^N \; \Pi^M) \; = \; i \; Y^M \frac{\partial}{\partial Y_N} \; - i \; Y^N \frac{\partial}{\partial Y_M} \tag{3.1}$$

where $\Pi^M = -i(\partial/\partial Y_A)$ is the canonical conjugate momentum variable to Y^M. Their commutators are

$$[Y^M, Y^N] \; = \; 0, \; [\Pi^M, \Pi^N] \; = \; 0, \; \; [Y^M, \Pi^N] \; = \; i \; \eta^{MN}, \; \; M, N = 1, 2, 3, 4, 5, 6 \tag{3.2}$$

The coordinates Y^M commute. The momenta Π^M also commute, and the canonical conjugate variables Y^M, Π^N obey the Weyl-Heisenberg algebra in $6D$.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting $4D$ spacetime coordinates x^μ, the noncommuting momenta p^μ, and the Lorentz $SO(5,1)$ algebra generators leading to the Yang algebra [2] is given by

$$x^\mu \; \leftrightarrow \; L_P \; J^{\mu 5} \; = \; - \; L_P \; (Y^\mu \; \Pi^5 \; - \; Y^5 \; \Pi^\mu) \tag{3.3a}$$

$$p^\mu \; \leftrightarrow \; \frac{1}{\mathcal{L}} \; J^{\mu 6} \; = \; - \; \frac{1}{\mathcal{L}} \; (Y^\mu \; \Pi^6 \; - \; Y^6 \; \Pi^\mu), \quad \mu, \nu = 1, 2, 3, 4 \tag{3.3b}$$

and which requires the introduction of an ultra-violet cutoff scale L_P given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$

[2] We choose a different signature than the one in the introduction
[3] Our choice differs by a minus sign from the conventional definition

the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [1] and Yang [2] algebras.[4]

The other generators are given by

$$\mathcal{N} \ \equiv \ J^{56} = -(Y^5 \ \Pi^6 - Y^6 \ \Pi^5), \ \ J^{\mu\nu} \ = \ - \ (Y^\mu \ \Pi^\nu - Y^\nu \ \Pi^\mu), \ \ \mu, \nu = 1, 2, 3, 4 \tag{3.4}$$

One can then verify that the Yang algebra is recovered after imposing the correspondence in eqs-(3.3, 3.4)

$$[x^\mu, \ x^\nu] \ = \ - \ i \ L_P^2 \ J^{\mu\nu}, \ \ [p^\mu, \ p^\nu] \ = \ - \ i \ (\frac{1}{\mathcal{L}})^2 \ J^{\mu\nu}, \ \ \eta^{55} \ = \ \eta^{66} = 1 \tag{3.5}$$

$$[x^\mu, \ J^{\nu\rho}] \ = \ i \ (\eta^{\mu\rho} \ x^\nu \ - \ \eta^{\mu\nu} \ x^\rho) \tag{3.6}$$

$$[p^\mu, \ J^{\nu\rho}] \ = \ i \ (\eta^{\mu\rho} \ p^\nu \ - \ \eta^{\mu\nu} \ p^\rho \ ) \tag{3.7}$$

$$[x^\mu, \ p^\nu] \ = \ - \ i \ \eta^{\mu\nu} \ \frac{L_P}{\mathcal{L}} \ \mathcal{N}, \ \ [J^{\mu\nu}, \mathcal{N}] = 0 \tag{3.8}$$

$$[x^\mu, \ \mathcal{N}] \ = \ i \ L_P \mathcal{L} \ p^\mu, \ \ [p^\mu, \ \mathcal{N}] \ = \ - \ i \ \frac{1}{L_P \mathcal{L}} \ x^\mu \tag{3.9}$$

and where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $SO(3,1)$ Lorentz algebra in $4D$. They are of the form

$$\begin{aligned} [\ J^{\mu_1\mu_2}, \ J^{\nu_1\nu_2} \ ] \ &= \ - \ i \ \eta^{\mu_1\nu_1} \ J^{\mu_2\nu_2} \ + i \ \eta^{\mu_1\nu_2} \ J^{\mu_2\nu_1} \ + \\ & i \ \eta^{\mu_2\nu_1} \ J^{\mu_1\nu_2} \ - i \ \eta^{\mu_2\nu_2} \ J^{\mu_1\nu_1}, \ \ \hbar = c = 1 \end{aligned} \tag{3.10}$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of eq-(1.10) since the commutator of two Hermitian operators is anti-Hermitian. The $4D$ spacetime metric is $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

Given the above correspondence (2.3), we were able to *extend* it further to the higher grade polyvector-valued coordinates and momenta operators in noncommutative Clifford phase spaces [14]. Given a Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$, a polyvector-valued coordinate is defined as $\mathbf{X} = X_M \Gamma^M$, and admits the following expansion in terms of the Clifford algebra generators in D-dimensions, $\mathbf{1}, \gamma^\mu, \gamma^{\mu_1} \wedge \gamma^{\mu_2}, \cdots, \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \cdots \wedge \gamma^{\mu_D}$, as follows

$$\begin{aligned} \mathbf{X} \ = \ X \ \mathbf{1} \ + \ X_\mu \ \gamma^\mu \ + \ X_{\mu_1\mu_2} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \ + \ X_{\mu_1\mu_2\mu_3} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \ + \ ...... \ + \\ X_{\mu_1\mu_2\mu_3......\mu_D} \ \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \ \gamma^{\mu_3} .... \ \wedge \gamma^{\mu_D} \end{aligned} \tag{3.11a}$$

The numerical combinatorial factors can be omitted by imposing the ordering prescription $\mu_1 < \mu_2 < \mu_3 \cdots < \mu_D$. In order to match physical units in each

[4] A simple inspection reveals that a correspondence of the form $\frac{x^\mu}{L_P} = a_1 J^{\mu 5} + b_1 J^{\mu 6}; \mathcal{L} p^\mu = a_2 J^{\mu 5} + b_2 J^{\mu 6}$ will automatically lead to $b_1 = 0, a_2 = 0$; or $b_2 = 0, a_1 = 0$ resulting from the antisymmetry of the commutators $[x^\mu, x^\nu], [p^\mu, p^\nu]$

term of (2.11a) a length scale parameter must be suitably introduced in the expansion in eq-(2.11a). In [7] we introduced the Planck scale as the expansion parameter in (2.11a), and which was set to unity, when one adopts the units $\hbar = c = G = 1$.

Similarly, the polyvector-valued momentum $\mathbf{P} = P_M \Gamma^M$ admits the following expansion in terms of the Clifford algebra generators in D-dimensions

$$\mathbf{P} \;=\; P\,\mathbf{1} \;+\; P_\mu\,\gamma^\mu \;+\; P_{\mu_1\mu_2}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2} \;+\; P_{\mu_1\mu_2\mu_3}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\gamma^{\mu_3} \;+\; ......\;+$$

$$P_{\mu_1\mu_2\mu_3......\mu_D}\,\gamma^{\mu_1}\wedge\gamma^{\mu_2}\wedge\;\gamma^{\mu_3}.......\;\wedge\gamma^{\mu_D} \qquad (3.11b)$$

The scalar, vectorial, antisymmetric tensorial coordinates X, X_μ, $X_{\mu_1\mu_2} = -X_{\mu_2\mu_1}, \cdots, X_{\mu_1\mu_2\cdots\mu_D}$ are the scalar, vector, bivector, trivector, $\cdots$ components of the polyvector-valued coordinates. The $X_{\mu_1\mu_2}$ bivector (antisymmetric tensor of rank 2) corresponds to an oriented area element. The trivector $X_{\mu_1\mu_2\mu_3}$ (antisymmetric tensor of rank 3) corresponds to an oriented volume element, and so forth.

Similarly, the scalar, vectorial, antisymmetric tensorial coordinates P, P_μ, $P_{\mu_1\mu_2} = -P_{\mu_2\mu_1}, \cdots, P_{\mu_1\mu_2\cdots\mu_D}$ are the scalar, vector, bivector, trivector, $\cdots$ components of the polyvector-valued momentum coordinates. The $P_{\mu_1\mu_2}$ bivector (antisymmetric tensor of rank 2) corresponds to an oriented areal-momentum element. The trivector $P_{\mu_1\mu_2\mu_3}$ (antisymmetric tensor of rank 3) corresponds to an oriented volume-momentum element, and so forth.

We constructed in [14] the corresponding non-vanishing commutators among the *noncommutative* antisymmetric tensors $X^{\mu_1\mu_2}, X^{\mu_1\mu_2\mu_3}, \cdots; P^{\mu_1\mu_2}, P^{\mu_1\mu_2\mu_3}, \cdots$ of different ranks. We coined such *extension* of the Yang algebra the Clifford-Yang algebra since it involves polyvector-valued coordinates and momenta associated with a Clifford algebra. The *noncommuting* bivector coordinates obey

$$[X^{\mu_1\mu_2}, X^{\nu_1\nu_2}] \;\sim\; iL_P^4\,\eta^{55}\,J^{\mu_1\mu_2|\nu_1\nu_2},\;\; J^{\mu_1\mu_2|\nu_1\nu_2} \;\equiv\; -(Y^{\mu_1\mu_2}\,\Pi^{\nu_1\nu_2} - Y^{\nu_1\nu_2}\,\Pi^{\mu_1\mu_2}) \qquad (3.12a)$$

$Y^{\mu_1\mu_2}$ is a bivector coordinate associated with the $Cl(5,1)$ algebra of the $6D$ flat spacetime. $\Pi^{\mu_1\mu_2} = -i(\partial/\partial Y_{\mu_1\mu_2})$ is the corresponding bivector canonical momentum conjugate. Their commutators are

$$[Y^{\mu_1\mu_2}, Y^{\nu_1\nu_2}] \;=\; 0,\;\; [\Pi^{\mu_1\mu_2}, \Pi^{\nu_1\nu_2}] \;=\; 0,\;\; [Y^{\mu_1\mu_2}, P^{\nu_1\nu_2}] \;=\; i\,\eta^{\mu_1\mu_2|\nu_1\nu_2} \qquad (3.12b)$$

and where the generalized metric involving bivector indices is defined as

$$\eta^{\mu_1\mu_2|\nu_1\nu_2} \;=\; \eta^{\nu_1\nu_2|\mu_1\mu_2} \;=\; \eta^{\mu_1\nu_1}\,\eta^{\mu_2\nu_2} \;-\; \eta^{\mu_1\nu_2}\,\eta^{\mu_2\nu_1} \qquad (3.12c)$$

The *noncommuting* bivector momenta obey

$$[P^{\mu_1\mu_2}, P^{\nu_1\nu_2}] \;\sim\; i\mathcal{L}^{-4}\,\eta^{66}\,J^{\mu_1\mu_2|\nu_1\nu_2} \qquad (3.12d)$$

And so forth. All the commutators have the same structural form of a generalized angular momentum algebra as follows

$$[J^{A(r_1)|B(r_2)},\, J^{C(s_1)|D(s_2)}] \;=\; -i\,\eta^{A(r_1)|C(s_1)}\,J^{B(r_2)|D(s_2)} + i\,\eta^{A(r_1)|D(s_2)}\,J^{B(r_2)|C(s_1)} \;+$$

$$i\ \eta^{B(r_2)|C(s_1)}\ J^{A(r_1)|D(s_2)}\ -i\ \eta^{B(r_2)|D(s_2)}\ J^{A(r_1)|C(s_1)},\quad \hbar = c = 1 \qquad (3.12e)$$

where the grades of the polyvector indices $A(r_1)B(r_2), C(s_1), D(s_2)$ appearing in the generators are r_1, r_2, s_1, s_2, respectively. The shorthand notation for $J^{a_1 a_2 \cdots a_{r_1}|b_1 b_2 \cdots b_{r_2}}$ is $J^{A(r_1)|B(r_2)}, \cdots$. The generalized metric tensor $\eta^{A|C} = 0$ if the grade of A is *not* equal to the grade of C. Similarly, $\eta^{A|D} = 0$ if the grade of A is *not* equal to the grade of $D, \cdots$. Also, $\eta^{\mu 5} = \eta^{\mu 6} = 0$ since the $6D$ metric is diagonal. The commutators (3.12e) will ensure that the Jacobi identities are satisfied. In addition, we found the spectrum of the quantum harmonic oscillator in noncommutative spaces in terms of the eigenvalues of the generalized angular momentum operators in higher dimensions, and discussed how to extend these results to higher grade polyvector-valued coordinates and momenta. For full details we refer to [14].

3.2 Realization of the Deformed Quaplectic Algebra and its Extensions

We have seen above how the *noncommutative* coordinates and momenta of the Yang-algebra in $4D$ can be realized in terms of the angular momentum operators in $6D$, and which in turn, are expressed in terms of the canonical-conjugate variables Y^M, Π^N in $6D$ shown in eqs-(3.3,3.4), and obeying the standard commutation relations displayed in eqs-(3.2). Inspired by this procedure we shall find next a realization of the $deformed$ Quaplectic algebra generators in terms of the canonical coordinate and momentum variables Y_a, Π_b, Y_5, Π_5 as follows

$$M_{ab} \ = \ M_{ba} \ = \ \frac{1}{2}(Y_a\ \Pi_b\ +\ \Pi_b\ Y_a)\ +\ \frac{1}{2}(Y_b\ \Pi_a\ +\ \Pi_a\ Y_b) \qquad (3.13a)$$

$$M_{a5} \ = \ M_{5a} \ = \ \frac{1}{2}(Y_a\,\Pi_5 + \Pi_5\,Y_a) + \frac{1}{2}(Y_5\,\Pi_a + \Pi_a\,Y_5),\ \ M_{55} \ = \ (Y_5\,\Pi_5 + \Pi_5\,Y_5) \qquad (3.13b)$$

$$L_{ab} \ = \ -\ L_{ba} \ = \ \frac{1}{2}(Y_a\ \Pi_b\ +\ \Pi_b\ Y_a)\ -\ \frac{1}{2}(Y_b\ \Pi_a\ +\ \Pi_a\ Y_b) \qquad (3.13c)$$

$$L_{a5} \ = \ -\ L_{5a} \ = \ \frac{1}{2}(Y_a\ \Pi_5\ +\ \Pi_5\ Y_a)\ -\ \frac{1}{2}(Y_5\ \Pi_a\ +\ \Pi_a\ Y_5) \qquad (3.13d)$$

From eqs-(3.12,3.13) one then finds an explicit realization of the generators $Z_{AB} = \frac{1}{2}(M_{AB} + L_{AB})$ of the deformed Quaplectic algebra, with $A, B = 1, 2, 3, 4, 5$, directly in terms of the canonical coordinate and momentum variables Y_a, Π_b, Y_5, Π_5, and obeying the following commutation relations

$$[Y_a,\ Y_b] \ = \ 0,\ \ [Y_a,\ Y_5] \ = \ 0,\ \ [\Pi_a,\ \Pi_b] \ = \ 0 \qquad (3.14a)$$

$$[\Pi_a,\ \Pi_5] \ = \ 0,\ \ [Y_a,\ \Pi_b] \ = \ i\ \eta_{ab},\ \ [Y_5,\ \Pi_5] \ = \ i\ \eta_{55} \qquad (3.14b)$$

From eqs-(3.14) one learns that when $a \neq b$, the generator M_{ab} reduces to $Y_a \Pi_b + Y_b \Pi_a$, and when $a = b$, $M_{aa} = Y_a \Pi_a + \Pi_a Y_a$. While the generator

$L_{ab} = Y_a \Pi_b - Y_b \Pi_a$. Similarly, M_{a5} reduces to $Y_a \Pi_5 + Y_5 \Pi_a$; $M_{55} = Y_5 \Pi_5 + \Pi_5 Y_5$, and $L_{a5} = Y_a \Pi_5 - Y_5 \Pi_a$

The difference between the Yang and the deformed Quaplectic algebra is that in the Yang algebra case one adds two additional coordinates and momenta Y^5, Y^6, Π^5, Π^6 in order to construct the $SO(5,1)$ algebra with 15 generators. Whereas in the deformed Quaplectic algebra case one adds one additional coordinate and momentum Y^5, Π^5, and the extra generators $M_{ab}, M_{a5}, M_{55} = \mathcal{I}$ in order to construct the $U(1,4)$ algebra with 25 generators. Furthermore, the construction of the Yang algebra requires the two length scales $L_P, \mathcal{L}$; whereas in the (deformed) Quaplectic algebra one has the length scale λ_l, and the momentum scale λ_p.

The antisymmetric rank-2 tensor coordinates and momenta operators extensions of the expressions in eqs-(3.12,3.13) are given by

$$M_{a_1 a_2 | b_1 b_2} \ = \ \frac{1}{2} (Y_{a_1 a_2} \ \Pi_{b_1 b_2} \ + \ \Pi_{b_1 b_2} \ Y_{a_1 a_2}) \ + \ \frac{1}{2} (Y_{b_1 b_2} \ \Pi_{a_1 a_2} \ + \ \Pi_{a_1 a_2} \ Y_{b_1 b_2}) \tag{3.15a}$$

$$L_{a_1 a_2 | b_1 b_2} \ = \ \frac{1}{2} (Y_{a_1 a_2} \ \Pi_{b_1 b_2} \ + \ \Pi_{b_1 b_2} \ Y_{a_1 a_2}) \ - \ \frac{1}{2} (Y_{b_1 b_2} \ \Pi_{a_1 a_2} \ + \ \Pi_{a_1 a_2} \ Y_{b_1 b_2}) \tag{3.15b}$$

where

$$M_{a_1 a_2 | b_1 b_2} \ = \ - \ M_{a_2 a_1 | b_1 b_2} \ = \ - \ M_{a_1 a_2 | b_2 b_1} \ = \ M_{b_1 b_2 | a_1 a_2} \tag{3.16a}$$

$$L_{a_1 a_2 | b_1 b_2} \ = \ - \ L_{a_2 a_1 | b_1 b_2} \ = \ - \ L_{a_1 a_2 | b_2 b_1} \ = \ - \ L_{b_1 b_2 | a_1 a_2} \tag{3.16b}$$

Given $M_{a_1 a_2 | b_1 b_2}, L_{a_1 a_2 | b_1 b_2}$ the generalization of the operator Z_{ab} is

$$Z_{a_1 a_2 | b_1 b_2} \ \equiv \ \frac{1}{2} \ (M_{a_1 a_2 | b_1 b_2} \ + \ L_{a_1 a_2 | b_1 b_2}) \tag{3.16c}$$

The generalization of the commutators in eqs-(2.8a,2.8b,2.8c) corresponding to the $M_{a_1 a_2 | b_1 b_2}, L_{a_1 a_2 | b_1 b_2}$ generators is given by

$$[L_{a_1 a_2 | b_1 b_2}, \ L_{c_1 c_2 | d_1 d_2}] \ = \ i \ \eta_{b_1 b_2 | c_1 c_2} \ L_{a_1 a_2 | d_1 d_2} \ - \ i \ \eta_{a_1 a_2 | c_1 c_2} \ L_{b_1 b_2 | d_1 d_2} \ - \ i \ \eta_{b_1 b_2 | d_1 d_2} \ L_{a_1 a_2 | c_1 c_2} \ + \ i \ \eta_{a_1 a_2 | d_1 d_2} \ L_{b_1 b_2 | c_1 c_2} \tag{3.17}$$

$$[M_{ab}, \ M_{cd}] \ = \ - \ i \ \eta_{b_1 b_2 | c_1 c_2} \ L_{a_1 a_2 | d_1 d_2} \ - \ i \ \eta_{a_1 a_2 | c_1 c_2} \ L_{b_1 b_2 | d_1 d_2} \ - \ i \ \eta_{b_1 b_2 | d_1 d_2} \ L_{a_1 a_2 | c_1 c_2} \ - \ i \ \eta_{a_1 a_2 | d_1 d_2} \ L_{b_1 b_2 | c_1 c_2} \tag{3.18}$$

$$[L_{ab}, \ M_{cd}] \ = \ i \ \eta_{b_1 b_2 | c_1 c_2} \ M_{a_1 a_2 | d_1 d_2} \ - \ i \ \eta_{a_1 a_2 | c_1 c_2} \ M_{b_1 b_2 | d_1 d_2} \ + \ i \ \eta_{b_1 b_2 | d_1 d_2} \ M_{a_1 a_2 | c_1 c_2} \ - \ i \ \eta_{a_1 a_2 | d_1 d_2} \ M_{b_1 b_2 | c_1 c_2} \tag{3.19}$$

where

$$\eta^{a_1 a_2 | b_1 b_2} \ \equiv \ \eta^{a_1 b_1} \ \eta^{a_2 b_2} \ - \ \eta^{a_1 b_2} \ \eta^{a_2 b_1} \tag{3.20}$$

From eqs-(3.16c,3.17-3.20) one finds that

$$[\ Z_{a_1a_2|b_1b_2},\ Z_{c_1c_2|d_1d_2}\ ]\ =\ -i\ (\ \eta_{b_1b_2|c_1c_2}\ Z_{a_1a_2|d_1d_2}\ -\ \eta_{a_1a_2|d_1d_2}\ Z_{c_1c_2|b_1b_2}\ ). \tag{3.21}$$

This is a result of the canonical antisymmetric rank-2 tensor coordinates and momenta variables $Y_{a_1a_2}, \Pi_{b_1b_2}$ obeying the following commutation relations (the generalization of eqs-(3.14))

$$[Y_{a_1a_2},\ Y_{b_1b_2}]\ =\ 0,\ \ [\Pi_{a_1a_2},\ \Pi_{b_1b_2}]\ =\ 0,\ \ [Y_{a_1a_2},\ \Pi_{b_1b_2}]\ =\ i\ \eta_{a_1a_2|b_1b_2} \tag{3.22}$$

The other *dimensionless* generators are[5]

$$M_{a_1a_2|5}\ =\ \frac{Y_{a_1a_2}}{\lambda_l^2}\ \frac{\Pi_5}{\lambda_p}\ +\ \frac{Y_5}{\lambda_l}\ \frac{\Pi_{a_1a_2}}{\lambda_p^2},$$

$$M_{5|a_1a_2}\ =\ \frac{Y_5}{\lambda_l}\ \frac{\Pi_{a_1a_2}}{\lambda_p^2}\ +\ \frac{Y_{a_1a_2}}{\lambda_l^2}\ \frac{\Pi_5}{\lambda_p} \tag{3.23}$$

$$L_{a_1a_2|5}\ =\ \frac{Y_{a_1a_2}}{\lambda_l^2}\ \frac{\Pi_5}{\lambda_p}\ -\ \frac{Y_5}{\lambda_l}\ \frac{\Pi_{a_1a_2}}{\lambda_p^2},$$

$$L_{5|a_1a_2}\ =\ \frac{Y_5}{\lambda_l}\ \frac{\Pi_{a_1a_2}}{\lambda_p^2}\ -\ \frac{Y_{a_1a_2}}{\lambda_l^2}\ \frac{\Pi_5}{\lambda_p} \tag{3.24}$$

such that

$$Z_{a_1a_2|5}\ =\ \frac{1}{2}\ (M_{a_1a_2|5}\ +\ L_{a_1a_2|5}),\ \ Z_{5|a_1a_2}\ =\ \frac{1}{2}\ (M_{5|a_1a_2}\ +\ L_{5|a_1a_2}) \tag{3.25}$$

and leading to the following generators

$$Z_{[a_1a_2]}\ \equiv\ \frac{1}{\sqrt{2}}\ (\frac{X_{a_1a_2}}{\lambda_l^2}\ -i\ \frac{P_{a_1a_2}}{\lambda_p^2})\ =\ \alpha\ Z_{a_1a_2|5}\ +\beta\ Z_{5|a_1a_2}, \tag{3.26a}$$

$$Z^\dagger_{[a_1a_2]}\ \equiv\ \frac{1}{\sqrt{2}}\ (\frac{X_{a_1a_2}}{\lambda_l^2}\ +i\ \frac{P_{a_1a_2}}{\lambda_p^2})\ =\ \alpha^*\ Z_{a_1a_2|5}\ +\beta^*\ Z_{5|a_1a_2} \tag{3.26b}$$

where α, β are suitable complex-valued coefficients chosen so that[6]

$$[Z_{[a_1a_2]},\ Z^\dagger_{[b_1b_2]}]\ =\ -(\ \eta_{a_1a_2|b_1b_2}\ \mathcal{I}\ +\ M_{a_1a_2|b_1b_2}\ ) \tag{3.27}$$

$$[Z_{[a_1a_2]},\ Z_{[b_1b_2]}]\ =\ [Z^\dagger_{[a_1a_2]},\ Z^\dagger_{[b_1b_2]}]\ =\ -i\ L_{a_1a_2|b_1b_2} \tag{3.28}$$

Finally, from eqs-(3.26,3.27,3.28) one arrives at the desired result

[5]Since $\lambda_l\lambda_p = 1$, in units of $\hbar = 1$, the powers of λ_l, λ_p decouple explicitly from eqs-(3.15)

[6]Note that one must not confuse $Z_{ab} \equiv \frac{1}{2}(M_{ab} + L_{ab})$ with $Z_{[a_1a_2]}$ defined by eq-(3.26a)

$$[\frac{X_{a_1a_2}}{\lambda_l^2}, \frac{P_{b_1b_2}}{\lambda_p^2}] \ = \ i \ (\eta_{a_1a_2|b_1b_2} \ \mathcal{I} + M_{a_1a_2|b_1b_2}) \tag{3.29}$$

$$[X_{a_1a_2}, X_{b_1b_2}] \ = \ -i \ (\lambda_l)^4 \ L_{a_1a_2|b_1b_2}; \ [P_{a_1a_2}, P_{b_1b_2}] \ = \ i \ (\lambda_p)^4 \ L_{a_1a_2|b_1b_2}; \tag{3.30}$$

The above construction can be *extended* to higher rank antisymmetric tensor coordinates and momenta $Y_{a_1a_2,a_3}, \Pi_{a_1a_2a_3}, \cdots$ leading to the generators $Z_{a_1a_2a_3|b_1b_2b_3}; Z_{a_1a_2a_3|5}; Z_{5|a_1a_2a_3}, \cdots$, and whose commutators are the extensions of the equations above. The end result is

$$[\frac{X_{a_1a_2\cdots a_n}}{\lambda_l^n}, \ \frac{P_{b_1b_2\cdots b_n}}{\lambda_p^n}] \ = \ i \ (\eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} \ \mathcal{I} + M_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}) \tag{3.31}$$

$$[X_{a_1a_2\cdots a_n}, \ X_{b_1b_2\cdots b_n}] \ = \ - \ i \ (\lambda_l)^{2n} \ L_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} \tag{3.32a}$$

$$[P_{a_1a_2\cdots a_n}, \ P_{b_1b_2\cdots b_n}] \ = \ i \ (\lambda_p)^{2n} \ L_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} \tag{3.32b}$$

where $\eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}$ can be written as the determinant of the $n \times n$ matrix whose entries are $\eta^{a_ib_j}$ with $i, j = 1, 2, \cdots, n$. The same occurs with $\delta^{a_1a_2\cdots a_n}_{b_1b_2\cdots b_n}$ where the entries are $\delta^{a_i}_{b_j}$. One finds that eqs-(3.31,32) do not differ too much from those corresponding equations of the Clifford-Yang algebra [14]. In the latter algebra, $\mathcal{I} = 2Z_{55} = M_{55}$ is replaced by $\mathcal{N} \equiv J^{56}$; there are no $M_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}$ terms, and λ_l, λ_p are replaced by $L_P, \mathcal{L}^{-1}$, respectively where $L_P, \mathcal{L}$ are the lower and upper length scales.

To sum up, all the commutation relations can be obtained from

$$[\ Z_{a_1a_2\cdots a_n|b_1b_2\cdots b_n}, \ Z_{c_1c_2\cdots c_n|d_1d_2\cdots d_n} \ ] \ =$$
$$-i \ (\ \eta_{b_1b_2\cdots b_n|c_1c_2\cdots c_n} \ Z_{a_1a_2\cdots a_n|d_1d_2\cdots d_n} \ - \ \eta_{a_1a_2\cdots a_n|d_1d_2\cdots d_n} \ Z_{c_1c_2\cdots c_n|b_1b_2\cdots b_n} \ ). \tag{3.33a}$$

$$[\ Z_{a_1a_2\cdots a_n|5}, \ Z_{5|b_1b_2\cdots b_n} \ ] \ = \ -i \ (\ \eta_{55} \ Z_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} - \eta_{a_1a_2\cdots a_n|b_1b_2\cdots b_n} \ Z_{55} \ ), \ \ \cdots \tag{3.33b}$$

4 Curved Phase Space due to Noncommutative Coordinates and Momenta

Noncommuting momentum operators are a reflection of the spacetime curvature after invoking the QM prescription $p_\mu \leftrightarrow -i\hbar\nabla_\mu$. By Born's reciprocity, noncommuting coordinates are a reflection of the momentum space curvature after invoking $x_\mu \leftrightarrow i\hbar\tilde{\nabla}_\mu$, where the tilde derivatives represent derivatives with respect to the momentum variables.

Having reviewed the basics of the Yang algebra of noncommutative phase spaces, Born Reciprocal Relativity, the extended Yang and (deformed) Quaplectic algebras, in this section we shall provide a solution for the exact analytical mapping of the non-commuting x^μ, p^μ operator variables (associated to an $8D$ curved phase space) into the canonical Y^A, Π^A operator variables of a flat $12D$ phase space. We explore the geometrical implications of this mapping which provides, in the *classical* limit, with the embedding functions $Y^A(x,p), \Pi^A(x,p)$ of an $8D$ curved phase space into a flat $12D$ phase space background. The latter embedding functions determine the functional forms of the base spacetime metric $g_{\mu\nu}(x,p)$, the fiber metric of the vertical space $h^{ab}(x,p)$, and the nonlinear connection $N_{a\mu}(x,p)$ associated with the $8D$ cotangent space of the $4D$ spacetime.

4.1 Mapping of x^μ, p^μ to the Y^A, Π^A variables in Flat Phase Space

The Y^5, Y^6, Π^5, Π^6 canonical coordinates and momenta (operators) in the flat 12-dim phase space are scalars from the point of view of the 8-dim curved phase space parametrized by the non-canonical coordinates x^μ and momenta p^μ. Therefore, Y^5, Y^6, Π^5, Π^6 must be functions of the Lorentz scalars

$$x^2 = \eta_{\mu\nu} x^\mu x^\nu, \;\; p^2 = \eta_{\mu\nu} p^\mu p^\nu, \;\; x \cdot p = \eta_{\mu\nu} x^\mu p^\mu, \;\; p \cdot x = \eta_{\mu\nu} p^\mu x^\nu, \;\; \mu, \nu = 1,2,3,4 \tag{4.1}$$

Setting $\alpha = \mathcal{L}^{-1}, \beta = L_P$, due to the Born reciprocity principle, one must have functions $f(z_1, z_2, z_3)$ of the arguments z_1, z_2, z_3 given by the following combination of Hermitian variables (operators)

$$z_1 \equiv (\alpha^2 x^2 \, + \, \beta^2 p^2), \;\; z_2 \equiv (x \cdot p + p \cdot x), \;\; z_3 \equiv i\,(x \cdot p - p \cdot x), \;\; \alpha = \mathcal{L}^{-1}, \; \beta = L_P \tag{4.2}$$

The arguments z_1, z_2, z_3 are invariant under $\alpha \leftrightarrow \beta$, $x \leftrightarrow p$, and $i \leftrightarrow -i$ if one wishes to implement Born's reciprocity symmetry. Therefore, one must have functions of the form

$$Y^5 \; = \; Y^5(z_1, z_2, z_3), \;\; Y^6 \; = \; Y^6(z_1, z_2, z_3), \;\; \Pi^5 = \Pi^5(z_1, z_2, z_3), \;\; \Pi^6 = \Pi^6(z_1, z_2, z_3) \tag{4.3}$$

For instance, one could have functions linear in z_1, z_2, z_3 defined as follows

$$Y^5(x,p) \; = \; a_1(\alpha^2 x^2 \; + \; \beta^2 p^2) \; + \; b_1(x \cdot p) \; + \; b_1^*(p \cdot x) \; + \; c_1 \tag{4.4a}$$

$$Y^6(x,p) \; = \; a_2(\alpha^2 x^2 \; + \; \beta^2 p^2) \; + \; b_2(x \cdot p) \; + \; b_2^*(p \cdot x) \; + \; c_2 \tag{4.4b}$$

$$\Pi^5(x,p) \; = \; a_3(\alpha^2 x^2 \; + \; \beta^2 p^2) \; + \; b_3(x \cdot p) \; + \; b_3^*(p \cdot x) \; + \; c_3 \tag{4.4c}$$

$$\Pi^6(x,p) \; = \; a_4(\alpha^2 x^2 \; + \; \beta^2 p^2) \; + \; b_4(x \cdot p) \; + \; b_4^*(p \cdot x) \; + \; c_4. \tag{4.4d}$$

where a_i, b_i, c_i $(i = 1, 2, 3, 4)$ are judicious numerical (dimensionful) coefficients. The units of the coefficients in eqs-(4.4a,4.4b) are those of length, while those in eqs-(4.4c,4.4d) are those of mass. Note that the b_i coefficients in eqs-(4.4) are complex-valued $b_i = \gamma_i + i\delta_i$. The reason is that the combination

$$b_i\ (x{\cdot}p) + b_i^*\ (p{\cdot}x)\ =\ \gamma_i\ (x{\cdot}p{+}p{\cdot}x) + i\ \delta_i\ (x{\cdot}p{-}p{\cdot}x)\ =\ \gamma_i\ z_2 + \delta_i\ z_3,\ \ i = 1,2,3,4 \qquad (4.4e)$$

ensures that eq-(4.4e) is Hermitian by construction. Eq-(4.4e) is also invariant under Born's reciprocity $x \leftrightarrow p$ and $i \leftrightarrow -i$. We shall show that eqs-(4.4) should, in principle, provide satisfactory solutions to the embedding problem defined below.

The $[x^\mu, p^\nu]$ commutator is defined as

$$[x^\mu, p^\nu]\ =\ x^\mu\ p^\nu\ -\ p^\nu\ x^\mu\ =\ i\ \gamma^{\mu\nu}(x,p) \qquad (4.5)$$

where $\gamma^{\mu\nu}(x,p)$ is a second rank tensor, not necessarily symmetric, that we refrain from identifying it to a metric tensor. The above commutator can also be expressed in terms of the $6D$ angular momenta variables displayed by eqs-(3.3,3.4) as

$$[x^\mu, p^\nu] =\ i\ \gamma^{\mu\nu}(x,p)\ =\ -\ i\ \alpha\beta\ J^{56}(x,p)\ \eta^{\mu\nu}\ =$$

$$i\ \alpha\beta\ [\ Y^5(x,p)\ \Pi^6(x,p)\ -\ Y^6(x,p)\ \Pi^5(x,p)\ ]\ \eta^{\mu\nu},\quad \alpha = \mathcal{L}^{-1}, \beta = L_P \qquad (4.6)$$

Therefore, from eqs-(4.5,4.6) one arrives at the following relation, after contracting both equations with $\eta_{\mu\nu}$,

$$\frac{1}{4i}\ \eta_{\mu\nu}\ (x^\mu\ p^\nu\ -\ p^\nu\ x^\mu)\ =\ \frac{1}{4i}\ (x \cdot p\ -\ p \cdot x)\ =$$

$$\alpha\beta\ (\ Y^5(x,p)\ \Pi^6(x,p)\ -\ Y^6(x,p)\ \Pi^5(x,p))\ )\ =\ -\ \alpha\beta\ \mathcal{N} \qquad (4.7)$$

Therefore, in this particular case, one finds that the tensor is symmetric $\gamma^{\mu\nu}(x,p) = \Phi(x,p)\eta^{\mu\nu}$ and such that the conformal factor $\Phi(x,p)$ is Hermitian and given by the left hand side of eq-(4.7). The r.h.s of (4.7) is Hermitian because J^{56} is Hermitian due to the canonical and Hermiticity nature of the $6D$ variables : $(Y^5\Pi^6)^\dagger = \Pi^6 Y^5 = Y^5\Pi^6$, and $(Y^6\Pi^5)^\dagger = \Pi^5 Y^6 = Y^6\Pi^5$ resulting from the commutators of the $6D$ canonical variables given by eq-(3.2).

From eqs-(3.3) one learnt that the $4D$ operators x^μ, p^μ admitted a $6D$ angular momentum realization of the form

$$x^\mu =\ \beta\ J^{\mu 5}\ =\ -\ \beta\ (Y^\mu\ \Pi^5\ -\ Y^5\ \Pi^\mu),\quad \beta = L_P \qquad (4.8)$$

$$p^\mu\ =\ \alpha\ J^{\mu 6}\ =\ -\ \alpha\ (Y^\mu\ \Pi^6\ -\ Y^6\ \Pi^\mu),\quad \alpha = \mathcal{L}^{-1} \qquad (4.9)$$

From eqs-(4.8, 4.9) one can deduce the relation

$$\mathcal{J}^{\mu\nu}\ =\ x^\mu\ p^\nu\ -\ x^\nu\ p^\mu\ =\ \alpha\beta\ J^{56}\ (Y^\mu\ \Pi^\nu\ -\ Y^\nu\ \Pi^\mu) \qquad (4.10)$$

where $J^{56} \equiv \mathcal{N}$ and $J^{\mu\nu}$ are given by eq-(3.4) explicitly in terms of the $6D$ canonical variables Y^A, Π^B.

One can *invert* the relations in eqs-(4.8,4.9) as follows. After multiplying eqs-(4.8 4.9) on the *right* by Π^6, Π^5, respectively, and subtracting the top equation from the bottom one, it yields

$$\beta^{-1}\ x^\mu\ \Pi^6\ -\ \alpha^{-1}\ p^\mu\ \Pi^5\ =\ \Pi^\mu\ \mathcal{N}\ =\ \mathcal{N}\ \Pi^\mu \qquad (4.11a)$$

due to the canonical nature of the $6D$ variables Y^A, Π^A described by the commutators in eqs-(3.2) and which allows us to re-order the relevant factors due to the commutativity.

And multiplying eqs-(4.8, 249) on the *right* by Y^6, Y^5, respectively, and subtracting the top equation from the bottom one, it yields

$$\beta^{-1}\ x^\mu\ Y^6\ -\ \alpha^{-1}\ p^\mu\ Y^5\ =\ Y^\mu\ \mathcal{N}\ =\ \mathcal{N}\ Y^\mu \qquad (4.11b)$$

We shall see next that the functional forms of $Y^5(x,p), Y^6(x,p), \Pi^5(x,p)$, $\Pi^6(x,p)$ provided eqs-(4.4) lead to solutions to eq-(4.7), and which in turn, yield automatically the solutions to eqs-(4.11a, 4.11b). And, in doing so, one has found the solutions to the embedding problem : $Y^\mu = Y^\mu(x,p); \Pi^\mu = \Pi^\mu(x,p)$, with $\mathcal{N}(x,p) \equiv J^{56}(x,p) = -(Y^5\Pi^6 - Y^6\Pi^5)(x,p)$, and where $[\mathcal{N}, Y^\mu] = [\mathcal{N}, \Pi^\mu] = 0$. The operator $\mathcal{N}$ appearing in the right hand side of eqs-(4.11) can be moved to the left hand side via the inverse $\mathcal{N}^{-1}$ operator, and that can be defined as a formal power series as follows $[1-(1-\mathcal{N})]^{-1} = 1 + (1-\mathcal{N}) + (1-\mathcal{N})^2 + \cdots$.

Thus, from eqs-(4.7,4.11) one can then construct the maps from the x^μ, p^μ noncanonical (operator) variables in $4D$ to the canonical (operator) variables Y^A, Π^A in $6D$. After a laborious but straightforward procedure we find the following family of solutions

$$Y^5(x,p)\ =\ \kappa_1\ \beta\ z_1\ +\ \kappa_2\ \beta\ z_2\ +\ \kappa_3\ \beta\ z_3\ +\ \kappa_4\ \beta \qquad (4.12a)$$

$$Y^6(x,p)\ =\ \kappa_1\ \beta\ z_1\ +\ \kappa_2\ \beta\ z_2\ +\ \kappa_3\ \beta\ z_3\ +\ (\kappa_4+1)\ \beta \qquad (4.12b)$$

$$\Pi^5(x,p)\ =\ \kappa_1\ \beta^{-1}\ z_1\ +\ \kappa_2\ \beta^{-1}\ z_2\ +\ \frac{5}{4}\ \kappa_3\ \beta^{-1}\ z_3\ +\ \kappa_4\ \beta^{-1} \qquad (4.12c)$$

$$\Pi^6(x,p)\ =\ \kappa_1\ \beta^{-1}\ z_1\ +\ \kappa_2\ \beta^{-1}\ z_2\ +\ \frac{5}{4}\ \kappa_3\ \beta^{-1}\ z_3\ +\ (\kappa_4+1)\ \beta^{-1} \qquad (4.12d)$$

where $\kappa_3 = (\alpha\beta)^{-1}$ and $\kappa_1, \kappa_2, \kappa_4$ are three arbitrary parameters. This is due to the nonlinearity of the equations that one is solving. These solutions (4.12) have the form $Y^6 = Y^5 + \beta; \Pi^5 = \Pi^6 - \beta^{-1}$ such that $\alpha\beta\ Y^{[5}\ \Pi^{6]} = -\frac{z_3}{4} = -\alpha\beta\mathcal{N}$ as required by eq-(4.7).

When one takes the classical limit, upon restoring $\hbar$ which was set to unity in the terms $\gamma_i z_2 \to \frac{\gamma_i}{\hbar} z_2$ of eqs-(4.4e), in order to match units, one can see that these terms are *singular* in the $\hbar \to 0$ limit. Whereas the terms $\frac{\delta_i}{\hbar} z_3 \to -4\delta_i$ are well behaved and yield constants.

For these reasons we shall just adhere to the following prescription in finding the *classical* limit of the embedding functions $Y^A(x,p), \Pi^A(x,p)$. We could simply drop the *singular* $\frac{1}{\hbar} z_2$ terms in eqs-(4.12) by setting the arbitrary constant κ_2 to zero $\kappa_2 = 0$; and set the $\frac{1}{\hbar} z_3$ terms to constants that can be reabsorbed into a redefinition of the κ_4 parameter in the explicit solutions for Y^5, Y^6, Π^5, Π^6 given by eqs-(4.12). In doing so one ends up with the following expressions in the *classical* limit

$$Y^5(z_1) \ = \ \kappa_1 \ \beta \ z_1 \ + \ \beta \ (\ \kappa_4 \ - \ 4(\alpha\beta)^{-1} \ ) \qquad (4.13a)$$

$$Y^6(z_1) \ = \ \kappa_1 \ \beta \ z_1 \ + \ \beta \ (\ \kappa_4 + 1 \ - \ 4(\alpha\beta)^{-1} \ ) \qquad (4.13b)$$

$$\Pi^5(z_1) \ = \ \kappa_1 \ \beta^{-1} \ z_1 \ + \ \beta^{-1} \ (\ \kappa_4 \ - \ 5(\alpha\beta)^{-1} \ ) \qquad (4.13c)$$

$$\Pi^6(z_1) \ = \ \kappa_1 \ \beta^{-1} \ z_1 \ + \ \beta^{-1} \ (\ \kappa_4 + 1 \ - \ 5(\alpha\beta)^{-1} \ ) \qquad (4.13d)$$

To conclude, one can finally obtain the explicit solutions for $Y^\mu, (z_1, x^\mu, p^\mu)$; $\Pi^\mu(z_1, x^\mu, p^\mu)$, in the classical limit, and given in terms of the functions $Y^5(z_1)$, $Y^6(z_1), \Pi^5(z_1), \Pi^6(z_1)$ in eqs-(4.13) (and x^μ, p^μ) as follows

$$\alpha \ x^\mu \ \Pi^6(z_1) \ - \ \beta \ p^\mu \ \Pi^5(z_1) \ = \ - \ \Pi^\mu(z_1, x^\mu, p^\mu) \qquad (4.14a)$$

$$\alpha \ x^\mu \ Y^6(z_1) \ - \ \beta \ p^\mu \ Y^5(z_1) \ = \ - \ Y^\mu(z_1, x^\mu, p^\mu) \qquad (4.14b)$$

where $z_1 \equiv \alpha^2 x^2 + \beta^2 p^2$, $\alpha = \mathcal{L}^{-1}; \beta = L_P$. Next we shall study the geometrical implications of the (classical) embedding solutions found in this section and provided by eqs-(4.13.4.14).

4.2 Embedding a $8D$ curved phase space into a $12D$ flat phase space

The previous section involved the use of coordinates and momenta *operators*. In this section we shall deal with *classical* variables (**c**-numbers) x, p. A more rigorous notation in the previous section would have been to assign "hats" to operators $\hat{x}^\mu, \hat{p}^\mu; \hat{Y}^A, \hat{\Pi}^A$. For the sake of simplicity we avoided it. The geometry of the cotangent bundle of spacetime (phase space) can be best explored within the context of Lagrange-Finsler, Hamilton-Cartan geometry [17], [18]. The line element in the $8D$ curved phase space is

$$(ds)^2 \ = \ g_{\mu\nu}(x,p) \, dx^\mu dx^\nu \ + \ h^{ab}(x,p) \ (dp_a + N_{a\mu}(x,p) dx^\mu) \ (dp_b + N_{b\nu}(x,p) dx^\nu) \qquad (4.15)$$

where $g_{\mu\nu}(x,p), h^{ab}(x,p)$ are the base spacetime and internal space metrics, respectively, with $a, b = 1,2,3,4$, $\mu, \nu = 1,2,3,4$, and $N_{a\mu}(x,p)$ is the nonlinear connection.

One should note that the metric tensor $g_{\mu\nu}$ is *not* the vertical Hessian of the square of a Finsler function, and h^{ab} is *not* the inverse of $g_{\mu\nu}$. h^{ab} represents, physically, the cotangent bundle's internal-space metric tensor which is independent from the base-spacetime metric tensor $g_{\mu\nu}$. The number of total components of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ is $10 + 10 + 16 = 36 = (8 \times 9)/2$.

The generalized (vacuum) gravitational field equations associated with the geometry of the $8D$ cotangent bundle differ considerably from the the standard (vacuum) Einstein field equations in $8D$ based on Riemannian geometry. Thus, for instance, by using a base-spacetime $g_{\mu\nu}$ metric to be *independent* from the internal-space metric h_{ab}, and a nonlinear connection $N_{\mu a}$, it might avoid the reduction of the solutions of the generalized gravitational field equations to the standard Schwarzschild (Tangherlini) solutions when radial symmetry is imposed.

For example, in [15] we studied a scalar-gravity model in curved phase spaces. After a very laborious procedure, the variation of the action S with respect to the fundamental fields

$$\frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} = 0, \quad \frac{\delta \mathcal{S}}{\delta h_{ab}} = 0, \quad \frac{\delta \mathcal{S}}{\delta N_{\mu a}} = 0, \quad \frac{\delta \mathcal{S}}{\delta \Phi} = 0 \tag{4.16}$$

leads to the very *complicated* field equations which differ considerably from the Einstein field equations. Exact nontrivial analytical solutions for the base-spacetime $g_{\mu\nu}$, the internal-space metric h_{ab} components, the nonlinear connection N_{ia}, and the scalar field Φ were found that obey the generalized gravitational field equations, in addition to satisfying the *zero* torsion conditions for *all* of the torsion components. See [15] for details.

The embedding of the $8D$ curved phase space into the 12-dim flat phase space is described by equating the $8D$ line interval ds^2 in (4.15) with the $12D$ one $ds^2 = \eta_{AB} dZ^A dZ^B$. After doing so, given $Z^A \equiv (Y^A, \Pi^A)$ one learns that

$$g_{\mu\nu} + h^{ab} N_{a\mu} N_{b\nu} = \eta_{AB} \frac{\partial Z^A}{\partial x^\mu} \frac{\partial Z^A}{\partial x^\nu} \tag{4.17}$$

$$h^{ab} = \eta_{AB} \frac{\partial Z^A}{\partial p_a} \frac{\partial Z^A}{\partial p_b} \tag{4.18}$$

$$h^{ab} N_{b\mu} = \eta_{AB} \frac{\partial Z^A}{\partial p_a} \frac{\partial Z^A}{\partial x^\mu} \quad A, B = 1, 2, \cdots, 5, 6 \tag{4.19}$$

Eqs-(4.17-4.19) *determine* the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ after one inserts the functional forms of the embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ found in the previous section, and by making the following replacement $p^\mu \to p_a$. We explained at the end of the previous section how the $x \cdot p, p \cdot x$ terms could *decouple* in the *classical* limit, by removing the singular terms $\frac{z_2}{\hbar}$, and where the $\frac{z_3}{\hbar}$ terms become constants, leaving only the terms $z_1 = w_1 = \alpha^2 x^2 + \beta^2 p^2$. Thus, after making the replacement $p^\mu \to p_a$ one has $\eta_{\mu\nu} p^\mu p^\nu \to \eta^{ab} p_a p_b$, and such that the indices will now match those appearing in eqs-(4.17-4.19).

To sum up, the (classical) embedding functions $Z^A(x,p) = Y^A(x,p), \Pi^A(x,p)$ obtained in the previous section in eqs-(4.13,4.14) determine the functional form of $g_{\mu\nu}, h^{ab}, N_{a\mu}$ in eqs-(4.17-4.19). The key question is whether or not the solutions found for $g_{\mu\nu}, h^{ab}, N_{a\mu}$ *also* solve the vacuum field equations. And if not, can one find the appropriate field/matter sources which are consistent with these solutions ?. It is natural to assume that quantum matter/fields could be the source of the noncommutativity of the spacetime coordinates and momenta. After all, quantum fields live in spacetime. If this were not the case, what then is the source of this phase space noncommutativity ? Is it space-time foam, dark matter, dark energy ? .... If one expects to have a space-time-matter unification then one has that matter curves space-time, and space-time back-reacts on matter curving momentum space, "curving matter".

5 Concluding Remarks

After a review of Born reciprocal relativity, and its physical implications, this work was mainly devoted to the Yang and the deformed Quaplectic algebras associated with noncommutative phase spaces, and to their extensions involving antisymmetric tensor coordinates and momenta of different ranks. Our approach to construct extended Yang algebras $differs$ from the study by [8]. We finalized with an analysis of the embedding an $8D$ curved phase space into a $12D$ flat phase space which provides a direct link between noncommutative curved phase spaces in lower dimensions to commutative flat phase spaces in higher dimensions. Left from our discussion was the role of Quantum groups, Hopf algebras, κ-deformed Poincare algebras, deformed special relativity [9], [10], [11], [12], [13]. This will be the subject of future investigations.

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