A COMPLETE PROOF OF THE CONJECTURE

\[ c < \text{rad}^{1.63}(abc) \]

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To the memory of my Father who taught me arithmetic,  
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen  
To Prof. A. Nitaj for his work on the \( abc \) conjecture

Abstract. In this paper, we consider the \( abc \) conjecture, we will give the proof that the conjecture \( c < \text{rad}^{1.63}(abc) \) is true. It constitutes the key to resolve the \( abc \) conjecture.

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1. Introduction and notations

Let \( a \) be a positive integer, \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[ a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1} \]

We denote:

\[ \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a) \]

The \( abc \) conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) \( \Box \). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the \( abc \) conjecture is given below:

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1
Conjecture 1.1. (\textit{abc Conjecture}): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:
\begin{equation}
    c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)
\end{equation}
where $K$ is a constant depending only of $\epsilon$.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912$ \cite{2}. It concerned the best example given by E. Reyssat \cite{2}:
\begin{equation}
    2 + 3^{10.109} = 23^5 \implies c < \text{rad}^{1.629912}(abc)
\end{equation}
A conjecture was proposed that $c < \text{rad}^2(abc)$ \cite{3}. In 2012, A. Nitaj \cite{4} proposed the following conjecture:

Conjecture 1.2. Let $a, b, c$ be positive integers relatively prime with $c = a + b$, then:
\begin{align}
    &c < \text{rad}^{1.63}(abc) \quad (5) \\
    &abc < \text{rad}^{1.42}(abc) \quad (6)
\end{align}

In this paper, we will give the proof of the conjecture given by \cite{5} that constitutes the key to obtain the proof of the \textit{abc} conjecture using classical methods with the help of some theorems from the field of the number theory.

2. The Proof of the Conjecture $c < \text{rad}^{1.63}(abc)$

Let $a, b, c$ be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = \text{rad}(abc)$, $c = \prod_{j' = 1}^{\#} c_{j'}^{\beta_{j'}}$, $\beta_{j'} \geq 1$, $c_{j'} \geq 2$ prime integers.

In the following, we will give the proof of the conjecture $c < \text{rad}^{1.63}(abc)$.

\textbf{Proof}: 

2.1. \textbf{Trivial cases}: - We suppose that $c < \text{rad}(abc)$, then we obtain:
\begin{equation*}
    c < \text{rad}(abc) < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\end{equation*}
and the condition \cite{5} is satisfied.

- We suppose that $c = \text{rad}(abc)$, then $a, b, c$ are not coprime, case to reject.

In the following, we suppose that $c > \text{rad}(abc)$ and $a, b$ and $c$ are not all prime numbers.

- We suppose $\mu_a \leq \text{rad}^{0.63}(a)$. We obtain:
\begin{equation*}
    c = a + b < 2a \leq 2\text{rad}^{1.63}(a) < \text{rad}^{1.63}(abc) \implies c < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\end{equation*}
Then \cite{5} is satisfied.

- We suppose $\mu_c \leq \text{rad}^{0.63}(c)$. We obtain:
\begin{equation*}
    c = \mu_c \text{rad}(c) \leq \text{rad}^{1.63}(c) < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\end{equation*}
and the condition \cite{5} is satisfied.
2.2. We suppose $\mu_c > rad^{0.63}(c)$ and $\mu_a > rad^{0.63}(a)$.

2.2.1. Case : $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ and $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$.

We can write:

$$\begin{align*}
\mu_c & \leq rad^{1.63}(c) \implies c \leq rad^{2.63}(c) \\
\mu_a & \leq rad^{1.63}(a) \implies a \leq rad^{2.63}(a)
\end{align*}$$

$$\implies ac \leq rad^{2.63}(ac) \implies a^2 < ac \leq rad^{2.63}(ac)

\implies a < rad^{1.315}(ac) \implies c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)

\implies c = a + b < R^{1.63}$$

2.2.2. Case : $rad^{1.63}(c) < \mu_c$ or $rad^{1.63}(a) < \mu_a$. I - We suppose that $rad^{1.63}(c) < \mu_c$ and $rad^{1.63}(a) < \mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$:

In this case $a = \mu_a$, $rad(a) \leq rad^3(a) \leq rad^{1.63}(a)rad^{1.37}(a) < rad^{1.63}(a).rad^{1.37}(c)$

$$\implies c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \implies c < R^{1.63}$$

I-2- Case $rad(c) < rad(a) < rad^{1.44}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c)$

$$\implies c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \implies c < R^{1.63}$$

I-3- Case $rad^{1.63}(c) < rad(a)$:

I-3-1- We suppose $rad^{1.63}(c) < \mu_c \leq rad^{2.26}(c)$, we obtain:

$$c \leq rad^{2.26}(c) \implies c \leq rad^{1.63}(c).rad^{1.63}(c) \implies c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(c).rad^{1.63}(a).rad^{1.63}(b) = R^{1.63} \implies c < R^{1.63}$$

I-3-2- We suppose $\mu_c > rad^{2.26}(c) \implies c > rad^{2.26}(c)$.

I-3-2-1- We consider the case $\mu_a = rad^3(a) \implies a = rad^3(a)$ and $c = a + 1$.

Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$(7) \quad X^3 + 1 = c$$

I-3-2-1-1- We suppose that $c = rad^n(c)$ with $n \geq 4$, we obtain the equation:

$$(8) \quad rad^n(c) - rad^3(a) = 1$$

But the solutions of the equation (8) are [5] :$(rad(c) = 3, n = 2, rad(a) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c = rad^n(c), n \geq 4$ is to reject.

I-3-2-1-2- In the following, we will study the cases $\mu_c = A.rad^n(c)$ with $rad(c) \nmid A, n \geq 0$. The above equation (9) can be written as :

$$(9) \quad (X + 1)(X^2 - X + 1) = c$$

Let $\delta$ one divisor of $c$ so that :

$$(10) \quad X + 1 = \delta$$

$$(11) \quad X^2 - X + 1 = \frac{c}{\delta} = m = \delta^2 - 3X$$
We recall that \( \text{rad}(a) > \text{rad}^{\frac{1}{44}}(c) \).

**I-3-2-1-2-1-** We suppose \( \delta = l . \text{rad}(c) \). We have \( \delta = l . \text{rad}(c) < c = \mu_c . \text{rad}(c) \) \( \implies l < \mu_c \). As \( \frac{c}{\delta} = \frac{\mu_c . \text{rad}(c)}{\text{rad}(c)} = \frac{\mu_c}{l} = m = \delta^2 - 3X \implies \mu_c = l . m = 1(\delta^2 - 3X) \). From \( m = \delta^2 - 3X \) and \( X = \text{rad}(a) \), we obtain:

\[
\frac{c}{\delta} = \frac{\mu_c . \text{rad}(c)}{\text{rad}(c)} = \frac{A . \text{rad}(c)}{\text{rad}^2(c)} = \frac{A}{\text{rad}(c)} \implies \text{rad}(c) | A
\]

It follows that \( a, c \) are not coprime, then the contradiction.

**B - Case m = 3 \implies \mu_c = 3l \implies c = 3. \text{rad}(c) = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = l . \text{rad}(c) = 3 \implies c = 3\delta = 9 = a + 1 \implies a = 8 \implies c = 9 < (2 \times 3)^{1.63} \approx 18.55 \), it is a trivial case and the conjecture is true.

**I-3-2-1-2-2-** We suppose \( \delta = l . \text{rad}^2(c), l \geq 2 \). If \( n = 0 \) then \( \mu_c = A \) and from the equation above (11):

\[
m = \frac{c}{\delta} = \frac{\mu_c . \text{rad}(c)}{\text{rad}^2(c)} = \frac{A . \text{rad}(c)}{\text{rad}^2(c)} = \frac{A}{\text{rad}(c)} \implies \text{rad}(c) | A
\]

It follows the contradiction with the hypothesis above \( \text{rad}(c) \not\mid A \).

**I-3-2-1-2-3-** We suppose \( \delta = l . \text{rad}^2(c), l \geq 2 \) and in the following \( n > 0 \). As \( m = \frac{c}{\delta} = \frac{\mu_c . \text{rad}(c)}{\text{rad}^2(c)} = \frac{\mu_c}{\text{rad}(c)} \), if \( \text{rad}(c) \not\mid \mu_c \) then the case is to reject. We suppose \( \text{rad}(c) \not\mid \mu_c \implies \mu_c = m. \text{rad}(c) \), with \( m, \text{rad}(c) \) not coprime, then \( \frac{c}{\delta} = m = \delta^2 - 3.\text{rad}(a) \).

**C - Case m = 1 \implies c/\delta \implies \delta^2 - 3.\text{rad}(a) = 1 \implies (\delta - 1)(\delta + 1) = 3.\text{rad}(a) = \text{rad}(a)(\delta + 1) \implies \delta = 2 = l . \text{rad}^2(c) \), then the contradiction.

**D - Case m = 3, we obtain 3(1 + \text{rad}(a)) = \delta^2 = 3\delta \implies \delta = 3 = l.\text{rad}^2(c) \). Then the contradiction.

**E - Case m \neq 1, 3, we obtain: 3.\text{rad}(a) = l^2.\text{rad}^4(c) - m \implies \text{rad}(a) and \text{rad}(c) are not coprime. Then the contradiction.**

**I-3-2-1-2-4-** We suppose \( \delta = l . \text{rad}^n(c), l \geq 2 \) with \( n \geq 3 \). \( c = \mu_c . \text{rad}(c) = l . \text{rad}^n(c) (\delta^2 - 3.\text{rad}(a)) \) and \( m = \delta^2 - 3.\text{rad}(a) = \delta^2 - 3X \).

**F - As seen above (paragraphs C,D), the cases m = 1 and m = 3 give contradictions, it follows the reject of these cases.**

**G - Case m \neq 1, 3. Let q be a prime that divides m (q can be equal to m), it follows q/|\mu_c = l.m \implies q = c_j \implies c_j^2 | \delta^2 \implies c_j^2 | 3.\text{rad}(a) \). Then \( \text{rad}(a) \)
and rad(c) are not coprime. It follows the contradiction.

\textbf{I-3-2-1-2-5-} We suppose \( \delta = \prod_{j \in J_1} \beta_j \), \( \beta_j \geq 1 \) with at least one \( j_0 \in J_1 \) with:

\begin{equation}
\beta_{j_0} \geq 2, \quad \mathrm{rad}(c) \nmid \delta
\end{equation}

We can write:

\begin{equation}
\delta = \mu_3 \cdot \mathrm{rad}(\delta), \quad \mathrm{rad}(c) = r \cdot \mathrm{rad}(\delta), \quad r > 1, \quad (r, \mu_3) = 1
\end{equation}

Then, we obtain:

\begin{equation}
c = \mu_c \cdot \mathrm{rad}(c) = \mu_c \cdot r \cdot \mathrm{rad}(\delta) = \delta(\delta^2 - 3X) = \mu_3 \cdot \mathrm{rad}(\delta)(\delta^2 - 3X) \implies
\end{equation}

\( r \cdot \mu_c = \mu_3(\delta^2 - 3X) \)

- We suppose \( \mu_c = \mu_3 \) \( \implies r = \delta^2 - 3X = (\mu_c \cdot \mathrm{rad}(\delta))^2 - 3X \). As \( \delta < \delta^2 - 3X \implies r > \delta \implies \mathrm{rad}(c) > r > (\mu_c \cdot \mathrm{rad}(\delta) = A \cdot \mathrm{rad}^n(c) \cdot \mathrm{rad}(\delta)) \implies 1 > A \cdot \mathrm{rad}^n - 1(\delta) \), then the contradiction.

- We suppose \( \mu_c < \mu_3 \). As \( \mathrm{rad}(a) = \delta - 1 = \mu_3 \cdot \mathrm{rad}(\delta) - 1 \), we obtain:

\begin{equation}
\mathrm{rad}(a) > \mu_c \cdot \mathrm{rad}(\delta) - 1 > 0 \implies \mathrm{rad}(ac) > c \cdot \mathrm{rad}(\delta) - \mathrm{rad}(c) > 0
\end{equation}

As \( c = 1 + a \) and we consider the cases \( c > \mathrm{rad}(ac) \), then:

\( c > \mathrm{rad}(ac) > c \cdot \mathrm{rad}(\delta) - \mathrm{rad}(c) > 0 \implies c > c \cdot \mathrm{rad}(\delta) - \mathrm{rad}(c) > 0 \implies
\begin{equation}
1 > \mathrm{rad}(\delta) - \frac{\mathrm{rad}(c)}{c} > 0, \quad \mathrm{rad}(\delta) \geq 2 \implies \text{The contradiction}
\end{equation}

- We suppose \( \mu_c > \mu_3 \). In this case, from the equation (14) and as \( (r, \mu_3) = 1 \), it follows we can write:

\begin{equation}
\mu_c = \mu_1 \cdot \mu_2, \quad \mu_1, \mu_2 > 1,
\end{equation}

\begin{equation}
c = \mu_c \cdot \mathrm{rad}(c) = \mu_1 \cdot \mu_2 \cdot \mathrm{rad}(\delta), r = \delta(\delta^2 - 3X),
\end{equation}

We do a choice so that \( \mu_2 = \mu_3 \), \( r \cdot \mu_1 = \delta^2 - 3X \implies \delta = \mu_2 \cdot \mathrm{rad}(\delta) \).

** I- ** We suppose \( (\mu_1, \mu_2) \neq 1 \), then \( \exists c_{j_0} \) so that \( c_{j_0} \mid \mu_1 \) and \( c_{j_0} \mid \mu_2 \). But \( \mu_3 = \mu_2 \implies r_{j_0} \mid \delta \). From \( 3X = \delta^2 - r \mu_1 \implies c_{j_0} \mid 3X \implies c_{j_0} \mid X \) or \( c_{j_0} = 3 \).

- If \( c_{j_0} \mid (X = \mathrm{rad}(a)) \), it follows the contradiction with \( (c, a) = 1 \).
- If \( c_{j_0} = 3 \). We have \( r \mu_1 = \delta^2 - 3X = 3(\delta - 1) \implies \delta^2 - 3\delta + 3 - r \mu_1 = 0 \).

As \( 3 \mid \mu_1 \implies \mu_1 = 3^k \mu_1', 3 \nmid \mu_1', k \geq 1 \), we obtain:

\begin{equation}
\delta^2 - 3\delta + 3(1 - 3^{k-1}r \mu_1') = 0
\end{equation}

** I-I- ** We consider the case \( k > 1 \implies 3 \nmid (1 - 3^{k-1}r \mu_1') \). Let us recall the Eisenstein criterion [6]:

\textbf{Theorem 2.1. (Eisenstein Criterion)} Let \( f = a_0 + \cdots + a_n X^n \) be a polynomial \( \in \mathbb{Z}[X] \). Suppose that \( \exists p \) a prime number so that \( p \nmid a_n \), \( p \nmid a_i, \) \( (0 \leq i \leq n - 1) \), and \( p^2 \nmid a_0 \), then \( f \) is irreducible in \( \mathbb{Q} \).

We apply Eisenstein criterion to the polynomial \( R(Z) \) given by:

\begin{equation}
R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}r \mu_1')
\end{equation}

then:

- \( 3 \nmid 1, - 3 \mid (-3), - 3 \mid 3(1 - 3^{k-1}r \mu_1') \), and \( - 3^2 \nmid 3(1 - 3^{k-1}r \mu_1') \).
It follows that the polynomial \( R(Z) \) is irreducible in \( \mathbb{Q} \), then, the contradiction with \( R(\delta) = 0 \).

** 1-2.** We consider the case \( k = 1 \), then \( \mu_1 = 3\mu_1' \) and \( (\mu_1',3) = 1 \), we obtain:

(18) \[ \delta^2 - 3\delta + 3(1 - r\mu_1') = 0 \]

** 1-2.1.** We consider that \( 3 \nmid (1 - r\mu_1') \), we apply the same Eisenstein criterion to the polynomial \( R'(Z) \) given by:

\[ R'(Z) = Z^2 - 3Z + 3(1 - r\mu_1') \]

and we find a contradiction with \( R'(\delta) = 0 \).

** 1-2.2.** We consider that:

(19) \[ 3 | (1 - r\mu_1') \implies r\mu_1' - 1 = 3^i.h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^* \]

\( \delta \) is an integer root of the polynomial \( R'(Z) \):

(20) \[ R'(Z) = Z^2 - 3Z + 3(1 - r\mu_1') = 0 \]

The discriminant of \( R'(Z) \) is:

\[ \Delta = 3^2 + 3^{i+1} \times 4.h \]

As the root \( \delta \) is an integer, it follows that \( \Delta = t^2 > 0 \) with \( t \) a positive integer. We obtain:

(21) \[ \Delta = 3^2(1 + 3^{i-1} \times 4h) = t^2 \]

(22) \[ \implies 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \]

We can write the equation (18) as:

(23) \[ \delta(\delta - 3) = 3^{i+1}.h \implies 3^3\mu_1' \frac{\text{rad}(\delta)}{3}.(\mu_1' \text{rad}(\delta) - 1) = 3^{i+1}.h \implies \]

(24) \[ \mu_1' \frac{\text{rad}(\delta)}{3}.(\mu_1' \text{rad}(\delta) - 1) = h \]

We obtain \( i = 2 \) and \( q^2 = 1 + 12h = 1 + 4\mu_1' \text{rad}(\delta)(\mu_1' \text{rad}(\delta) - 1) \). Then, \( q \) satisfies:

(25) \[ q^2 - 1 = 12h = 4\mu_1' \text{rad}(\delta)(\mu_1' \text{rad}(\delta) - 1) \implies \]

(26) \[ \frac{(q-1)(q+1)}{2} = 3h = (\mu_1' \text{rad}(\delta) - 1).\mu_1' \text{rad}(\delta) \implies \]

(27) \[ q - 1 = 2\mu_1' \text{rad}(\delta) - 2 \]

(28) \[ q + 1 = 2\mu_1' \text{rad}(\delta) \]

It follows that \( (q = x, 1 = y) \) is a solution of the Diophantine equation:

(29) \[ x^2 - y^2 = N \]

with \( N = 4\mu_1' \text{rad}(\delta)(\mu_1' \text{rad}(\delta) - 1) = 12h > 0 \). Let \( Q(N) \) be the number of the solutions of (29) and \( \tau(N) \) is the number of suitable factorization of \( N \), then we announce the following result concerning the solutions of the Diophantine equation (29) (see theorem 27.3 in [7]):

- If \( N \equiv 2 \pmod{4} \), then \( Q(N) = 0 \).
- If \( N \equiv 1 \) or \( N \equiv 3 \pmod{4} \), then \( Q(N) = \lceil \tau(N)/2 \rceil \).
- If \( N \equiv 0 \pmod{4} \), then \( Q(N) = \lceil \tau(N/4)/2 \rceil \).
[x] is the integral part of x for which [x] ≤ x < [x] + 1.

As \( N = 4\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) \Rightarrow N \equiv 0 \pmod{4} \Rightarrow Q(N) = [\tau(N/4)/2]. \)

As \((q, 1)\) is a couple of solutions of the Diophantine equation \((29)\), then \(\exists d, d'\) positive integers with \(d > d'\) and \(N = d.d'\) so that:

\[
\begin{align*}
(30) & \quad d + d' = 2q \\
(31) & \quad d - d' = 2.1 = 2
\end{align*}
\]

** 1-2-2-1** As \(N > 1\), we take \(d = N\) and \(d' = 1\). It follows:

\[
\begin{cases}
N + 1 = 2q \\
N - 1 = 2
\end{cases} \Rightarrow N = 3 \Rightarrow \text{then the contradiction with } N \equiv 0 \pmod{4}.
\]

** 1-2-2-2** Now, we consider the case \(d = 2\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1)\) and \(d' = 2\). It follows:

\[
\begin{cases}
2\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) + 2 = 2q \\
2\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) - 2 = 2
\end{cases} \Rightarrow 2\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) = q + 1
\]

As \(q + 1 = 2\mu_1 \text{rad}(\delta)\), we obtain \(\mu_1 \text{rad}(\delta) = 2\), then the contradiction with \(3|\delta\).

** 1-2-2-3** Now, we consider the case \(d = \mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1)\) and \(d' = 4\). It follows:

\[
\begin{cases}
\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) + 4 = 2q \\
\mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) - 4 = 2 \Rightarrow \mu_1 \text{rad}(\delta)(\mu_1 \text{rad}(\delta) - 1) = 6
\end{cases}
\]

As \(\mu_1 \text{rad}(\delta) > 0 \Rightarrow \mu_1 \text{rad}(\delta) = 3 \Rightarrow \mu_1 = 1, \text{ rad}(\delta) = 3\) and \(q = 5\).

From \(q^2 = 1 + 12h\), we obtain \(h = 2\). Using the relation \((19)\) \(r\mu_1^2 - 1 = 3'h\) as \(\mu_1 = 1, i = 2, h = 2\), it gives \(r - 1 = 9h = 18\). As \(\delta\) is the positive root of the equation \((18)\):

\[
Z^2 - 3Z + 3(1 - r) = 0 \Rightarrow \delta = 9 = 3^2
\]

But \(\delta = 1 + X = 1 + \text{rad}(a) \Rightarrow \text{rad}(a) = 8 = 2^3\), then the contradiction.

** 1-2-2-4** Now, let \(c_{j_0}\) be a prime integer so that \(c_{j_0}|\text{rad}\delta\), we consider the case \(d = \mu_1 \frac{\text{rad}(\delta)}{c_{j_0}}(\mu_1 \text{rad}(\delta) - 1)\) and \(d' = 4c_{j_0}\). It follows:

\[
\begin{cases}
\mu_1 \frac{\text{rad}(\delta)}{c_{j_0}}(\mu_1 \text{rad}(\delta) - 1) + 4c_{j_0} = 2q \\
\mu_1 \frac{\text{rad}(\delta)}{c_{j_0}}(\mu_1 \text{rad}(\delta) - 1) - 4c_{j_0} = 2 \Rightarrow \mu_1 \frac{\text{rad}(\delta)}{c_{j_0}}(\mu_1 \text{rad}(\delta) - 1) = 2(1 + 2c_{j_0}) \Rightarrow
\end{cases}
\]

Then the contradiction as the left member is greater than the right member \(2(1 + 2c_{j_0})\).

** 1-2-2-5** Now, we consider the case \(d = 4\mu_1 \text{rad}(\delta)\) and \(d' = (\mu_1 \text{rad}(\delta) - 1)\).

It follows:

\[
\begin{cases}
4\mu_1 \text{rad}(\delta) + (\mu_1 \text{rad}(\delta) - 1) = 2q \\
4\mu_1 \text{rad}(\delta) - (\mu_1 \text{rad}(\delta) - 1) = 2 \Rightarrow 3\mu_1 \text{rad}(\delta) = 1 \Rightarrow \text{Then the contradiction.}
\end{cases}
\]
**1.2.2.6** Now, we consider the case \( d = 2\mu'_1 \text{rad}(\delta) \) and \( d' = 2(\mu'_1 \text{rad}(\delta) - 1) \). It follows:

\[
\begin{cases}
2\mu'_1 \text{rad}(\delta) + 2(\mu'_1 \text{rad}(\delta) - 1) = 2q \\
2\mu'_1 \text{rad}(\delta) - 2(\mu'_1 \text{rad}(\delta) - 1) = 2
\end{cases}
\Rightarrow 2 = 2
\]

It follows that this case presents no contradictions a priori.

**1.2.2.7** \( \mu'_1 \text{rad}(\delta) \) and \( \mu'_1 \text{rad}(\delta) - 1 \) are coprime, let \( \mu'_1 \text{rad}(\delta) - 1 = \prod_{j=1}^{j=j'} \lambda_j^\delta \), we consider the case \( d = 2\lambda'_j \mu'_1 \text{rad}(\delta) \) and \( d' = 2\mu'_1 \text{rad}(\delta) - 1 \). It follows:

\[
\begin{cases}
2\lambda'_j \mu'_1 \text{rad}(\delta) + 2\mu'_1 \text{rad}(\delta) - 1 = 2q \\
2\lambda'_j \mu'_1 \text{rad}(\delta) - 2\mu'_1 \text{rad}(\delta) - 1 = 2
\end{cases}
\]

**1.2.2.7-1** We suppose that \( \gamma_j = 1 \). We consider the case \( d = 2\lambda_j \mu'_1 \text{rad}(\delta) \) and \( d' = 2\mu'_1 \text{rad}(\delta) - 1 \). It follows:

\[
\begin{cases}
2\lambda'_j \mu'_1 \text{rad}(\delta) + 2\mu'_1 \text{rad}(\delta) - 1 = 2q \\
2\lambda'_j \mu'_1 \text{rad}(\delta) - 2\mu'_1 \text{rad}(\delta) - 1 = 2
\end{cases}
\Rightarrow 4\lambda'_j \mu'_1 \text{rad}(\delta) = 2(q + 1) \Rightarrow 2\lambda'_j \mu'_1 \text{rad}(\delta) = q + 1
\]

But from the equation \( [25] \), \( q + 1 = 2\mu'_1 \text{rad}(\delta) \), then \( \lambda'_j = 1 \), it follows the contradiction.

**1.2.2.7-2** We suppose that \( \gamma_j \geq 2 \). We consider the case \( d = 2\lambda_j^{\gamma_j - r_j} \mu'_1 \text{rad}(\delta) \) and \( d' = 2\mu'_1 \text{rad}(\delta) - 1 \). It follows:

\[
\begin{cases}
2\lambda_j^{\gamma_j - r_j} \mu'_1 \text{rad}(\delta) + 2\mu'_1 \text{rad}(\delta) - 1 = 2q \\
2\lambda_j^{\gamma_j - r_j} \mu'_1 \text{rad}(\delta) - 2\mu'_1 \text{rad}(\delta) - 1 = 2
\end{cases}
\Rightarrow 4\lambda_j^{\gamma_j - r_j} \mu'_1 \text{rad}(\delta) = 2(q + 1) \Rightarrow 2\lambda_j^{\gamma_j - r_j} \mu'_1 \text{rad}(\delta) = q + 1
\]

As above, it follows the contradiction. It is trivial that the other cases for more factors \( \prod_{j'=j}^\delta \lambda_j^{\gamma_j - r_j} \) give also contradictions.
**1-2.2-8** Now, we consider the case $d = 4(\mu'_1 \text{rad}(\delta) - 1)$ and $d' = \mu'_1 \text{rad}(\delta)$, we have $d > d'$. It follows:

\[
\begin{align*}
4(\mu'_1 \text{rad}(\delta) - 1) + \mu'_1 \text{rad}(\delta) &= 2q \\
4(\mu'_1 \text{rad}(\delta) - 1) - \mu'_1 \text{rad}(\delta) &= 2
\end{align*}
\]

Then the contradiction as $3|\delta$.

**1-2.2-9** Now, we consider the case $d = 4u(\mu'_1 \text{rad}(\delta) - 1)$ and $d' = \frac{\mu'_1 \text{rad}(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu'_1 \text{rad}(\delta)$. We have $d > d'$ and:

\[
\begin{align*}
4u(\mu'_1 \text{rad}(\delta) - 1) + \frac{\mu'_1 \text{rad}(\delta)}{u} &= 2q \\
4u(\mu'_1 \text{rad}(\delta) - 1) - \frac{\mu'_1 \text{rad}(\delta)}{u} &= 2
\end{align*}
\]

Then the contradiction as $\mu'_1 \text{rad}(\delta)$ and $(\mu'_1 \text{rad}(\delta) - 1)$ are coprime.

In conclusion, we have found only one case (**1-2.2-6** above) where there is no contradictions *a priori*. As $\tau(N)$ is large and also $|\tau(N)/4|/2]$, it follows the contradiction with $Q(N) \leq 1$ and the hypothesis $(\mu_1, \mu_2) \neq 1$ is false.

**2-** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $\tau\mu_1 = \delta^2 - 3X$ and the condition $\text{rad}(a) = X > \text{rad}^{1.63/1.37}(c) \iff \delta - 1 = X > \text{rad}^{1.19}(c)$, we obtain the following inequality:

\[
\delta - 1 > (r.\text{rad}(\delta))^{1.19} \implies -3(\delta - 1) < -3r.\text{rad}(\delta).(r.\text{rad}(\delta))^{0.19} \implies \\
\tau\mu_1 = \delta^2 - 3(\delta - 1) < (r.\text{rad}(\delta))^2 - 3r.\text{rad}(\delta).(r.\text{rad}(\delta))^{0.19} \implies \\
\mu_1 < r.\text{rad}^2(\delta) - 3.\text{rad}(\delta).(r.\text{rad}(\delta))^{0.19} \implies
\]

(32)

\[
\mu_1 < r.\text{rad}^2(\delta) \left(1 - \frac{3}{(r.\text{rad}(\delta))^{0.81}}\right)
\]

As $a = \text{rad}^3(a) < c$, we can write:

\[
\text{rad}^3(a) < \mu_1 \mu_2 \text{rad}(c) < \mu_2.\text{rad}(\delta).\text{rad}^2(c) \left(1 - \frac{3}{(r.\text{rad}(\delta))^{0.81}}\right)
\]

but $(r, \text{rad}(\delta)) = 1$, $r.\text{rad}(\delta) \geq 6 \implies (r.\text{rad}(\delta))^{0.81} \geq 6^{0.81} \approx 4.26$ and $\delta = \mu_2.\text{rad}(\delta)$, it follows:

$\text{rad}^3(a) < \mu_1 \mu_2 \text{rad}(c) < \mu_2.\text{rad}(\delta).\text{rad}^2(c) \implies \text{rad}^3(a) < \delta.\text{rad}^2(c) < 1.6\text{rad}(a).\text{rad}^2(c)$

As $\text{rad}(a) > (\text{rad}^{1.62/1.37}(c) = \text{rad}^{1.19}(c)) \implies \text{rad}^{1.19}(c) < \text{rad}(a) < 1.27\text{rad}(c)$, then we obtain:

$\text{rad}^{1.19}(c) < 1.27\text{rad}(c) \implies \text{rad}(c) < 3.5 \implies \text{rad}(c) \leq 3$, but $\text{rad}(c) = r.\text{rad}(\delta) \geq 6$

Then the contradiction.

It follows that the case $c > \text{rad}^{2.26}(c) \implies c > \text{rad}^{3.26}(c)$ and $a = \text{rad}^3(a)$ is impossible.
We consider the case $\mu_a = \text{rad}^3(a) \implies a = \text{rad}^3(a)$ and $c = a + b$. Then, we obtain that $X = \text{rad}(a)$ is a solution in positive integers of the equation:

$$(33)\quad X^3 + 1 = \tilde{c}$$

with $\tilde{c} = c - b + 1 = a + 1 \implies (\tilde{c}, a) = 1$. We obtain the same result as of the case **I-3-2-1** studied above considering $\text{rad}(a) > \text{rad}^{1423}((\tilde{c})$.

**I-3-2-3** - We suppose $\mu_c > \text{rad}^2.26(c) \implies c > \text{rad}^2.26(c)$ and $c$ large and $\mu_a < \text{rad}^2(a)$, we consider $c = a + b, b \geq 1$. Then $c = \text{rad}^3(c) + h, h > \text{rad}^3(c), h$ a positive integer and we can write $a + l = \text{rad}^3(a)$, $l > 0$. Then we obtain :

$$(34)\quad \text{rad}^3(c) + h = \text{rad}^3(a) - l + b \implies \text{rad}^3(a) - \text{rad}^3(c) = h + l - b > 0$$

as $\text{rad}(a) > \text{rad}^{1423}(c)$. We obtain the equation:

$$(35)\quad \text{rad}^3(a) - \text{rad}^3(c) = h + l - b = m > 0$$

Let $X = \text{rad}(a) - \text{rad}(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

$$(36)\quad H(X) = X^3 + 3\text{rad}(ac)X - m = 0$$

To resolve the above equation, we denote $X = u + v$. It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

$$(37)\quad G(t) = t^2 - mt - \text{rad}^3(ac) = 0$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4\text{rad}^3(ac) = \alpha^2, \alpha > 0$. As $m = \text{rad}^3(a) - \text{rad}^3(c) > 0$, we obtain that $\alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0$, then from the expression of the discriminant, it follows that the couple $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$(38)\quad x^2 - y^2 = N$$

with $N = 4\text{rad}^3(ac) = 4\text{rad}^3(a).\text{rad}^3(c) > 0$. Here, we will use the same method that is given in the above sub-paragraph **I-2-2** of the paragraph **I-3-2-1-2-5**. We have the two terms $\text{rad}^3(a)$ and $\text{rad}^3(c)$ coprime. As $(\alpha, m)$ is a couple of solutions of the Diophantine equation (38) and $\alpha > m$, then $\exists d, d'$ positive integers with $d > d'$ and $N = d.d'$ so that :

$$(39)\quad d + d' = 2\alpha$$
$$(40)\quad d - d' = 2m$$

**I-3-2-3** - Let us consider the case $d = 2\text{rad}^3(a), d' = 2\text{rad}^3(c)$. It follows:

$$\left\{\begin{array}{l}
2\text{rad}^3(a) + 2\text{rad}^3(c) = 2\alpha \implies \alpha = \text{rad}^3(a) + \text{rad}^3(c) \\
2\text{rad}^3(a) - 2\text{rad}^3(c) = 2m \implies m = \text{rad}^3(a) - \text{rad}^3(c)
\end{array}\right.$$

It follows that this case presents a priori no contradictions.

**I-3-2-2** - Now, we consider for example, the case $d = 4\text{rad}^3(a)$ and $d' = \text{rad}^3(c) \implies d > d'$. We rewrite the equations (39, 40):

$$4\text{rad}^3(a) + \text{rad}^3(c) = 2(\text{rad}^3(a) + \text{rad}^3(c)) \implies 2\text{rad}^3(a) = \text{rad}^3(c)$$
$$4\text{rad}^3(a) - \text{rad}^3(c) = 2(\text{rad}^3(a) - \text{rad}^3(c)) \implies 2\text{rad}^3(a) = -\text{rad}^3(c)$$
Then the contradiction.

**I-3-2-3-3**- We consider the case \( d = 4\text{rad}^3(c)\text{rad}^3(a) \) and \( d' = 1 \implies d > d' \). We rewrite the equations (39-40):

\[
\begin{align*}
4\text{rad}^3(c)\text{rad}^3(a) + 1 &= 2(\text{rad}^3(c) + \text{rad}^3(a)) \\
2(2\text{rad}^3(c)\text{rad}^3(a) - \text{rad}^3(c) - \text{rad}^3(a)) &= -1 \implies \text{a contradiction}
\end{align*}
\]

Then the contradiction.

**I-3-2-3-4**- Let \( c_1 \) be the first factor of \( \text{rad}(c) \). We consider the case \( d = 4c_1\text{rad}^3(a) \) and \( d' = \frac{\text{rad}^3(c)}{c_1} \implies d > d' \). We rewrite the equation (39):

\[
\begin{align*}
4c_1\text{rad}^3(a) + \frac{\text{rad}^3(c)}{c_1} &= 2(\text{rad}^3(a) + \text{rad}^3(c)) \\
2\text{rad}^3(a)(2c_1 - 1) &= \frac{\text{rad}^3(c)}{c_1}(2c_1 - 1) \implies 2\text{rad}^3(a) = \text{rad}^3(c)\frac{\text{rad}(c)}{c_1}
\end{align*}
\]

\( c_1 = 2 \) or not, there is a contradiction with \( a, c \) coprime.

The other cases of the expressions of \( d \) and \( d' \) not coprime so that \( N = d.d' \) give also contradictions.

Let \( Q(N) \) be the number of the solutions of (38), as \( N \equiv 0 \pmod{4} \), then \( Q(N) = \lceil (\tau(N)/4)/2 \rceil \). From the study of the cases above, we obtain that \( Q(N) \leq 1 \) is \( \ll \lceil (\tau(N)/4)/2 \rceil \). It follows the contradiction.

Then the cases \( \mu_a \leq \text{rad}^2(a) \) and \( c > \text{rad}^{3.26}(c) \) are impossible.

**II**- We suppose that \( \text{rad}^{1.63}(c) < \mu_c \leq \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \):

**II-1**- Case \( \text{rad}(c) < \text{rad}(a) \): As \( c \leq \text{rad}^3(c) = \text{rad}^{1.63}(c).\text{rad}^{1.37}(c) \implies c < \text{rad}^{1.63}(c).\text{rad}^{1.37}(a) < \text{rad}^{1.63}(ac) < \text{rad}^{1.63}(abc) \implies \boxed{c < R^{1.63}} \).

**II-2**- Case \( \text{rad}(a) < \text{rad}(c) < \text{rad}^{1.63}(a) \): As \( c \leq \text{rad}^3(c) \leq \text{rad}^{1.63}(c).\text{rad}^{1.37}(c) \implies c < \text{rad}^{1.63}(c).\text{rad}^{1.63}(a) < \text{rad}^{1.63}(abc) \implies \boxed{c < R^{1.63}} \).

**II-3**- Case \( \text{rad}^{1.63}(a) < \text{rad}(c) \):

**II-3-1**- We suppose \( \text{rad}^{1.63}(a) < \mu_a \leq \text{rad}^{2.26}(a) \implies a \leq \text{rad}^{1.63}(a).\text{rad}^{1.63}(a) \implies a < \text{rad}^{1.63}(a).\text{rad}^{1.37}(c) \implies c = a + b < 2a < \text{rad}^{1.63}(a).\text{rad}^{1.63}(c) < \text{rad}^{1.63}(abc) \implies \boxed{c < R^{1.63}} \).

**II-3-2**- We suppose \( \mu_a > \text{rad}^{2.26}(a) \implies a > \text{rad}^{3.26}(a) \) and \( \mu_c \leq \text{rad}^2(c) \). Using the same method as it was explicated in the paragraphs **I-3-2**- (permuting \( a, c \) see in Appendix **II-3-2**-), we arrive at a contradiction. It follows
that the cases $\mu_c \leq \text{rad}^2(c)$ and $\mu_c > \text{rad}^{2.63}(a)$ are impossible.

2.2.3. Case $\mu_a > \text{rad}^{1.63}(a)$ and $\mu_c > \text{rad}^{1.63}(c)$: Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^{1.63}(a)$,
- $\mu_a > \text{rad}^2(a)$ and $\mu_c > \text{rad}^{1.63}(c)$.

III- We suppose $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^{1.63}(a) \implies c > \text{rad}^3(c)$ and $a > \text{rad}^{2.63}(a)$. We can write $c = \text{rad}^3(c) + h$ and $a = \text{rad}^3(a) + l$ with $h$ a positive integer and $l \in \mathbb{Z}$.

III-1- We suppose $\text{rad}(c) < \text{rad}(a)$. We obtain the equation:
\begin{equation}
\text{rad}^3(a) - \text{rad}^3(c) = h - l - b = m > 0
\end{equation}
Let $X = \text{rad}(a) - \text{rad}(c)$, from the above equation, $X$ is a real root of the polynomial:
\begin{equation}
H(X) = X^3 + 3\text{rad}(ac)X - m = 0
\end{equation}
As above, to resolve $H(X)$, we denote $X = u + v$, It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by :
\begin{equation}
G(t) = t^2 - mt - \text{rad}^3(ac) = 0
\end{equation}
The discriminant of $G(t)$ is:
\begin{equation}
\Delta = m^2 + 4\text{rad}^3(ac) = \alpha^2, \quad \alpha > 0
\end{equation}
As $m = \text{rad}^3(a) - \text{rad}^3(c) > 0$, we obtain that $\alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0$, then from the equation $H(X)$, it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:
\begin{equation}
x^2 - y^2 = N
\end{equation}
with $N = 4\text{rad}^3(ac) > 0$. Let $Q(N)$ be the number of the solutions of $\text{rad}(a)$ and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-2-3- above, we obtain a contradiction.

III-2- We suppose $\text{rad}(a) < \text{rad}(c)$. We obtain the equation:
\begin{equation}
\text{rad}^3(c) - \text{rad}^3(a) = b + l - h = m > 0
\end{equation}
Let $X$ be the variable $X = \text{rad}(c) - \text{rad}(a)$, we use the similar calculations as in the paragraph above I-3-2-3- permuting $c, a$, we find a contradiction.

It follows that the case $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^{1.63}(a)$ is impossible.

IV - We suppose $\mu_a > \text{rad}^2(a)$ and $\mu_c > \text{rad}^{1.63}(c)$, we obtain $a > \text{rad}^3(a)$ and $c > \text{rad}^{2.63}(c)$. We can write $a = \text{rad}^3(a) + h$ and $c = \text{rad}^3(c) + l$ with $h$ a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in the cases of the paragraph III. We obtain a contradiction.
It follows that the case \( \mu_c > \text{rad}^{1.63}(c) \) and \( \mu_a > \text{rad}^2(a) \) is impossible.

All possible cases are discussed. \( \square \)

We can state the following important theorem:

**Theorem 2.2.** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then \( c < \text{rad}^{1.63}(abc) \).

From the theorem above, we can announce also:

**Corollary 2.2.1.** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then the conjecture \( c < \text{rad}^2(abc) \) is true.

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**Appendix**

**II'-3-2-** We suppose \( \mu_a > \text{rad}^{2.26}(a) \Rightarrow a > \text{rad}^{3.26}(a) \).

**II'-3-2-1-** We consider the case \( \mu_c = \text{rad}^2(c) \Rightarrow c = \text{rad}^3(c) \) and \( c = a + 1 \).

Then, we obtain that \( Y = \text{rad}(c) \) is a solution in positive integers of the equation:

\[
Y^3 - 1 = a
\]

**II'-3-2-1-1-** We suppose that \( a = \text{rad}^n(a) \) with \( n \geq 4 \), we obtain the equation:

\[
\text{rad}^3(c) - \text{rad}^n(a) = 1
\]

But the solutions of the Catalan equation \( x^p - y^q = 1 \) where the unknowns \( x, y, p \) and \( q \) take integer values, all \( \geq 2 \), has only one solution \( (x, y, p, q) = (3, 2, 2, 3) \), but the solution of the equation \( (48) \) are \( (\text{rad}(c) = 3, \text{rad}(a) = 2, 3 \neq 2, n \geq 4) \), it follows the contradiction with \( n \geq 4 \) and the case \( a = \text{rad}^n(a), n \geq 4 \) is to reject.

**II'-3-2-1-2-** In the following, we will study the cases \( \mu_a = A.\text{rad}^n(a) \) with \( \text{rad}(a) \nmid A, n \geq 0 \). The above equation \( (47) \) can be written as :

\[
(Y - 1)(Y^2 + Y + 1) = a
\]

Let \( \delta \) one divisor of \( a \) so that :

\[
Y - 1 = \delta
\]

\[
Y^2 + Y + 1 = \frac{a}{\delta} = m = \delta^2 + 3Y
\]
We recall that \( \text{rad}(c) > \text{rad}^{\frac{144}{25}}(a) \).

**II’-3-2-1-2-1-** We suppose \( l = \text{rad}(a) \). We have \( l = \text{rad}(a) < a = \mu_a, \text{rad}(a) \Rightarrow l < \mu_a \). As \( l \) is a divisor of \( a \), then \( l \) is a divisor of \( \mu_a, \frac{a}{\delta} = \frac{\mu_a, \text{rad}(a)}{l, \text{rad}(a)} = \frac{\mu_a}{l} = m = \delta^2 + 3Y \), then \( \mu_a = l.m \). From \( \mu_a = l(\delta^2 + 3Y) \), we obtain:

\[
m = l^2 \text{rad}^2(a) + 3 \text{rad}(c) \Rightarrow 3 \text{rad}(c) = m - l^2 \text{rad}^2(a)
\]

A’ - Case \( 3|m \Rightarrow m = 3m', m' > 1 \): As \( \mu_a = ml = 3m'l \Rightarrow 3|\text{rad}(a) \) and \((\text{rad}(a), m') \) not coprime. We obtain:

\[
\text{rad}(c) = m' - l^2 \text{rad}(a). \frac{\text{rad}(a)}{3}
\]

It follows that \( a, c \) are not coprime, then the contradiction.

B’ - Case \( m = 3 \Rightarrow \mu_a = 3l \Rightarrow a = 3l \text{rad}(a) = 3\delta = \delta(\delta^2 + 3Y) \Rightarrow \delta^2 = 3(1 - Y) = -3\delta < 0 \), then the contradiction.

**II’-3-2-1-2-2-** We suppose \( \delta = l \text{rad}(a), l \geq 2 \). If \( n = 0 \) then \( \mu_a = A \) and from the equation above (1): \[ m = \frac{a}{\delta} = \frac{\mu_a, \text{rad}(a)}{l, \text{rad}(a)} = \frac{A, \text{rad}(a)}{l, \text{rad}(a)} \Rightarrow \text{rad}(a) | A \]

It follows the contradiction with the hypothesis above \( \text{rad}(a) \nmid A \).

**II’-3-2-1-2-3-** We suppose \( \delta = l \text{rad}^2(a), l \geq 2 \) and in the following \( n > 0 \).

As \( m = \frac{a}{\delta} = \frac{\mu_a, \text{rad}(a)}{l, \text{rad}(a)} = \frac{\mu_a}{l, \text{rad}(a)} \), if \( \text{rad}(a) \nmid \mu_a \) then the case is to reject.

We suppose \( \text{rad}(a) | \mu_a \Rightarrow \mu_a = ml \text{rad}(a) \), with \( m, \text{rad}(a) \) not coprime, then \( \frac{a}{\delta} = m = \delta^2 + 3 \text{rad}(c) \).

C’ - Case \( m = 1 = a/\delta \Rightarrow \delta^2 + 3 \text{rad}(c) = 1 \), then the contradiction.

D’ - Case \( m = 3 \), we obtain \( 3(1 - \text{rad}(c)) = \delta^2 \Rightarrow \delta^2 < 0 \). Then the contradiction.

E’ - Case \( m \neq 1, 3 \), we obtain: \( 3 \text{rad}(c) = m - l^2 \text{rad}^4(a) \Rightarrow \text{rad}(a) \) and \( \text{rad}(c) \) are not coprime. Then the contradiction.

**II’-3-2-1-2-4-** We suppose \( \delta = l \text{rad}^n(a), l \geq 2 \) with \( n \geq 3 \). From \( a = \mu_a, \text{rad}(a) = l \text{rad}^n(a)(\delta^2 + 3 \text{rad}(c)) \), we denote \( m = \delta^2 + 3 \text{rad}(c) = \delta^2 + 3Y \).

F’ - As seen above (paragraphs C’), the cases \( m = 1 \) and \( m = 3 \) give contradictions, it follows the reject of these cases.

G’ - Case \( m \neq 1, 3 \). Let \( q \) be a prime that divides \( m \) \( (q \) can be equal to \( m \) \), it follows \( q | \mu_a \Rightarrow q = a_{j_0} \Rightarrow a_{j_0} | \delta^2 \Rightarrow a_{j_0} | 3 \text{rad}(c) \). Then \( \text{rad}(a) \) and
rad(c) are not coprime. It follows the contradiction.

\[ \Pi' - 3.2.1.2.5- \] We suppose \( \delta = \prod_{j \in J_1} \alpha_j^{\beta_j} \), \( \beta_j \geq 1 \) with at least one \( j_0 \in J_1 \) with:

(52) \[ \beta_{j_0} \geq 2, \quad \text{rad}(a) \nmid \delta \]

We can write:

(53) \[ \delta = \mu_\delta \text{rad}(\delta), \quad \text{rad}(a) = r \cdot \text{rad}(\delta), \quad r > 1, \quad (r, \text{rad}(\delta)) = 1 \Rightarrow (r, \mu_\delta) = 1 \]

Then, we obtain:

(54) \[ a = \mu_a \cdot \text{rad}(a) = \mu_a \cdot r \cdot \text{rad}(\delta) = \delta (\delta^2 + 3Y) = \mu_\delta \cdot \text{rad}(\delta) (\delta^2 + 3Y) \implies \]

- We suppose \( \mu_a = \mu_\delta \Rightarrow r = \delta^2 + 3Y = (\mu_a \cdot \text{rad}(\delta))^2 + 3Y \). As \( \delta < \delta^2 + 3Y \implies r > \delta \implies \text{rad}(a) > r > (\mu_a \cdot \text{rad}(\delta) = A \cdot \text{rad}(\text{rad}(\delta)) \implies 1 > A \cdot \text{rad}(\text{rad}(\delta)) \), then the contradiction.

- We suppose \( \mu_a < \mu_\delta \). As \( \text{rad}(c) = \mu_a \cdot \text{rad}(\delta) + 1 \), we obtain:

(55) \[ \text{rad}(c) > \mu_a \cdot \text{rad}(\delta) + 1 > 0 \implies \text{rad}(ac) > a \cdot \text{rad}(\delta) + \text{rad}(a) > 0 \]

As \( c = 1 + a \) and we consider the cases \( c > \text{rad}(ac) \), then:

(56) \[ c > \text{rad}(ac) > a \cdot \text{rad}(\delta) + \text{rad}(a) > 0 \implies a + 1 > a \cdot \text{rad}(\delta) + \text{rad}(a) > 0 \implies a \cdot \text{rad}(\delta) + \text{rad}(a) \implies 1 \geq \text{rad}(\delta) + \frac{\text{rad}(a)}{a} > 0, \quad \text{rad}(\delta) \geq 2 \implies \text{The contradiction} \]

- We suppose \( \mu_a > \mu_\delta \). In this case, from the equation (14) and as \( (r, \mu_\delta) = 1 \), it follows we can write:

(57) \[ a = \mu_a \cdot \text{rad}(a) = \mu_1 \cdot \mu_2 \cdot r \cdot \text{rad}(\delta) = \delta (\delta^2 + 3Y) \]

\[ r \cdot \mu_a = \mu_\delta \implies \delta = \mu_\delta \cdot \text{rad}(\delta) \]

** I- If we suppose \( (\mu_1, \mu_2) \neq 1 \), then \( \exists a_{j_0} \) so that \( a_{j_0} \mid \mu_1 \) and \( a_{j_0} \mid \mu_2. \) But \( \mu_\delta = \mu_2 \Rightarrow a_j^3 \mid \delta. \) From \( 3Y = r \mu_1 - \delta^2 \Rightarrow a_j \mid 3Y \Rightarrow a_{j_0} \mid Y \) or \( a_{j_0} = 3. \)

- If \( a_{j_0} \mid (Y = \text{rad}(c)) \), it follows the contradiction with \( (c, a) = 1. \)
- If \( a_{j_0} = 3. \) We have \( r \mu_1 = \delta^2 + 3Y = \delta^2 + 3(\delta + 1) \Rightarrow \delta^2 + 3\delta + 3 - \mu_1 = 0. \)

As \( 3 \nmid \mu_1 \Rightarrow \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1 \), we obtain:

(58) \[ \delta^2 + 3\delta + 3(1 - 3^{k-1}r \mu'_1) = 0 \]

** II- If we consider the case \( k > 1 \Rightarrow 3 \nmid (1 - 3^{k-1}r \mu'_1). \) Let us recall the Eisenstein criterion [6].

**Theorem 2.3. (Eisenstein Criterion)** Let \( f = a_0 + \cdots + a_n X^n \) be a polynomial \( \in \mathbb{Z}[X] \). We suppose that \( \exists p \) a prime number so that \( p \nmid a_n, p \nmid a_i, (0 \leq i \leq n - 1) \), and \( p^2 \nmid a_0, \) then \( f \) is irreducible in \( \mathbb{Q} \).

We apply Eisenstein criterion to the polynomial \( R(Z) \) given by:

(59) \[ R(Z) = Z^2 + 3Z + 3(1 - 3^{k-1}r \mu'_1) \]

then:

- \( 3 \nmid 1, - 3(1 + 3), - 3 \nmid 3(1 - 3^{k-1}r \mu'_1) \), and \( 3^2 \nmid 3(1 - 3^{k-1}r \mu'_1). \)
It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

**1.2.** We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 + 3\delta + 3(1 - r\mu'_1) = 0 \tag{60}$$

**1.2.1.** We consider that $3 \nmid (1 - r\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 + 3Z + 3(1 - r\mu'_1) \tag{62}$$

and we find a contradiction with $R'(\delta) = 0$.

**1.2.2.** We consider that:

$$3 \mid (1 - r\mu'_1) = \Rightarrow r\mu'_1 - 1 = 3^i.h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^* \tag{61}$$

As $h$ is an integer root of the polynomial $R'(Z)$:

$$\Delta = 3^2 (1 + 3^{i-1} \times 4h) = t^2 \tag{63}$$

$$\Rightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{64}$$

As $\mu_2 = 3\mu_2$ and $3|\mu_2 = \mu_2 = 3\mu'_2$, then we can write the equation (60) as:

$$\delta(\delta + 3) = 3^{i+1}.h = \Rightarrow 3^{i+1}.h = \mu'_2 \frac{\text{rad}(\delta)}{3} (\mu'_2 \text{rad}(\delta) + 1) = 3^{i+1}.h \tag{65}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_2 \text{rad}(\delta)(\mu'_2 \text{rad}(\delta) + 1)$. Then, $q$ satisfies:

$$q^2 - 1 = 12h = 4\mu'_2 \text{rad}(\delta)(\mu'_2 \text{rad}(\delta) + 1) \Rightarrow \tag{66}$$

$$\frac{(q-1)(q+1)}{2} = 3h = \mu'_2 \text{rad}(\delta)(\mu'_2 \text{rad}(\delta) + 1). \Rightarrow \tag{67}$$

$$q + 1 = 2\mu'_2 \text{rad}(\delta) + 2 \tag{68}$$

$$q - 1 = 2\mu'_2 \text{rad}(\delta) \tag{69}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{71}$$

with $N = 4\mu'_2 \text{rad}(\delta)(\mu'_2 \text{rad}(\delta) + 1) = 12h > 0$. Let $Q(N)$ be the number of the solutions of (71) and $\tau(N)$ is the number of suitable factorization of $N$, then we announce the following result concerning the solutions of the Diophantine equation (71) (see theorem 27.3 in [7]):

- If $N \equiv 2(\text{mod } 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\text{mod } 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\text{mod } 4)$, then $Q(N) = [\tau(N/4)/2]$. 

As $N = 4\mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) \implies N \equiv 0 \pmod{4} \implies Q(N) = \lfloor \frac{\tau(N/4)}{2} \rfloor$.

As $(q, 1)$ is a couple of solutions of the Diophantine equation (71), then $\exists$ $d, d'$ positive integers with $d > d'$ and $N = d \cdot d'$ so that:

(72) 
$$d + d' = 2q$$

(73) 
$$d - d' = 2.1 = 2$$

** 1-2-2-1** As $N > 1$, we take $d = N$ and $d' = 1$. It follows:

$$
\begin{cases}
 N + 1 = 2q \\
 N - 1 = 2
\end{cases}
\implies N = 3 \implies \text{then the contradiction with } N \equiv 0 \pmod{4}.
$$

** 1-2-2-2** Now, we consider the case $d = 2\mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1)$ and $d' = 2$. It follows:

$$
\begin{cases}
 2\mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) + 2 = 2q \\
 2\mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) - 2 = 2
\end{cases}
\implies \mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) = q - 1
$$

As $q - 1 = 2\mu_2\text{rad}(\delta)$, we obtain $\mu_2\text{rad}(\delta) = 1$, then the contradiction.

** 1-2-2-3** Now, we consider the case $d = \mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1)$ and $d' = 4$. It follows:

$$
\begin{cases}
 \mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) + 4 = 2q \\
 \mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) - 4 = 2 \implies \mu_2\text{rad}(\delta)(\mu_2\text{rad}(\delta)+1) = 6
\end{cases}
$$

As $\mu_2\text{rad}(\delta) \geq 2 \implies \mu_2\text{rad}(\delta) = 2 \implies \mu_2 = 1 \implies \mu_2 = 3 = \mu_3$ and $\text{rad}(\delta) = 2$ but $3 \nmid 2$, then the contradiction.

** 1-2-2-4** Now, let $a_{j_0}$ be a prime integer so that $a_{j_0}\mid \text{rad}\delta$, we consider the case $d = \frac{\mu_2\text{rad}(\delta)}{a_{j_0}}(\mu_2\text{rad}(\delta)+1)$ and $d' = 4a_{j_0}$. It follows:

$$
\begin{cases}
 \mu_2\text{rad}(\delta)\left(\frac{\mu_2\text{rad}(\delta)}{a_{j_0}}(\mu_2\text{rad}(\delta)+1) + 4a_{j_0} = 2q \\
 \mu_2\text{rad}(\delta)\left(\frac{\mu_2\text{rad}(\delta)}{a_{j_0}}(\mu_2\text{rad}(\delta)+1) - 4a_{j_0} = 2
\end{cases}
\implies \mu_2\text{rad}(\delta)\left(\frac{\mu_2\text{rad}(\delta)}{a_{j_0}}(\mu_2\text{rad}(\delta)+1) = 2(1 + 2a_{j_0})
$$

Then the contradiction as the left member is greater than the right member $2(1 + 2a_{j_0})$.

** 1-2-2-5** Now, we consider the case $d = 4\mu_2\text{rad}(\delta)$ and $d' = (\mu_2\text{rad}(\delta)+1)$.

It follows:

$$
\begin{cases}
 4\mu_2\text{rad}(\delta) + (\mu_2\text{rad}(\delta)+1) = 2q \\
 4\mu_2\text{rad}(\delta) - (\mu_2\text{rad}(\delta)+1) = 2
\end{cases}
\implies 3\mu_2\text{rad}(\delta) = 3 \implies \text{Then the contradiction.}
$$

** 1-2-2-6** Now, we consider the case $d = 2(\mu_2\text{rad}(\delta)+1)$ and $d = 2\mu_2\text{rad}(\delta)$.

It follows:

$$
\begin{cases}
 2(\mu_2\text{rad}(\delta)+1) + 2\mu_2\text{rad}(\delta) = 2q \implies 2\mu_2\text{rad}(\delta) + 1 = q \\
 2(\mu_2\text{rad}(\delta)+1) - 2\mu_2\text{rad}(\delta) = 2 \implies 2 = 2
\end{cases}
$$

It follows that this case presents no contradictions a priori.
As above, it follows the contradiction. It is trivial that the other cases for $d$ and $q$

But from the equation (28), we have

$$2\lambda_{j'}\mu_2\text{rad}(\delta) + 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}} = 2q$$

and

$$2\lambda_{j'}\mu_2\text{rad}(\delta) - 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}} = 2$$

** 1-2-2.7 We suppose that $\gamma_{j'} = 1$. We consider the case $d = 2\lambda_{j'}\mu_2\text{rad}(\delta)$ and $d' = 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}}$. It follows:

$$\begin{cases}
2\lambda_{j'}\mu_1\text{rad}(\delta) + 2\frac{\mu_1\text{rad}(\delta) - 1}{\lambda_{j'}} = 2q \\
2\lambda_{j'}\mu_1\text{rad}(\delta) - 2\frac{\mu_1\text{rad}(\delta) - 1}{\lambda_{j'}} = 2
\end{cases}$$

But from the equation (28), $q + 1 = 2\mu_1\text{rad}(\delta)$, then $\lambda_{j'} = 1$, it follows the contradiction.

** 1-2-2.7-1 We suppose that $\gamma_{j'} \geq 2$. We consider the case $d = 2\lambda_{j'}\gamma_{j'} - \gamma_{j'}\gamma_{j'}\mu_2\text{rad}(\delta)$ and $d' = 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}\gamma_{j'}}$. It follows:

$$\begin{cases}
2\lambda_{j'}\gamma_{j'} - \gamma_{j'}\mu_2\text{rad}(\delta) + 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}\gamma_{j'}} = 2q \\
2\lambda_{j'}\gamma_{j'} - \gamma_{j'}\mu_2\text{rad}(\delta) - 2\frac{\mu_2\text{rad}(\delta) + 1}{\lambda_{j'}\gamma_{j'}} = 2
\end{cases}$$

As above, it follows the contradiction. It is trivial that the other cases for more factors $\prod_{j'} \lambda_{j'}^{\gamma_{j'} - \gamma_{j'}}$ give also contradictions.

** 1-2-2.8 Now, we consider the case $d = 4(\mu_2\text{rad}(\delta) + 1)$ and $d' = \mu_2\text{rad}(\delta)$, we have $d > d'$. It follows:

$$\begin{cases}
4(\mu_2\text{rad}(\delta) + 1) + \mu_2\text{rad}(\delta) = 2q \Rightarrow 5\mu_2\text{rad}(\delta) = 2(q + 2) \\
4(\mu_2\text{rad}(\delta) + 1) - \mu_2\text{rad}(\delta) = 2 \Rightarrow \mu_2\text{rad}(\delta) = 2
\end{cases}$$

Then the contradiction as $3|\delta$.

** 1-2-2.9 Now, we consider the case $d = 4u(\mu_2\text{rad}(\delta) + 1)$ and $d' = \frac{\mu_2\text{rad}(\delta)}{u}$, where $u > 1$ is an integer divisor of $\mu_2\text{rad}(\delta)$. We have $d > d'$
and:
\[
\begin{cases}
4u\mu_2^2\text{rad}(\delta) + 1 + \frac{\mu_2^2\text{rad}(\delta)}{u} = 2q \\
4u\mu_2^2\text{rad}(\delta) + 1 - \frac{\mu_2^2\text{rad}(\delta)}{u} = 2
\end{cases}
\Rightarrow 2u(\mu_2^2\text{rad}(\delta)+1) = \mu_2^2\text{rad}(\delta)+1 \Rightarrow 2u = 1
\]

Then the contradiction.

In conclusion, we have found only one case (** 1-2-2-6 above) where there is no contradictions a priori. As \(\tau(N)\) is large and also \([\tau(N/4)]/2\), it follows the contradiction with \(Q(N) \leq 1\) and the hypothesis \((\mu_1, \mu_2) \neq 1\) is false.

** 2. We suppose that \((\mu_1, \mu_2) = 1\).

We recall that \(\text{rad}(c) = Y > \text{rad}^{1.63/1.37}(a)\), \(\delta + 1 = Y\), \(\text{rad}(a) = r\cdot\text{rad}(\delta), (r, \text{rad}(\delta)) = 1, \delta = \mu_2\text{rad}(\delta)\) and \(r\mu_1 = \delta^2 + 3X\), it follows:

\[
(74) \quad U(\delta) = \delta^2 + 3\delta + 3 - r\mu_1 = 0
\]

** 2-1. We suppose \(3|(3 - r\mu_1)\) and \(3^2 \not| (3 - r\mu_1)\), then we use the Eisenstein criterion \([6]\) to the polynomial \(U(\delta)\) given by the equation \((74)\), and the contradiction.

** 2-2. We suppose \(3|(3 - r\mu_1)\) and \(3^2|(3 - r\mu_1)\). From \(3|(3 - r\mu_1) \Rightarrow 3|r\mu_1 \Rightarrow 3|r \text{ or } 3|\mu_1\).
- If \(3|r \Rightarrow (3, \text{rad}\delta) = 1 \Rightarrow 3 \not| \delta\). Then the contradiction with \(3|\delta^2\) by the equation \((74)\).
- If \(3|\mu_1 \Rightarrow 3 \not| \mu_2 \Rightarrow 3 \not| \delta\), it follows the contradiction with \(3|\delta^2\) by the equation \((74)\).

** 2-3. We suppose \(3 \not| (3 - r\mu_1) \Rightarrow 3 \not| r\mu_1 \Rightarrow 3 \not| r \text{ and } 3 \not| \mu_1\). From the equation \((74)\) \(U(\delta) = 0 \Rightarrow r\mu_1 \equiv \delta^2 (\text{mod } 3)\), as \(\delta^2\) is a square then \(\delta^2 \equiv 1 (\text{mod } 3) \Rightarrow r\mu_1 \equiv 1 (\text{mod } 3)\), but this result is not all verified. Then the contradiction.

It follows that the case \(\mu_a > \text{rad}^{2.26}(a) \Rightarrow a > \text{rad}^{3.26}(a)\) and \(c = \text{rad}^3(c)\) is impossible.

**I'-3-2-2.** We consider the case \(\mu_c = \text{rad}^2(c) \Rightarrow c = \text{rad}^3(c)\) and \(c = a + b\). Then, we obtain that \(Y = \text{rad}(c)\) is a solution in positive integers of the equation:

\[
(75) \quad Y^3 + 1 = \tilde{c}
\]

with \(\tilde{c} = a + b + 1 = c + 1 \Rightarrow (\tilde{c}, c) = 1\). We obtain the same result as of the case **I-3-2-1**- studied above considering \(\text{rad}(\tilde{c}) > \text{rad}^{1.63}(c)\).

**I'-3-2-3.** We suppose \(\mu_a > \text{rad}^{2.26}(a) \Rightarrow a > \text{rad}^{3.26}(a)\) and \(c\) large and \(\mu_c < \text{rad}^2(c)\), we consider \(c = a + b, b \geq 1\). Then \(a = \text{rad}^3(a) + h, h > 0\), \(h\) a positive integer and we can write \(c + l = \text{rad}^3(c), l > 0\). As \(\text{rad}(c) >\)
\[ \text{rad}^{1.63}(a) \Rightarrow \text{rad}(c) > \text{rad}(a) \Rightarrow h + l + b = m > 0, \text{ it follows:} \]
\[ (76) \]
\[ \text{rad}^3(c) - l = \text{rad}^3(a) + h + b > 0 \Rightarrow \text{rad}^3(c) - \text{rad}^3(a) = h + l + b = m > 0 \]

We obtain the same result (a contradiction) as of the case 1-3-2-3- studied above considering \( \text{rad}(c) > \text{rad}^{1.63}(a) \). Then, this case is to reject.

Then the cases \( \mu_c \leq \text{rad}^2(c) \) and \( a > \text{rad}^{3.26}(a) \) are impossible.

References

[1] M. Waldschmidt, *On the abc Conjecture and some of its consequences*, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6–9, (2013)


