Consistent Supergravity Theories in Diverse Dimensions

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Abstract

The scientific article reviews some properties of the low-energy effective actions for consistent supergravity models. We summarize the current state of knowledge regarding quantum gravity theories with minimal supersymmetry. We provide an elegant extension of the theory and give a deffinitions of the anomaly-free models in advanced supergravity constructions. The deep relation between anomalies and inconsistency is emphasized in this research. The conditions for anomaly cancellation in these supergravity theories typically constitute determined types of equations. For completeness of theoretical framework, we are including anomaly-free models, which are consistent supergravity theories.

1 Introduction

The main theories of interest in this scientific article are supergravities with a minimal amount of supersymmetry and all possible anomaly-free models. In the first part of our work we use the symplectic structure of four-dimensional minimal supergravities to study the possibility of gauged axionic shift symmetries. This leads to the introduction of generalized Chern-Simons terms, and a Green-Schwarz cancellation mechanism for gauge anomalies. Similarly, we study the possibility of adding higher order derivative corrections to the two-derivative action, leading to a cancellation of the mixed gauge-gravitational anomalies. Our models constitute the supersymmetric framework for string compactifications with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies. We discuss the presence of gauge and gravitational anomalies in theories with $N = 1$ local supersymmetry and a conventional gauging. We present a Green-Schwarz mechanism that involves Peccei-Quinn terms, generalized Chern-Simons terms, higher order derivative corrections and appropriate gauge transformations of the scalar fields. We discuss the mutual consistency conditions for all these ingredients, such that the theory is anomaly-free. We construct the complete coupling of (1, 0) supergravity in six dimensions to tensor multiplets, extending previous results to all orders in the fermi fields. We then add couplings to vector multiplets, as dictated by the generalized Green-Schwarz mechanism. The resulting theory embodies factorized gauge and supersymmetry anomalies, to be disposed of by fermion loops, and is determined by corresponding Wess-Zumino consistency conditions, aside from a quartic coupling for the gaugini. In addition, we show how to revert to a supersymmetric formulation in terms of covariant field equations that embody corresponding covariant anomalies. The subsequent work of some authors has developed the consistent formulation, but one can actually revert to a covariant formulation, at the price of having non-integrable field equations. The relation between the two sets of equations is one more instance of the link between covariant and consistent anomalies in field theory. This is a remarkable laboratory for current algebra, where one can play explicitly with anomalous symmetries and their consequences. The supersymmetry algebra contains a corresponding extension that plays a crucial role for the consistency of the construction. Whereas gauge and supersymmetry anomalies occur in theories with global or local supersymmetry, mixed anomalies are specific for gauged supergravities. They manifest themselves as a non-invariance of the effective action under local Lorentz transformations. Mixed anomaly usually refers to a mixture of a gauge and gravitational anomaly. These structures appear in gauge current anomalies and lead to inconsistency unless cancelled. It is common practice to attempt to cancel them by adding new fermions to the model. We have found new anomaly structures which involve scalar fields, and these can require new independent cancellation conditions. The issue of anomaly freedom in chiral supergravities can be examined not only in the usual context of superstring theories, but also in the context of lower-dimensional supergravities, many of which arise from superstring and M-theory compactifications. Anomaly freedom is generally independent of the existence of superstring theory. The anomaly freedom of our theory strongly suggests the deep significance of such interactions, and may lead to a more fundamental theory of extended objects which are not necessarily superstrings. We will extend our results about gauge anomaly cancellation to the framework of specific supergravity theories. We will see that anomaly cancellation requires the modification of the representation constraint, and we will interpret the original constraint as the condition for anomaly freedom. In our work we take an important step forward that goal, and we unravel the intricate gauge structure of these theories. We find an intricate interplay between the gaugings and certain quantum aspects of the theory. More precisely, we obtain the general cancellation conditions for quantum anomalies, using a Green-Schwarz mechanism. The main purpose of our work is to identify the necessary and sufficient conditions, such that generic $N = 1$ theories with chiral matter couplings are free from the above anomalies. We also take a first step towards the identification of consistent supergravity theories. The anomaly-free models in this publication are based on consistency conditions on low-energy supergravity theories, and do not depend upon a specific completion such as superstring theory. The results naturally lead to the question of whether it is possible to place stronger bounds on the set of consistent theories than those understood from anomaly cancellation and other known constraints. We identify a number of models which obey all known low-energy consistency conditions, but which have no known string theory realization. Many of these models contain novel matter representations, suggesting possible new superstring theory constructions. We hope that the variety of new apparently consistent supergravity models identified in the thesis will stimulate some further understanding of new string realizations or will help to generate new constraints on quantum theories of gravity.

2 Supergravity Corrections in $D = 4$

We generalize our treatment to the full $\mathcal{N} = 1$, $D = 4$ supergravity theory. We check supersymmetry and gauge invariance of the supergravity action and show that no extra GCS terms have to be included to obtain supersymmetry or gauge invariance. The simplest way to go from rigid supersymmetry to supergravity makes use of the superconformal tensor calculus. Compared to the rigid theory, the additional fields reside in a Weyl multiplet, the gauge multiplet of the superconformal algebra, and a compensating multiplet. The Weyl multiplet contains the vierbein, the gravitino ψ_μ and an auxiliary vector, which will not be important for us. The compensating multiplet enlarges the set of chiral multiplets in the theory by one. The full set of fields in the chiral multiplets is now $(X^I,\ \Omega^I,\ H^I)$, which denote complex scalars, fermions and complex auxiliary fields. The physical chiral multiplets (z^i, χ^i, h^i) form a subset of these such that I runs over one more value than i. As our final results depend only on the vector multiplet, this addition will not be very important for us, and we do not have to discuss how the physical ones are embedded in the full set of chiral multiplets.

When going from rigid supersymmetry to supergravity, extra terms appear in the action are proportional to the gravitino ψ_μ. The integrand of specific equation is replaced by the so-called density formula, which is rather simple due to the use of the superconformal calculus

$$S_f = \int \mathrm{d}^4x\, e\, \mathrm{Re}\left[h(fW^2) + \bar{\psi}_{\mu R}\gamma^\mu \chi_L(fW^2) + \tfrac{1}{2}\bar{\psi}_{\mu R}\gamma^{\mu\nu}\psi_{\nu R} z(fW^2)\right] , \tag{1}$$

where e is the determinant of the vierbein. For completeness, we give the component expression of (1). It can be found by plugging in the relations, where we replace the fields of the chiral multiplets with an index i by the larger set I, into the density formula (1). The result is

$$\begin{aligned}\hat{S}_f = \int \mathrm{d}^4x\, e\Big[&\,\mathrm{Re}\, f_{AB}(X)\left(-\tfrac{1}{4}\mathcal{F}^A_{\mu\nu}\mathcal{F}^{\mu\nu\, B} - \tfrac{1}{2}\bar{\lambda}^A\gamma^\mu\hat{\mathcal{D}}_\mu\lambda^B + \tfrac{1}{2}D^A D^B\right.\\
&\left.+\tfrac{1}{8}\bar{\psi}_\mu\gamma^{\nu\rho}\left(\mathcal{F}^A_{\nu\rho} + \hat{\mathcal{F}}^A_{\nu\rho}\right)\gamma^\mu\lambda^B\right) + \tfrac{1}{4}\mathrm{i}\,\mathrm{Im}\, f_{AB}(X)\mathcal{F}^A_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu B}\\
&+\tfrac{1}{4}\mathrm{i}\left(\hat{\mathcal{D}}_\mu\,\mathrm{Im}\, f_{AB}(X)\right)\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B + \Big\{\tfrac{1}{2}\partial_I f_{AB}(X)\left[\bar{\Omega}^I_L\left(-\tfrac{1}{2}\gamma^{\mu\nu}\hat{\mathcal{F}}^A_{\mu\nu} + \mathrm{i}D^A\right)\lambda^B_L\right.\\
&\left.-\tfrac{1}{2}\left(H^I + \bar{\psi}_{\mu R}\gamma^\mu\Omega^I_L\right)\bar{\lambda}^A_L\lambda^B_L\right] + \tfrac{1}{4}\partial_I\partial_J f_{AB}(X)\,\bar{\Omega}^I_L\Omega^J_L\bar{\lambda}^A_L\lambda^B_L + \mathrm{h.c.}\Big\}\Big],\end{aligned} \tag{2}$$

where the hat denotes full covariantization with respect to gauge and local supersymmetry,

$$\hat{\mathcal{F}}^A_{\mu\nu} = \mathcal{F}^A_{\mu\nu} + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^A\,. \tag{3}$$

Note that we use already the derivative $\hat{\mathcal{D}}_\mu \operatorname{Im} f_{AB}(X)$, covariant with respect to the shift symmetries, as explained around current literature. Therefore, we denote this action as $\hat{S}_f$ as we did for rigid supersymmetry.

The kinetic matrix f_{AB} is now a function of the scalars X^I. We thus have in the superconformal formulation

$$\delta_C f_{AB} = \partial_I f_{AB} \delta_C X^I = \mathrm{i} C_{AB,C} + \dots . \tag{4}$$

Let us first consider the supersymmetry variation of (2). Compared with supergravity action, the supersymmetry variation of (2) can only get extra contributions that are proportional to the C-tensor. These extra contributions come from the variation of H^I and Ω^I in covariant objects that are now also covariantized with respect to the supersymmetry transformations and from the variation of e and λ^A in the gauge covariantization of the $(\hat{\mathcal{D}}_\mu \operatorname{Im} f_{AB})$-term. Let us list in more detail the parts of the action that give these extra contributions.

First there is a coupling of Ω^I with a gravitino and gaugini, coming from the exceptional term $-\frac{1}{4} e \partial_I f_{AB} \bar{\Omega}^I_L \gamma^{\mu\nu} \hat{\mathcal{F}}^A_{\mu\nu} \lambda^B_L$:

$$\begin{aligned} S_1 &= \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{4} \partial_I f_{AB} \bar{\Omega}^I_L \gamma^{\mu\nu} \lambda^B_L \bar{\psi}_{[\mu} \gamma_{\nu]} \lambda^A + \text{h.c.} \Big] \\ &\to \delta(\epsilon) S_1 = \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{8} \mathrm{i} C_{AB,C} W^C_\rho \bar{\lambda}^B_L \gamma^{\mu\nu} \gamma^\rho \epsilon_R \bar{\psi}_\mu \gamma_\nu \lambda^A + \dots + \text{h.c.} \Big]. \end{aligned} \tag{5}$$

We used the expression (3) for $\hat{\mathcal{F}}^A_{\mu\nu}$ where $\mathcal{D}_\mu X^I$ is now also covariantized with respect to the supersymmetry transformations and we have $\hat{\mathcal{D}}_\mu X^I$. There is another coupling between Ω^I, a gravitino and gaugini that we will treat separately

$$\begin{aligned} S_2 &= \int \mathrm{d}^4 x\, e \Big[\tfrac{1}{4} \partial_I f_{AB} \bar{\Omega}^I_L \gamma^\mu \psi_{\mu R} \bar{\lambda}^A_L \lambda^B_L + \text{h.c.} \Big] \\ &\to \delta(\epsilon) S_2 = \int \mathrm{d}^4 x\, e \Big[\tfrac{1}{8} \mathrm{i} C_{AB,C} W^C_\rho \bar{\epsilon}_R \gamma^\rho \gamma^\mu \psi_{\mu R} \bar{\lambda}^A_L \lambda^B_L + \dots + \text{h.c.} \Big]. \end{aligned} \tag{6}$$

A third contribution comes from the variation of the auxiliary field H^I in S_3, where

$$S_3 = \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{4} \partial_I f_{AB} H^I \bar{\lambda}^A_L \lambda^B_L + \text{h.c.} \Big] . \tag{7}$$

The variation is of the form

$$\delta_\epsilon H^I = \bar{\epsilon}_R \gamma^\mu \mathcal{D}_\mu \Omega^I_L + \dots = -\tfrac{1}{2} \bar{\epsilon}_R \gamma^\mu \gamma^\nu \hat{\mathcal{D}}_\nu X^I \psi_{\mu R} + \dots = \tfrac{1}{2} \delta_C X^I W^C_\nu \bar{\epsilon}_R \gamma^\mu \gamma^\nu \psi_{\mu R} + \dots . \tag{8}$$

Therefore we obtain the action

$$\begin{aligned} S_3 &= \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{4} \partial_I f_{AB} H^I \bar{\lambda}^A_L \lambda^B_L + \text{h.c.} \Big] \\ &\to \delta(\epsilon) S_3 = \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{8} \mathrm{i} C_{AB,C} W^C_\nu \bar{\epsilon}_R \gamma^\mu \gamma^\nu \psi_{\mu R} \bar{\lambda}^A_L \lambda^B_L + \dots + \text{h.c.} \Big]. \end{aligned} \tag{9}$$

Finally, we need to consider the variation of the vierbein e and the gaugini in a part of the covariant derivative on $\operatorname{Im} f_{AB}$:

$$\begin{aligned} S_4 &= \int \mathrm{d}^4 x\, e \Big[\tfrac{1}{4} \mathrm{i} C_{AB,C} W^C_\mu \bar{\lambda}^A \gamma^\mu \gamma_5 \lambda^B \Big] \\ &\to \delta(\epsilon) S_4 = \int \mathrm{d}^4 x\, e \Big[-\tfrac{1}{4} \mathrm{i} C_{AB,C} W^C_\rho \Big(\bar{\lambda}^A_R \gamma^\mu \lambda^B_L \bar{\epsilon}_R \gamma^\rho \psi_{\mu L} + \tfrac{1}{4} \bar{\epsilon}_R \gamma^\rho \gamma^\mu \gamma^\nu \psi_{\nu L} \bar{\lambda}^A_L \gamma_\mu \lambda^B_R \\ &\qquad + \tfrac{1}{4} \bar{\epsilon}_R \gamma^\rho \gamma^\mu \psi_{\mu R} \bar{\lambda}^A_L \lambda^B_L \Big) \\ &\qquad + \tfrac{1}{4} \mathrm{i} C_{AB,C} W^{\mu C} \bar{\psi}_{\mu R} \epsilon_R \bar{\lambda}^A_L \lambda^B_L + \dots + \text{h.c.} \Big] . \end{aligned} \tag{10}$$

It requires some careful manipulations to obtain the given result for $\delta(\epsilon)S_4$. One needs the variation of the determinant of the vierbein, gamma matrix identities and Fierz relations.

In the end, we find that

$$\delta(\epsilon)\,(S_1+S_2+S_3+S_4)=0\,. \tag{11}$$

This means that all extra contributions that were not present in the supersymmetry variation of the original supergravity action vanish without the need of extra terms. We should also remark here that the variation of the GCS terms themselves is not influenced by the transition from rigid supersymmetry to supergravity because it depends only on the vectors W_μ^A, whose supersymmetry transformations have no gravitino corrections in $\mathcal{N}=1$. Let us check now the gauge invariance of terms proportional to the gravitino. Neither terms involving the real part of the gauge kinetic function, $\mathrm{Re}\,f_{AB}$, nor its derivatives violate the gauge invariance of $\hat{S}_f$. The only contributions to gauge non-invariance come from the pure imaginary parts, $\mathrm{Im}\,f_{AB}$, of the gauge kinetic function. On the other hand, no extra $\mathrm{Im}\,f_{AB}$ terms appear when one goes from rigid supersymmetry to supergravity and, hence, the gauge variation of $\hat{S}_f$ does not contain any gravitini. This is consistent with our earlier result that neither $\delta(\epsilon)\hat{S}_f$ nor $S_{\rm CS}$ contain gravitini. Consequently, the general $\mathcal{N}=1$ action contains just the extra terms, and we can add them to the original action.

3 Dyonic Strings in Gauged Supergravity

We find a new family of supersymmetric vacuum solutions in the six-dimensional chiral gauged $N=(1,0)$ supergravity theory. We shall be focusing on the simplest example, for which the field content comprises a graviton multiplet with bosonic fields (g_{MN},B^+_{MN}) and chiral gravitino superpartner ψ_M, a tensor multiplet with bosonic field B^-_{MN} and chiral superpartner χ, and a vector multiplet with bosonic field A_M and chiral superpartner λ.

The bosonic sector of the six-dimensional $N=(1,0)$ gauged supergravity is elegantly described by the Lagrangian

$$\mathcal{L}=R{*}\mathbb{1}-\tfrac14{*}d\phi\wedge d\phi-\tfrac12 e^{\phi}\,{*}H_{(3)}\wedge H_{(3)}-\tfrac12 e^{\frac12\phi}\,{*}F_{(2)}\wedge F_{(2)}-8g^2\,e^{-\frac12\phi}\,{*}\mathbb{1}\,, \tag{12}$$

where $F_{(2)}=dA_{(1)}$, $H_{(3)}=dB_{(2)}+\frac12 F_{(2)}\wedge A_{(1)}$, and g is the gauge-coupling constant. This leads to the bosonic equations of motion

$$\begin{aligned}
R_{MN} &= \tfrac14\partial_M\phi\,\partial_N\phi+\tfrac12 e^{\frac12\phi}\,(F^2_{MN}-\tfrac18 F^2\,g_{MN})+\tfrac14 e^{\phi}\,(H^2_{MN}-\tfrac16 H^2\,g_{MN})\\
&\quad +2g^2\,e^{-\frac12\phi}\;g_{MN}\,,\\
\Box\,\phi &= \tfrac14 e^{\frac12\phi}\,F^2+\tfrac16 e^{\phi}\,H^2-8g^2\,e^{-\frac12\phi}\,,\\
&\quad d(e^{\frac12\phi}\,{*}F_{(2)})=e^{\phi}\,{*}H_{(3)}\wedge F_{(2)}\,,\qquad d(e^{\phi}\,{*}H_{(3)})=0\,.
\end{aligned} \tag{13}$$

The transformations rules for the fermionic fields are given by

$$\begin{aligned}
\delta\psi_M &\equiv \tilde{D}_M\epsilon=[D_M+\tfrac{1}{48}e^{\frac12\phi}\,H^+_{NPQ}\,\Gamma^{NPQ}\,\Gamma_M]\epsilon\,,\\
\delta\chi &\equiv -\tfrac14\Delta_\phi\epsilon=-\tfrac14[\Gamma^M\partial_M\phi-\tfrac16 e^{\frac12\phi}\,H^-_{MNP}\,\Gamma^{MNP}]\epsilon\,,\\
\delta\lambda &\equiv \tfrac{1}{4\sqrt2}\Delta_F\epsilon=\tfrac{1}{4\sqrt2}[e^{\frac14\phi}\,F_{MN}\,\Gamma^{MN}-8\mathrm{i}\,g\,e^{-\frac14\phi}]\epsilon\,,
\end{aligned} \tag{14}$$

where D_M is the gauge-covariant derivative, $D_M\epsilon\equiv(\nabla_M-\mathrm{i}\,g\,A_M)\epsilon$. The $\pm$ superscripts appearing on the 3-form H_{MNP} in these expressions are redundant, since the chirality of ϵ already implies projections onto the self-dual or anti-self-dual parts, but we include them for convenience, to emphasise which projection occurs in which transformation rule. We shall show that once the F and H field equations and Bianchi identities, and the Killing spinor conditions are satisfied by our ansatz, the remaining Einstein and dilaton field equations are automatically satisfied as well as a consequence of the Killing spinor integrability conditions. As a byproduct we will determine the full Killing spinor integrability conditions and observe that the first order Killing spinor equations by themselves are in general insufficient to guarantee that all the equations of motion are satisfied.

We now determine the Killing spinor integrability conditions. For the gravitino variation, we may take the usual commutator of generalized covariant derivatives. After considerable algebra, we obtain

$$\begin{aligned}\Gamma^N[\tilde D_M,\tilde D_N] = &-\tfrac12[R_{MN}-\tfrac14\partial_M\phi\partial_N\phi-\tfrac12 e^{\frac12\phi}(F^2_{MN}-\tfrac18 g_{MN}F^2)\\ &-\tfrac14 e^{\phi}(H^2_{MN}-\tfrac16 g_{MN}H^2)-2g^2e^{-\frac12\phi}g_{MN}]\Gamma^N-\tfrac1{48}e^{\frac12\phi}\left(\partial_{[N}H_{PQR]}-\tfrac34 F_{[NP}F_{QR]}\right)\Gamma^{NPQR}\Gamma_M\\ &-\tfrac1{16}e^{-\frac12\phi}\nabla^N(e^{\phi}H_{NPQ})\Gamma^{PQ}\Gamma_M-\tfrac18(\partial_M\phi+\tfrac1{12}e^{\frac12\phi}H_{NPQ}\Gamma^{NPQ}\Gamma_M)\Delta_\phi+\tfrac18 e^{\frac14\phi}F_{MN}\Gamma^N\Delta_F\\ &-\tfrac1{64}\Gamma_M(e^{\frac14\phi}F_{NP}\Gamma^{NP}+8\mathrm{i}\,g\,e^{-\frac14\phi})\Delta_F\,,\end{aligned}\tag{15}$$

where the last two lines vanish when acting on Killing spinors. The quantities Δ_ϕ and Δ_F are defined in (14) and are supersymmetry transformations on χ and λ, up to unimportant numerical factors. We see that once the H field equation, Bianchi identity and the Killing spinor conditions are satisfied, and given that the Ricci tensor is diagonal, the Einstein equation is then satisfied as well.

Additional integrability conditions may be derived from the $\delta\chi$ and $\delta\lambda$ variations. For the tensor multiplet, we find

$$\begin{aligned}\Gamma^M[\tilde D_M,\Delta_\phi] = &[\Box\,\phi-\tfrac14 e^{\frac12\phi}F^2-\tfrac16 e^{\phi}H^2+8g^2e^{-\frac12\phi}]\\ &-\tfrac16 e^{\frac12\phi}\left(\partial_{[M}H_{NPQ]}-\tfrac34 F_{[MN}F_{PQ]}\right)\Gamma^{MNPQ}-\tfrac12 e^{-\frac12\phi}\nabla^M(e^{\phi}H_{MNP})\Gamma^{NP}\\ &+\tfrac1{24}e^{\frac12\phi}H_{MNP}\Gamma^{MNP}\Delta_\phi-\tfrac18\left(e^{\frac14\phi}F_{MN}\Gamma^{MN}+8\mathrm{i}\,g\,e^{-\frac14\phi}\right)\Delta_F\,.\end{aligned}\tag{16}$$

This shows once the H field equation and Bianchi identity and the Killing spinor conditions are satisfied, then the dilaton field equation is satisfied as well.

From the Killing spinor condition coming form the Maxwell multiplet we find

$$\begin{aligned}\Gamma^M[\tilde D_M,\Delta_F] = &\,e^{\frac14\phi}\partial_{[M}F_{NP]}\Gamma^{MNP}+2e^{-\frac14\phi}[\nabla^M(e^{\frac12\phi}F_{MP})\\ &-\tfrac12 e^{\phi}H_{MNP}F^{MN}]\Gamma^P-\tfrac14\Gamma^M\partial_M\phi\Delta_F+\tfrac12 e^{\frac14\phi}F_{MN}\Gamma^{MN}\Delta_\phi+\tfrac14[\Delta_\phi,\Delta_F]\,,\end{aligned}\tag{17}$$

which is automatically satisfied as a result of the F field equation and the Killing spinor conditions. The Killing spinor integrability conditions presented above can also be used to analyze in more general situations the extent to which they imply the field equations.

We shall make a convenient choice for the coordinate gauge function h and exhibit the explicit form of the dyonic string solution, and study its salient properties such as its behavior in various limits. In particular, we choose h so that the solutions for ϕ and c will be identical for the gauge dyonic string. This is achieved by making the gauge choice

$$h=-\frac{2a^2\,b\,c^2}{r^3}\,,\tag{18}$$

and defining $\phi_\pm\equiv\phi\pm4\log c$, whereupon the equations become diagonalised, with

$$\hat\phi_+=\frac{4P}{r^3}\,e^{\frac12\phi_+}\,,\qquad \hat\phi_-=-\frac{4Q}{r^3}\,e^{-\frac12\phi_-}\,.\tag{19}$$

The solutions can be written as

$$e^{-\frac{1}{2}\phi_+} = P_0 + \frac{P}{r^2}\,, \qquad e^{\frac{1}{2}\phi_-} = Q_0 + \frac{Q}{r^2}\,, \tag{20}$$

and hence

$$e^{\phi} = \Big(Q_0 + \frac{Q}{r^2}\Big)\Big(P_0 + \frac{P}{r^2}\Big)^{-1}\,, \qquad c^{-4} = \Big(Q_0 + \frac{Q}{r^2}\Big)\Big(P_0 + \frac{P}{r^2}\Big)\,. \tag{21}$$

Were we indeed looking for the dyonic superstring solutions in the ungauged theory, we would then solve for a and b, with $g = 0$. In our present case, however, we already have the algebraic equations, which came from solving the F_2 field equation and the $\delta\lambda = 0$ supersymmetry condition respectively. Both these conditions would have been vacuous in the dyonic string solutions in the ungauged theory. Thus our solution is simply given by (21), together with the expressions for a and b. Collecting the above results, we find that the dyonic string solution of the six-dimensional gauged $N = (1, 0)$ supergravity is given by

$$\begin{aligned} ds^2 &= H_P^{-\frac{1}{2}} H_Q^{-\frac{1}{2}}\, dx^\mu\, dx_\mu + \frac{k^2\, P}{4g^2 r^6} H_P^{-\frac{5}{2}} H_Q^{\frac{1}{2}}\, dr^2 + \frac{k}{4g} H_P^{-\frac{1}{2}} H_Q^{\frac{1}{2}} \Big(\sigma_1^2 + \sigma_2^2 + \frac{4g\, P}{k}\, \sigma_3^2\Big)\,, \\ H_{(3)} &= P\, \sigma_1 \wedge \sigma_2 \wedge \sigma_3 - d^2x \wedge dH_Q^{-1}\,, \qquad F_{(2)} = k\, \sigma_1 \wedge \sigma_2\,, \qquad e^{\phi} = H_Q/H_P\,, \end{aligned} \tag{22}$$

where

$$H_Q = Q_0 + \frac{Q}{r^2}\,, \qquad H_P = P_0 + \frac{P}{r^2}\,. \tag{23}$$

The $H_{(3)}$ and $F_{(2)}$ charges must satisfy the algebraic constraint, namely $4g\, P = k(1 - 2g\, k)$.

Before turning to the properties of this solution, we may examine its relation to the dyonic superstring of the ungauged theory. To highlight the similarities, we may reexpress the metric as

$$ds^2 = (H_P H_Q)^{-\frac{1}{2}} dx^\mu\, dx_\mu + (H_P H_Q)^{\frac{1}{2}} \left[4\xi^2 \Xi^{-3}\, dr^2 + \Xi^{-1}\, r^2(\xi(\sigma_1^2 + \sigma_2^2) + \sigma_3^2)\right]\,, \tag{24}$$

where

$$\xi = \frac{k}{4gP} = (1 - 2g\, k)^{-1}\,, \qquad \Xi = 1 + \left(\frac{P_0}{P}\right) r^2\,. \tag{25}$$

To obtain the ungauged theory, we may take the limit $g \to 0$, $k \to 0$ with the $H_{(3)}$ magnetic charge $P \to k/4g$ held fixed, so that $\xi \to 1$. The metric, (24), then approaches that of the dyonic string in the ungauged theory, provided $\Xi \to 1$. This latter condition is somewhat surprising, as this restriction is absent in the ungauged theory. Its origin is apparently related to the nature of turning on both $F_{(2)}$ and $H_{(3)}$ flux over the squashed S^3.

We now return to the gauged theory, and consider the properties of the dyonic string solution, (22). For small r, the functions H_P and H_Q blow up as $1/r^2$ (we take both P and Q positive). Furthermore, in this limit, we see that $\Xi \to 1$. Hence the near-horizon limit of the string may be read off from the metric (24) by taking $\Xi = 1$ and retaining ξ as a squashing parameter. We see that this limit in fact precisely yields the AdS_3 times squashed 3-sphere family of solutions. Turning to the asymptotics away from the horizon, we note that some care must be involved in handling the constant P_0. For $P_0 > 0$, $H_P \to$ const as $r \to \infty$. However $\Xi \sim r^2$ in this limit, and this drastically modifies the asymptotics. In particular, $ds^2 \sim dr^2/r^6$ at large r, so that the r interval has a finite range. On the other hand, for $P_0 = 0$, the function Ξ is identically 1, and in this fashion we are able to recover large distance asymptotics. For $P_0 < 0$, the function H_P goes through zero when $r^2 = |P/P_0|$, thus putting a natural limit on the coordinate, $r \in (0, |P/P_0|^{1/2})$.

In fact, for either $P_0 = 0$ or $P_0 < 0$, the large distance asymptotics originate when $H_P \to 0$. Both cases may be treated simultaneously by changing to a new radial coordinate ρ, related to r by

$$P_0 + \frac{P}{r^2} = \frac{P^2}{\rho^4}\,. \tag{26}$$

We also replace the constants Q_0 and Q by

$$\widetilde{Q}_0 \equiv Q_0 - \frac{Q\,P_0}{P}\,, \qquad \widetilde{Q}^2 \equiv Q\,P\,. \tag{27}$$

In terms of these redefined quantities, dyonic superstring solution of (22) becomes

$$\begin{aligned} ds^2 &= \frac{\rho^2}{P\,\mathcal{H}^{\frac{1}{2}}}\,dx^\mu\,dx_\mu + 16\,\xi^2\mathcal{H}^{\frac{1}{2}}\,d\rho^2 + \mathcal{H}^{\frac{1}{2}}\,\rho^2\,(\xi(\sigma_1^2+\sigma_2^2)+\sigma_3^2)\,, \\ H_{(3)} &= P\,\sigma_1\wedge\sigma_2\wedge\sigma_3 - d^2x\wedge d\mathcal{H}^{-1}\,, \qquad F_{(2)} = k\,\sigma_1\wedge\sigma_2\,, \\ e^\phi &= \frac{\mathcal{H}\,\rho^4}{P^2}\,, \qquad \mathcal{H}\equiv \widetilde{Q}_0 + \frac{\widetilde{Q}^2}{\rho^4}\,.\langle 48 \end{aligned} \tag{28}$$

If $\widetilde{Q}_0$ is positive, these solutions describe everywhere non-singular dyonic string. If $\widetilde{Q}_0$ is negative, there is a naked singularity at the value of ρ for which $\mathcal{H}$ vanishes. The previously noted horizon at $r=0$ corresponds here to $\rho=0$, and in the near-horizon limit where $\rho\longrightarrow 0$, the metric approaches

$$ds^2 \sim (PQ)^{\frac{1}{2}}\left(16\,\xi^2\frac{d\rho^2}{\rho^2} + \frac{\rho^4}{P^2\,Q}\,dx^\mu\,dx_\mu\right) + (PQ)^{\frac{1}{2}}\,(\xi(\sigma_1^2+\sigma_2^2)+\sigma_3^2)\,,\langle 49 \tag{29}$$

while the dilaton approaches a constant; $e^\phi\longrightarrow Q/P$. In this limit the tensor multiplet is frozen, and the projection condition is lost. As a result, the supersymmetry at the horizon is restored to $\frac{1}{2}$ of the original supersymmetry.

At large distance, $\rho\to\infty$, the metric approaches

$$ds^2 \sim 16\xi^2\,\widetilde{Q}_0^{\frac{1}{2}}\left(d\rho^2 + \frac{1}{16\xi^2}\,\rho^2\,(\xi(\sigma_1^2+\sigma_2^2)+\sigma_3^2) + \frac{P\,g^2}{k^2\widetilde{Q}_0}\,\rho^2\,dx^\mu\,dx_\mu\right), \tag{30}$$

which describes a cone over the product of a squashed S^3 and the (Minkowski)$_2$ string worldsheet metric. Unlike the gauge dyonic string, or for that matter typical string solitons, the present dyonic string is not asymptotic to the usual vacuum attributed to this gauged $N=(1,0)$ theory, namely (Minkowski)$_4\times S^2$. In fact, looking at the dilaton, one finds $e^\phi\sim\rho^4(\widetilde{Q}_0/P^2)$, which blows up asymptotically. This is not necessarily surprising, since the potential is of a single-exponential form, which suggests the possibility of a domain-wall type solution with runaway dilaton. We note, however, that when the dilaton is active, the $\delta\chi=0$ condition requires non-vanishing $H_{(3)}$ for the preservation of supersymmetry. This indicates that domain wall solutions with (Minkowski)$_5$ symmetry do not occur, and that the (Minkowski)$_2$ times squashed S^3 geometry above is perhaps the most symmetric that may be obtained for a domain wall configuration.

4 Poincaré Supergravity

4.1 Off-shell Poincaré Action

The off-shell $(1,0)$ supergravity action has been constructed by means of a superconformal tensor calculus in which the off-shell so-called dilaton Weyl multiplet with independent fields

$$\{\,e_\mu{}^a\,,\psi_\mu^i, B_{\mu\nu}\,,\mathcal{V}_\mu^{ij}\,,b_\mu\,,\psi^i\,,\sigma\,\} \tag{31}$$

and Weyl weights $(-1,-1/2,0,0,0,5/2,2)$, respectively, is coupled to an off-shell linear multiplet consisting of the fields

$$\{\,E_{\mu\nu\rho\sigma}\,,L^{ij}\,,\varphi^i\,\}\,, \tag{32}$$

with Weyl weights $(0, 4, 9/2)$, respectively. The fields $(\psi^i_\mu, \psi^i, \varphi^i)$ are symplectic Majorana–Weyl spinors labelled by a $\mathrm{Sp}(1)_R$ doublet index, the fields B and E are two- and four-forms with tensor gauge symmetries, respectively, b_μ is the dilatation gauge field and L_{ij} are three real scalars. An appropriate set of gauge choices for obtaining off-shell supergravity with the Einstein–Hilbert term, namely $\mathcal{L} = eR + \cdots$, is given by

$$L_{ij} = \frac{1}{\sqrt{2}}\,\delta_{ij}\ , \quad \varphi^i = 0\ , \quad b_\mu = 0 \tag{33}$$

which fixes the dilatations, conformal boost and special supersymmetry transformations. Moreover, the first of the gauge choices in (33) breaks $\mathrm{Sp}(1)_R$ down to $U(1)_R$. This set of gauge choices leads to an off-shell multiplet containing $48+48$ degrees of freedom described by the fields

$$e_\mu{}^a\ (15)\ ,\quad \mathcal{V}'_\mu{}^{ij}\ (12)\ ,\quad \mathcal{V}_\mu\ (5)\ ,\quad B_{\mu\nu}\ (10)\ ,\quad \sigma\ (1)\ ,\quad E_{\mu\nu\rho\sigma}\ (5)\ ;\quad \psi_\mu{}^i\ (40)\ ,\quad \psi^i\ (8)\ . \tag{34}$$

The field $\mathcal{V}_\mu$ is the gauge field of the surviving $U(1)_R$ gauge symmetry. It arises in the decomposition

$$\mathcal{V}^{ij}_\mu = \mathcal{V}'^{ij}_\mu + \frac{1}{2}\delta^{ij}\mathcal{V}_\mu\ , \qquad \mathcal{V}'^{ij}_\mu\,\delta_{ij} = 0\ , \tag{35}$$

where the traceless part $\mathcal{V}'^{ij}_\mu$ has no gauge symmetry. The Lagrangian up to quartic fermion terms is given by

$$\begin{aligned} e^{-1}\mathcal{L}_R\big|_{L=1} \ =\ & \frac{1}{2}R - \frac{1}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{24}\sigma^{-2}F_{\mu\nu\rho}(B)F^{\mu\nu\rho}(B) + \mathcal{V}'_{\mu ij}\mathcal{V}'^{\mu ij} \\ & -\frac{1}{4}E^\mu E_\mu + \frac{1}{\sqrt{2}}E^\mu\mathcal{V}_\mu - \frac{1}{4\sqrt{2}}E_\rho\bar{\psi}^i_\mu\gamma^{\rho\mu\nu}\psi^j_\nu\delta_{ij} \\ & -\frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho - 2\sigma^{-2}\bar{\psi}\gamma^\mu D'_\mu(\omega)\psi + \sigma^{-2}\bar{\psi}_\nu\gamma^\mu\gamma^\nu\psi\,\partial_\mu\sigma \\ & -\frac{1}{48}\sigma^{-1}F_{\mu\nu\rho}(B)\left(\bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + 4\sigma^{-1}\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\psi - 4\sigma^{-2}\bar{\psi}\gamma^{\mu\nu\rho}\psi\right)\ . \end{aligned} \tag{36}$$

The indication $L = 1$ in the left-hand side indicates all the gauge choices (33). Here we have defined the field strength for the 2-form potential and the dual of the field strength for the 4-form potentials as follows

$$F_{\mu\nu\rho}(B) \ =\ 3\partial_{[\mu}B_{\nu\rho]}\ , \tag{37}$$

$$E^\mu \ =\ \frac{1}{24}e^{-1}\varepsilon^{\mu\nu_1\cdots\nu_5}\partial_{[\nu_1}E_{\nu_2\cdots\nu_5]}\ . \tag{38}$$

The $U(1)_R$ covariant derivatives $D_\mu(\omega)$ and the SU(2) covariant derivatives $D'_\mu(\omega)$ are given by

$$D_\mu(\omega)\psi^i_\nu \ =\ (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi^i_\nu - \frac{1}{2}\mathcal{V}_\mu\delta^{ij}\psi_{\nu j}\ , \tag{39}$$

$$D'_\mu(\omega)\psi^i \ =\ (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\psi^i - \frac{1}{2}\mathcal{V}_\mu\delta^{ij}\psi_j + \mathcal{V}_\mu{}'^i{}_j\psi^j\ , \tag{40}$$

where $\omega_{\mu ab}$ is the standard torsion-free connection. Note that the symmetric traceless field $\mathcal{V}'_\mu{}^{ij}$, occurring in the decomposition (35), is absent in the covariant derivative of the gravitino. This is a consequence of having broken the SU(2) symmetry present in the dilaton Weyl multiplet by the gauge choices (33). In the above formula, and throughout the paper the spin connection $\omega_{\mu ab}$

is the standard one associated with the Christoffel symbol, and as such, it does not depend on fermionic or bosonic torsion. The supersymmetry transformations, up to cubic fermion terms are

$$
\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\
\delta\psi^i_\mu &= D_\mu(\omega)\epsilon^i + \frac{1}{48}\sigma^{-1}\gamma\cdot F(B)\gamma_\mu\epsilon^i - \mathcal{V}'^{ij}_\mu\epsilon_j + \gamma_\mu\eta^i\,, \\
\delta B_{\mu\nu} &= -\sigma\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} - \bar{\epsilon}\gamma_{\mu\nu}\psi\,, \\
\delta\psi^i &= \frac{1}{48}\gamma\cdot F(B)\epsilon^i + \frac{1}{4}\partial\sigma\epsilon^i - \sigma\eta^i\,, \\
\delta\sigma &= \bar{\epsilon}\psi\,, \\
\delta E_{\mu\nu\rho\sigma} &= 2\sqrt{2}\bar{\psi}_{[\mu}{}^i\gamma_{\nu\rho\sigma]}\epsilon^j\delta_{ij}\,, \\
\delta\mathcal{V}^{ij}_\mu &= \frac{1}{2}\bar{\epsilon}^{(i}\gamma^\nu R_{\mu\nu}{}^{j)}(Q) + \frac{1}{8}\sigma^{-1}\bar{\epsilon}^{(i}\gamma^\nu\Big(F_{[\mu}{}^{ab}(B)\gamma_{ab}\psi^{j)}_{\nu]}\Big) + \frac{1}{24}\sigma^{-1}\bar{\epsilon}^{(i}\gamma\cdot F(B)\psi^{j)}_\mu \\
&\quad + \frac{1}{2}\sigma^{-1}\bar{\epsilon}^{(i}\gamma_\mu D'(\omega)\psi^{j)} - \frac{1}{8}\sigma^{-1}\bar{\epsilon}^{(i}\gamma_\mu\gamma^\rho\partial\sigma\psi_\rho{}^{j)} - \frac{1}{48}\sigma^{-2}\bar{\epsilon}^{(i}\gamma_\mu\gamma\cdot F(B)\psi^{j)} + 2\bar{\eta}^{(i}\psi^{j)}_\mu\,,
\end{aligned}
\tag{41}
$$

where $D_\mu(\omega)\epsilon^i$ is defined as in (39), $R_{\mu\nu}{}^i(Q)$ is the gravitino curvature and η_i is the effective contribution from the S-supersymmetry in the superconformal algebra:

$$
\begin{aligned}
D_\mu(\omega)\epsilon^i &= (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab})\epsilon^i - \frac{1}{2}\mathcal{V}_\mu\delta^{ij}\epsilon_j\,, \\
R_{\mu\nu}{}^i(Q) &= 2D_{[\mu}(\omega)\psi^i_{\nu]} - 2\mathcal{V}'^{ij}_{[\mu}\psi_{\nu]j}\,,
\end{aligned}
\tag{42}
$$

$$
\eta_k = \frac{1}{4}\Big(\gamma^\mu\mathcal{V}_\mu{}'^{(i}{}_l\delta^{j)l}\epsilon_j - \frac{1}{2\sqrt{2}}E_\mu\gamma^\mu\epsilon^i\Big)\delta_{ik}\,. \tag{43}
$$

The latter equation gives the compensating special supersymmetry transformation parameter in the gauge $\varphi^i = 0$. Note that the $U(1)_R$ part of $\mathcal{V}^{ij}_\mu$ has dropped out in this expression. The surviving $U(1)_R$ symmetry of the Lagrangian $\mathcal{L}_R$ is gauged by the auxiliary gauge field $\mathcal{V}_\mu$, which acts as follows

$$
\delta(\lambda)\mathcal{V}_\mu = \partial_\mu\lambda\,, \qquad \delta(\lambda)\psi_\mu{}^i = \frac{1}{2}\delta^{ij}\lambda\psi_{\mu j}\,, \qquad \delta(\lambda)\psi^i = \frac{1}{2}\delta^{ij}\lambda\psi_j\,, \tag{44}
$$

with λ being the parameter of the gauged symmetry.

We now wish to introduce a gauge multiplet, whose vector is not auxiliary, to gauge the $U(1)$ R-symmetry. The present gauging by $\mathcal{V}_\mu$ is undesirable since $\mathcal{V}_\mu$ has no standard kinetic term. To obtain this non-trivial gauging we add to $\mathcal{L}_\mathrm{R}$ the kinetic terms for an abelian vector multiplet $\mathcal{L}_\mathrm{V}$. The multiplet consists of the fields $(W_\mu, Y_{ij}, \Omega_i)$, being a physical gauge field, an auxiliary SU(2) triplet, and a physical fermion. They transform under dilatations with Weyl weights $(0, 2, 3/2)$, respectively. We add the coupling $g\mathcal{L}_\mathrm{VL}$ of the vector multiplet to the compensating linear multiplet. Prior to fixing any of the conformal symmetries, these Lagrangians, up to quartic fermion terms, are given by

$$
\begin{aligned}
e^{-1}\mathcal{L}_\mathrm{V} &= \sigma\Big(-\frac{1}{4}F_{\mu\nu}(W)F^{\mu\nu}(W) - 2\bar{\Omega}\gamma^\mu D'_\mu(\omega)\Omega + Y^{ij}Y_{ij}\Big) \\
&\quad -\frac{1}{16}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}F_{\rho\sigma}(W)F_{\lambda\tau}(W) - 4\bar{\Omega}^i\psi^jY_{ij} \\
&\quad +\frac{1}{2}\left(\sigma\bar{\Omega}\gamma^\mu\gamma\cdot F(W)\psi_\mu + 2\bar{\Omega}\gamma\cdot F(W)\psi\right) + \frac{1}{12}\bar{\Omega}\gamma\cdot F(B)\Omega\,,
\end{aligned}
\tag{45}
$$

$$
e^{-1}\mathcal{L}_\mathrm{VL} = Y_{ij}L^{ij} + 2\bar{\Omega}\varphi - L^{ij}\bar{\psi}_{\mu i}\gamma^\mu\Omega_j + \frac{1}{2}W_\mu E^\mu\,, \tag{46}
$$

where $D'_\mu(\omega)\Omega^i$ is defined as in (40). This action has the full SU(2) symmetry.

The coupling of the vector multiplet to supergravity is then achieved by considering the Lagrangian

$$\mathcal{L}_1 = \Big(\mathcal{L}_R + \mathcal{L}_V + g\mathcal{L}_{VL}\Big)\Big|_{L=1} \ , \tag{47}$$

where as before '$L = 1$' refers to the set of gauges given in (33). This formula, up to quartic fermion terms, yields the result

$$\begin{aligned}
e^{-1}\mathcal{L}_1 &= \frac{1}{2}R - \frac{1}{2}\sigma^{-2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{\sqrt{2}}g\delta^{ij}Y_{ij} - \frac{1}{24}\sigma^{-2}F_{\mu\nu\rho}(B)F^{\mu\nu\rho}(B) \\
&+\mathcal{V}'^{\,ij}_\mu\mathcal{V}'^\mu{}_{ij} - \frac{1}{4}E^\mu E_\mu + \frac{1}{\sqrt{2}}E^\mu(\mathcal{V}_\mu + \frac{1}{\sqrt{2}}gW_\mu) + \sigma Y^{ij}Y_{ij} \\
&-\frac{1}{4}\sigma F_{\mu\nu}(W)F^{\mu\nu}(W) - \frac{1}{16}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}F_{\rho\sigma}(W)F_{\lambda\tau}(W) \\
&-\frac{1}{2}\bar{\psi}_\rho\gamma^{\mu\nu\rho}D_\mu(\omega)\psi_\nu - 2\sigma^{-2}\bar{\psi}\gamma^\mu D'_\mu(\omega)\psi + \sigma^{-2}\bar{\psi}_\nu\gamma^\mu\gamma^\nu\psi\partial_\mu\sigma \\
&-\frac{1}{48}\sigma^{-1}F_{\mu\nu\rho}(B)\Big(\bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + 4\sigma^{-1}\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\psi - 4\sigma^{-2}\bar{\psi}\gamma^{\mu\nu\rho}\psi\Big) \\
&-\frac{1}{4\sqrt{2}}E_\rho\psi^i_\mu\gamma^{\rho\mu\nu}\psi^j_\nu\delta_{ij} - \frac{1}{\sqrt{2}}g\delta^{ij}\bar{\Omega}_i\gamma^\mu\psi_{\mu j} - 2\sigma\bar{\Omega}\gamma^\mu D'_\mu(\omega)\Omega - 4Y^{ij}\bar{\Omega}_i\psi_j \\
&+\frac{1}{2}F_{\mu\nu}(W)\left(\sigma\bar{\Omega}\gamma^\lambda\gamma^{\mu\nu}\psi_\lambda + 2\bar{\Omega}\gamma^{\mu\nu}\psi\right) + \frac{1}{12}F_{\mu\nu\rho}(B)\bar{\Omega}\gamma^{\mu\nu\rho}\Omega \ .
\end{aligned} \tag{48}$$

The action corresponding to the Lagrangian $\mathcal{L}_1$ is invariant under the supersymmetry transformations (41) supplemented by the supersymmetry transformations of the components of the off-shell vector multiplet. The transformations of the latter are given up to fermion terms by

$$\begin{aligned}
\delta W_\mu &= -\bar{\epsilon}\gamma_\mu\Omega \ , \\
\delta\Omega^i &= \frac{1}{8}\gamma\cdot F(W)\epsilon^i - \frac{1}{2}Y^{ij}\epsilon_j \ , \\
\delta Y^{ij} &= -\frac{1}{2}\bar{\epsilon}^i\gamma^\mu\left(D'_\mu(\omega)\Omega^j - \frac{1}{8}\gamma\cdot F(W)\psi^j_\mu + \frac{1}{2}Y^{jk}\psi_{\mu k}\right) + \bar{\eta}^i\Omega^j + (i\leftrightarrow j) \ ,
\end{aligned} \tag{49}$$

where η is as defined in (43). The Lagrangian $\mathcal{L}_1$ also has a manifest $U(1)_R \times U(1)$ symmetry with transformations parametrized by λ and η

$$\begin{aligned}
\delta\mathcal{V}_\mu &= \partial_\mu\lambda\,, \qquad \delta W_\mu = \partial_\mu\eta, \\
\delta\psi^i_\mu &= \frac{1}{2}\lambda\delta^{ij}\psi_{\mu j} \ , \qquad \delta\psi^i = \frac{1}{2}\lambda\delta^{ij}\psi_{\mu j} \ , \qquad \delta\Omega^i = \frac{1}{2}\lambda\delta^{ij}\Omega_j \ ,
\end{aligned} \tag{50}$$

where $(\lambda\,,\eta)$ are the parameters of the $\big(U(1)_R\,,U(1)\big)$ symmetry, respectively.

4.2 Explicit Construction of the Action

We need an expression constructed from the components of the linear multiplet that can be used as a superconformal action. The density formula is given for the product of a vector multiplet W_μ, Ω^i, Y^{ij} and a linear multiplet L^{ij}, φ^i, E_a :

$$e^{-1}\mathcal{L}_{VL} = Y_{ij}L^{ij} + 2\bar{\Omega}\varphi - L^{ij}\bar{\psi}_{\mu i}\gamma^\mu\Omega_j + \frac{1}{4}F_{\mu\nu}(W)E^{\mu\nu}. \tag{51}$$

The next step is then to construct a vector multiplet from the components of the linear multiplet

$$\Omega^i = \Omega^i(L^{ij},\varphi^i,E_a)\,, \qquad Y^{ij} = Y^{ij}(L^{ij},\varphi^i,E_a)\,, \qquad \hat{F}_{\mu\nu} = \hat{F}_{\mu\nu}(L^{ij},\varphi^i,E_a)\,, \tag{52}$$

and to plug in these components into (51) to obtain a superconformal action for the linear multiplet. Note that $\hat{F}_{\mu\nu} = F_{\mu\nu} + 2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega$ is the supercovariant field strength of W_μ. We also define

$$L = \left(L_{ij}L^{ij}\right)^{1/2}. \tag{53}$$

Using (51) and retaining only the bosonic terms we obtain

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bos}} &= -\Delta_a\Delta^a L + L^{-1}\Delta_a L^{ij}\Delta^a L_{ij} + L^{-3}L_{ij}L_{kl}\Big(\Delta^a L^{k(i}\Delta_a L^{j)l} - \Delta_a L^{ij}\Delta^a L^{kl}\Big) \\ &+\frac{1}{4}L^{-1}E_aE^a - \frac{1}{3}LD - L^{-3}L_{ij}E_aL^{k(i}\Delta^a L^{j)}{}_k \\ &+\frac{1}{4}E^{\mu\nu}\Big(-2\partial_\mu L^{ij}L^k{}_j\partial_\nu L_{ik}L^{-3} - 2\partial_{[\mu}\left(E_{\nu]}L^{-1} - 2\mathcal{V}_{\nu]ij}L^{ij}L^{-1}\right)\Big)\,. \end{aligned} \tag{54}$$

We also know from specific equation that

$$E^a = e_\mu{}^a\Delta_\nu E^{\mu\nu} = e_\mu{}^a\partial_\nu E^{\mu\nu} + \text{fermionic terms.} \tag{55}$$

After partial integration and using the identity, we can write the bosonic action as

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bos}} &= -\Delta_a\Delta^a L + \frac{1}{2}L^{-1}\Delta_a L^{ij}\Delta^a L_{ij} \\ &-\frac{1}{3}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu\mathcal{V}_{\mu ij}L^{ij}L^{-1} - \frac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L^k_j\partial_\nu L_{ik}L^{-3}. \end{aligned} \tag{56}$$

We will discuss the gauge fixing procedure. To fix the S-gauge we will choose $\varphi = 0$. Hence, in order to clarify calculations, we will drop all terms directly proportional to φ already at this stage of the calculation. To write down the fermionic part of the superconformal Lagrangian we need the fermionic terms of $F_{\mu\nu}(L)$. We obtain

$$\begin{aligned} F_{\mu\nu}(L) &= (\text{bosonic terms}) \\ &-8L^{-1}L_{ij}\bar{\psi}^i_{[\mu}\phi^j_{\nu]} - \frac{4}{3}L^{-1}L_{ij}\bar{\psi}^i_{[\mu}\gamma_{\nu]}\chi^j + 2L^{-1}\bar{\psi}_{[\mu}\gamma_{\nu]}\mathcal{D}\varphi - 2L^{-1}\bar{\psi}_{[\mu}\Delta_{\nu]}\varphi \\ &+\frac{1}{2}L^{-1}\bar{\psi}_{[\mu}\gamma^b\psi_{\nu]}E_b - 2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega(L^{ij},\varphi^i,E_a), \end{aligned} \tag{57}$$

where the first term in the third line is obtained by use of the solutions of the conventional constraints. By use of the theoretical construction we can write

$$-2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega(L^{ij},\varphi^i,E_a) \;=\; -2L^{-1}\bar{\psi}_{[\mu}\gamma_{\nu]}\mathcal{D}\varphi + \frac{4}{3}L^{-1}\bar{\psi}^i_{[\mu}\gamma_{\nu]}L_{ij}\chi^j. \tag{58}$$

Filling in the above equation into equation (57) and using (**??**) we find that all fermionic terms of $F_{\mu\nu}(L)$ drop. Using the density formula (51), we obtain the fermionic part of the action which, at this stage of the calculation, looks like

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{ferm}} &= -\frac{1}{4}L^{-1}\bar{\psi}^i_\rho\gamma^{\mu\nu\rho}\psi^j_\nu L_{ij}E_\mu + \frac{1}{2}L^{-1}L^{ij}L_{kl}\bar{\psi}^k_\rho\gamma^{\mu\nu\rho}\psi^l_\nu\mathcal{V}_{\mu ij} - \frac{1}{3}L\bar{\psi}_\mu\gamma^\mu\chi \\ &-L^{-1}L_{ij}\bar{\psi}^i_\mu\gamma^\mu\gamma^\nu\Big(-\frac{1}{2}\mathcal{D}L^{jk}\psi_{\nu k} + \frac{1}{4}\gamma^\rho E_\rho\psi_\nu{}^j + 4L^{jk}\phi_{\nu k}\Big). \end{aligned} \tag{59}$$

Note that the two first terms come from the step done in the construction of equation (55).

We will now prepare for the gauge fixing procedure by writing out the covariant derivatives and dependent fields. The bosonic superconformal Lagrangian in (56) can be rewritten in a form that is most convenient for the gauge fixing procedure

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bos}} &= L^{-3}L_{ij}L_{kl}\Delta_a L^{kl}\Delta^a L^{ij} - L^{-1}\Delta_a L_{ij}\Delta^a L^{ij} - L^{-1}L_{ij}\Delta_a\Delta^a L^{ij} + \frac{1}{2}L^{-1}\Delta_a L^{ij}\Delta^a L_{ij} \\ &-\frac{1}{3}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu\mathcal{V}_{\mu ij}L^{ij}L^{-1} - \frac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L^k{}_j\partial_\nu L_{ik}L^{-3}\,. \end{aligned} \tag{60}$$

The fermionic terms in these covariant derivatives will be collected in a new fermionic Lagrangian, which contains not only the terms in (59), but also terms from the bosonic part (60) due to terms quadratic in fermions in covariant derivatives or dependent bosonic fields.

We mentioned that the choice for the S-gauge will be $\varphi = 0$. We dropped already terms proportional to φ. Note that these terms should be restored to get a full superconformal action, but that is not the aim of this paper. As the only fermionic terms in $\Delta_a L^{ij}$ are proportional to φ, the first two terms on the RHS of (60) will not contribute any new fermionic terms when we write out the covariant derivatives. Instead we take a closer look at the third term on the RHS of (60). Note that $\varphi = 0$ does not imply the vanishing of $\Delta_\mu \varphi^i$. We obtain

$$\begin{aligned}
-L^{-1}L_{ij}\Delta_a\Delta^a L^{ij} &= -L^{-1}L_{ij}e^{a\mu}\partial_\mu\Delta_a L^{ij} - L^{-1}L_{ij}e^{a\mu}\omega_{\mu a}{}^b\Delta_b L^{ij} \\
&\quad -2L^{-1}L_{ij}\mathcal{V}^{a(i}{}_k\Delta_a L^{j)k} + 8Lf_a{}^a \\
&\quad +L^{-1}L_{ij}\bar{\psi}^{ai}\Big(-\frac{1}{2}\mathcal{D}L^{jk}\psi_{ak} + \frac{1}{4}\gamma^b E_b\psi_a^j + 4L^{jk}\phi_{ak}\Big) \\
&\quad +\frac{1}{6}L\bar{\psi}^a\gamma_a\chi\,. \qquad (61)
\end{aligned}$$

The last two lines will thus be added to $\mathcal{L}_{\rm ferm}$. Using the first relation of useful expression, the first two terms in (61) combine into $-L^{-1}L_{ij}e^{-1}\partial_\mu(e\Delta^\mu L^{ij})$ and there is a term $-\frac{1}{2}L^{-1}L_{ij}\bar{\psi}^a\gamma_a\psi^b\Delta_b L^{ij}$ that needs to be added to the fermionic Lagrangian. The bosonic Lagrangian thus becomes

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm bos} &= L^{-3}L_{ij}L_{kl}\Delta_a L^{kl}\Delta^a L^{ij} - \frac{1}{2}L^{-1}\Delta_a L_{ij}\Delta^a L^{ij} - L^{-1}L_{ij}e^{-1}\partial_\mu(e\Delta^\mu L^{ij}) \\
&\quad -2L^{-1}L_{ij}\mathcal{V}^{a(i}{}_k\Delta_a L^{j)k} + 8Lf_a{}^a \\
&\quad -\frac{1}{3}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu\mathcal{V}_{\mu ij}L^{ij}L^{-1} - \frac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L_j^k\partial_\nu L_{ik}L^{-3}. \qquad (62)
\end{aligned}$$

Writing out the covariant derivatives $\Delta^a L_{ij}$, dropping the fermionic terms and terms proportional to b_μ. We can write the bosonic Lagrangian as

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm bos} &= 8Lf_a{}^a + \frac{1}{2}L^{-1}\partial_a L^{ij}\partial^a L_{ij} - 2L^{-1}L^k{}_i\mathcal{V}^{aij}\partial_a L_{kj} + 2L^{-1}\mathcal{V}_a{}^{(l}{}_k L^{j)k}\mathcal{V}^a{}_{il}L^i{}_j \\
&\quad -\frac{1}{3}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu\mathcal{V}_{\mu ij}L^{ij}L^{-1} - \frac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L_j^k\partial_\nu L_{ik}L^{-3}\,. \qquad (63)
\end{aligned}$$

The fermionic action looks at this point as

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm ferm} &= -\frac{1}{4}L^{-1}\bar{\psi}_\rho^i\gamma^{\mu\nu\rho}\psi_\nu^j L_{ij}E_\mu + \frac{1}{2}L^{-1}L^{ij}L_{kl}\bar{\psi}_\rho^k\gamma^{\mu\nu\rho}\psi_\nu^l\mathcal{V}_{\mu ij} - \frac{1}{3}L\bar{\psi}_\mu\gamma^\mu\chi \\
&\quad -L^{-1}L_{ij}\bar{\psi}_\mu^i\gamma^{\mu\nu}\Big(-\frac{1}{2}\mathcal{D}L^{jk}\psi_{\nu k} + \frac{1}{4}\gamma^\rho E_\rho\psi_\nu{}^j + 4L^{jk}\phi_{\nu k}\Big) \\
&\quad +\frac{1}{6}L\bar{\psi}^a\gamma_a\chi - \frac{1}{2}L^{-1}L_{ij}\bar{\psi}^a\gamma_a\psi^b\Delta_b L^{ij}. \qquad (64)
\end{aligned}$$

Note that this fermionic Lagrangian differs from the one in (59) by the two last terms, and the $\gamma^\mu\gamma^\nu$ being replaced by $\gamma^{\mu\nu}$, which originate from the last two lines of (61).

The bosonic terms in the expression of $f_a{}^a$ lead to

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm bos} &= \frac{2}{5}LR + \frac{1}{2}L^{-1}\partial_a L^{ij}\partial^a L_{ij} - 2L^{-1}L^k{}_i\mathcal{V}^{aij}\partial_a L_{kj} \\
&\quad +2L^{-1}\mathcal{V}_a{}^{(l}{}_k L^{j)k}\mathcal{V}^a{}_{il}L^i{}_j - \frac{2}{15}LD - \frac{1}{4}L^{-1}E_aE^a + E^\mu\mathcal{V}_{\mu ij}L^{ij}L^{-1} \\
&\quad -\frac{1}{2}E^{\mu\nu}\partial_\mu L^{ij}L_j^k\partial_\nu L_{ik}L^{-3}. \qquad (65)
\end{aligned}$$

The fermionic terms of $f_a{}^a$ lead to terms modifying $\mathcal{L}_{\rm ferm}$ to

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm ferm} &= \frac{1}{2}L^{-1}L^{ij}L_{kl}\bar{\psi}^k_\rho\gamma^{\mu\nu\rho}\psi^l_\nu\mathcal{V}_{\mu ij} + \frac{1}{2}L^{-1}L_{ij}\bar{\psi}^i_\mu\gamma^{\mu\nu}\gamma^\rho D_\rho L^{jk}\psi_{\nu k} \\
&\quad -L\bar{\psi}_\mu\gamma^\mu\left(2\gamma^\nu\phi_\nu + \frac{1}{3}\chi\right) - \frac{4}{5}L\bar{\psi}^\mu\gamma^\nu\hat{D}_{[\nu}\psi_{\mu]} + \frac{1}{5}L\bar{\psi}_b\gamma_c\psi_a T^{-abc} \\
&\quad -\frac{1}{2}\bar{\psi}^a\gamma_a\psi^b\partial_b L.
\end{aligned} \tag{66}$$

We used here the SU(2)-covariant derivative,

$$D_\mu L^{ij} = \partial_\mu L^{ij} + 2\mathcal{V}_\mu{}^{(i}{}_k L^{j)k}, \tag{67}$$

where we already put $b_\mu = 0$ in view of the gauge fixing of special conformal transformations that we will adopt soon.

To arrive at the super Poincaré group, the redundant symmetries of the superconformal algebra need to be broken. The special conformal transformations are fixed by the condition $b_\mu = 0$. The dilatation gauge is fixed by $L = 1$. The SU(2) symmetry cannot be completely broken. A gauge choice $L^{ij} = \sqrt{\tfrac{1}{2}}\delta^{ij}$ still leaves a remaining $U(1)$ symmetry which will be gauged by the auxiliary $\mathcal{V}_{\mu ij}\delta^{ij}$. For the bosonic part of the gauge-fixed action, we write

$$2\mathcal{V}_a{}^{(\ell}{}_k L^{j)k}\mathcal{V}^a{}_{i\ell}L^i{}_j = \mathcal{V}'^{\,ij}_a\,\mathcal{V}'_{aij}\,. \tag{68}$$

Here $\mathcal{V}'_{aij}$ is the traceless part of $\mathcal{V}_{aij}$:

$$\mathcal{V}'_{aij} = \mathcal{V}_{aij} - \tfrac{1}{2}\delta_{ij}\delta^{k\ell}\mathcal{V}_{ak\ell}. \tag{69}$$

Applying the gauge fixing in (65), we obtain the bosonic part of the gauge fixed action

$$e^{-1}\mathcal{L}_{\rm bos} = \frac{2}{5}R + \mathcal{V}'^{\,ij}_a\mathcal{V}'^a{}_{ij} - \frac{2}{15}D - \frac{1}{4}E_aE^a + \frac{1}{\sqrt{2}}E^\mu\mathcal{V}_{\mu ij}\delta^{ij}. \tag{70}$$

We still need to fix the S-gauge. This can be done by demanding $\varphi = 0$. The fermionic part of the resulting action after gauge fixing is then

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm ferm} &= -\frac{1}{4}\mathcal{V}_\rho{}^{kl}\Big(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \epsilon_{kj}\epsilon_{li}\Big)\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}\psi_\nu{}^j \\
&\quad -\frac{1}{2}\delta_{ij}\bar{\psi}^i_\mu g^{\rho[\mu}\gamma^{\nu]}\mathcal{V}_\rho{}^j{}_l\delta^{kl}\psi_{\nu k} - \frac{1}{2}\bar{\psi}^i_\mu g^{\rho[\mu}\gamma^{\nu]}\mathcal{V}_\rho{}^k{}_i\psi_{\nu k} \\
&\quad -\bar{\psi}_\mu\gamma^\mu\Big(2\gamma^\nu\phi_\nu + \frac{1}{3}\chi^i\Big) - \frac{4}{5}\bar{\psi}^\mu\gamma^\nu\hat{D}_{[\nu}\psi_{\mu]} + \frac{1}{5}\bar{\psi}_b\gamma_c\psi_a T^{-abc},
\end{aligned} \tag{71}$$

This can still be simplified using $\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk} = \varepsilon^{ik}\varepsilon^{jl}$ and the fact that $\delta_{ij}\bar{\psi}^i_{[\mu}\gamma^a\psi^j_{\nu]} = 0$:

$$e^{-1}\mathcal{L}_{\rm ferm} = -\frac{1}{2}\mathcal{V}_{\rho ij}\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}\psi_\nu{}^j - \bar{\psi}_\mu\gamma^\mu\Big(2\gamma^\nu\phi_\nu + \frac{1}{3}\chi^i\Big) - \frac{4}{5}\bar{\psi}^\mu\gamma^\nu\hat{D}_{[\nu}\psi_{\mu]} + \frac{1}{5}\bar{\psi}_b\gamma_c\psi_a T^{-abc}\,. \tag{72}$$

Next we use specific expression to write

$$e^{-1}\mathcal{L}_{\rm ferm} = -\frac{1}{2}\mathcal{V}_{\rho ij}\bar{\psi}^i_\mu\gamma^{\mu\nu\rho}\psi_\nu{}^j - \frac{2}{5}\bar{\psi}_\rho\gamma^{\mu\nu\rho}\hat{D}_\mu\psi_\nu + \frac{1}{5}\bar{\psi}_b\gamma_c\psi_a T^{-abc} - \frac{2}{15}\bar{\psi}_\mu\gamma^\mu\chi. \tag{73}$$

As can be seen from the Lagrangians (70) and (73) the matter fields of the Weyl 1 multiplet, D, χ^i and T^-_{abc}, have no kinetic terms. The field equation for D gives an inconsistency. This problem can be solved by using the Weyl 2 multiplet instead. This can be done by plugging in specific expressions into the Lagrangians. We first write out the full expressions for χ^i and D. We find

$$\begin{aligned}
\chi^i &= \frac{15}{4}\sigma^{-1}\gamma^\mu\big(D_\mu\psi^i - \frac{1}{48}\gamma\cdot\hat{F}\psi^i_\mu - \frac{1}{4}\hat{D}\sigma\psi^i_\mu\big)\\
&\quad +\frac{3}{4}\gamma^{\mu\nu}\big(D_\mu{\psi_\nu}^i - \frac{1}{48}\sigma^{-1}\gamma\cdot\hat{F}\gamma_\nu\psi^i_\mu\big) - \frac{5}{32}\sigma^{-2}\gamma\cdot\hat{F}\psi^i,
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
D &= \frac{15}{4}\sigma^{-1}\Big[e^{-1}\partial_\mu(e\hat{D}^\mu\sigma) + \frac{1}{2}\bar{\psi}^a\gamma_a\psi^b\hat{D}_b\sigma - \frac{1}{5}\sigma R + \frac{1}{12}\sigma^{-1}\hat{F}\cdot\hat{F}\\
&\quad +\frac{2}{5}\sigma\bar{\psi}^\mu\gamma^\nu\big(D_{[\nu}\psi_{\mu]} - \frac{1}{48}\sigma^{-1}\gamma\cdot\hat{F}\gamma_{[\mu}\psi_{\nu]}\big) - \frac{1}{20}\bar{\psi}_b\gamma_c\psi_a\hat{F}^{-abc}\\
&\quad -\bar{\psi}^\mu\big(D_\mu\psi - \frac{1}{48}\gamma\cdot\hat{F}\psi_\mu - \frac{1}{4}\hat{D}\sigma\psi_\mu\big) + \frac{1}{48}\sigma^{-1}\bar{\psi}\gamma\cdot\hat{F}\gamma^\mu\psi_\mu\\
&\quad +\bar{\psi}\gamma^{\mu\nu}\big(D_\mu\psi_\nu - \frac{1}{48}\sigma^{-1}\gamma\cdot\hat{F}\gamma_\nu\psi_\mu\big) - \frac{1}{6}\sigma^{-2}\bar{\psi}\gamma\cdot\hat{F}\psi\\
&\quad +4\sigma^{-1}\bar{\psi}\gamma^\mu\big(D_\mu\psi - \frac{1}{48}\gamma\cdot\hat{F}\psi_\mu - \frac{1}{4}\hat{D}\sigma\psi_\mu\big)\Big],
\end{aligned} \tag{75}$$

where the $D_\mu\psi^i_\nu$ and $D_\mu\psi^i$ are Lorentz and SU(2) covariant derivatives, while $\hat{D}_\mu\sigma$ is a supercovariant derivative:

$$\begin{aligned}
D_\mu\psi^i_\nu &\equiv \nabla_\mu\psi^i_\nu + {\mathcal{V}_\mu}^i{}_j\psi^j_\nu = \big(\partial_\mu + \frac{1}{4}{\omega_\mu}^{ab}\gamma_{ab}\big)\psi^i_\nu + {\mathcal{V}_\mu}^i{}_j\psi^j_\nu\,,\\
D_\mu\psi^i &\equiv \nabla_\mu\psi^i + {\mathcal{V}_\mu}^i{}_j\psi^j = \big(\partial_\mu + \frac{1}{4}{\omega_\mu}^{ab}\gamma_{ab}\big)\psi^i + {\mathcal{V}_\mu}^i{}_j\psi^j\,,\\
\hat{D}_\mu\sigma &= \partial_\mu\sigma - \bar{\psi}_\mu\psi.
\end{aligned} \tag{76}$$

The expressions for χ^i and D can still be rewritten by using several gamma matrix manipulations:

$$\begin{aligned}
\chi^i &= \frac{15}{4}\sigma^{-1}\gamma^\mu D_\mu\psi^i + \frac{3}{4}\gamma^{\mu\nu}D_\mu\psi^i_\nu - \frac{15}{16}\sigma^{-1}\gamma^\mu\hat{D}\sigma\psi^i_\mu\\
&\quad -\frac{3}{32}\sigma^{-1}\gamma^{\mu\rho\sigma\chi}\hat{F}_{\rho\sigma\chi}\psi^i_\mu - \frac{3}{16}\sigma^{-1}\gamma^{\sigma\chi}\hat{F}_{\rho\sigma\chi}\psi^{\rho i} - \frac{5}{32}\sigma^{-2}\gamma\cdot\hat{F}\psi^i
\end{aligned} \tag{77}$$

and

$$\begin{aligned}
D &= \frac{15}{4}\sigma^{-1}\Big[e^{-1}\partial_\mu(e\hat{D}^\mu\sigma) + \frac{1}{2}\bar{\psi}^a\gamma_a\psi^b\hat{D}_b\sigma - \frac{1}{5}\sigma R + \frac{1}{12}\sigma^{-1}\hat{F}\cdot\hat{F}\\
&\quad +\frac{2}{5}\sigma\bar{\psi}^\mu\gamma^\nu D_{[\nu}\psi_{\mu]} - \frac{1}{40}\bar{\psi}^\rho\gamma^{\mu\sigma\chi}\hat{F}_{\rho\sigma\chi}\psi_\mu - \bar{\psi}^\mu D_\mu\psi + \frac{1}{40}\bar{\psi}^\mu\gamma\cdot\hat{F}\psi_\mu\\
&\quad +\bar{\psi}\gamma^{\mu\nu}D_\mu\psi_\nu - \frac{1}{6}\sigma^{-2}\bar{\psi}\gamma\cdot\hat{F}\psi - \frac{1}{8}\sigma^{-1}\bar{\psi}\gamma^{\mu\nu\rho\sigma}\hat{F}_{\nu\rho\sigma}\psi_\mu - \frac{1}{8}\sigma^{-1}\bar{\psi}\gamma^{\rho\sigma}\hat{F}_{\nu\rho\sigma}\psi^\nu\\
&\quad +4\sigma^{-1}\bar{\psi}D\psi - \sigma^{-1}\bar{\psi}\gamma^\mu\hat{D}\sigma\psi_\mu\Big],
\end{aligned} \tag{78}$$

where we also used

$$\gamma_\rho\Phi^{-\mu\nu\rho} = \frac{1}{2}\gamma_\rho\big(\Phi^{\mu\nu\rho} + \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\lambda\chi}\Phi_{\sigma\lambda\chi}\big) = \frac{1}{2}\big(\gamma_\rho\Phi^{\mu\nu\rho} + \frac{1}{6}\gamma^{\mu\nu\sigma\lambda\chi}\gamma_*\Phi_{\sigma\lambda\chi}\big). \tag{79}$$

This equation can be proven using the duality relation (**??**).

Filling in the expressions for D and χ^i into (70) and (73) and distributing the respective terms over the bosonic and fermionic Lagrangians we get

$$e^{-1}\mathcal{L}_{\rm bos} = \frac{1}{2}R + \mathcal{V}'^{ij}_a \mathcal{V}'^a{}_{ij} - \frac{1}{4}E_a E^a + \frac{1}{\sqrt{2}}E^\mu \mathcal{V}_{\mu ij}\delta^{ij} - \frac{1}{2}\sigma^{-2}\partial_a\sigma\partial^a\sigma - \frac{3}{8}\sigma^{-2}\partial_{[\mu}B_{\nu\rho]}\partial^\mu B^{\nu\rho}\,, \quad (80)$$

where we used the definition of $\hat{F}(B)$. We then perform gamma matrix manipulations and write out the covariant derivatives (76). After dropping a total derivative, the fermionic Lagrangian becomes

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm ferm} = &-\frac{1}{2}\bar\psi_\rho\gamma^{\mu\nu\rho}\nabla_\mu\psi_\nu - 2\sigma^{-2}\bar\psi D\psi + \sigma^{-2}\bar\psi\gamma^\nu\gamma^\mu\psi_\nu\partial_\mu\sigma + \frac{1}{16}\sigma^{-1}\bar\psi_\mu\gamma^{\mu\nu\rho\sigma\chi}\psi_\nu\partial_\rho B_{\sigma\chi} \\
&-\frac{3}{8}\sigma^{-1}\bar\psi^\mu\gamma^\nu\psi^\rho\partial_{[\mu}B_{\nu\rho]} + \frac{1}{4}\sigma^{-2}\bar\psi_\mu\gamma^{\mu\rho\sigma\chi}\psi\partial_\rho B_{\sigma\chi} - \frac{3}{4}\sigma^{-2}\bar\psi^\mu\gamma^{\nu\rho}\psi\partial_{[\mu}B_{\nu\rho]} \\
&+\frac{1}{4}\sigma^{-3}\bar\psi\gamma^{\mu\nu\rho}\psi\partial_\mu B_{\nu\rho} + \frac{1}{32}\bar\psi_\mu\gamma^{\mu\nu\rho\sigma\chi}\psi_\nu\bar\psi_\rho\gamma_\sigma\psi_\chi - \frac{3}{32}\bar\psi_{[\mu}\gamma_\nu\psi_{\rho]}\bar\psi^\mu\gamma^\nu\psi^\rho \\
&-\frac{1}{4}\sigma^{-1}\bar\psi^\mu\gamma_\mu\psi^\nu\bar\psi^\rho\gamma_{\rho\nu}\psi + \frac{1}{8}\sigma^{-1}\bar\psi_\mu\gamma^\nu\psi_\sigma\bar\psi_\nu\gamma^{\mu\sigma}\psi + \frac{1}{16}\sigma^{-1}\bar\psi_\mu\gamma^{\mu\nu\rho\sigma\chi}\psi_\nu\bar\psi_\rho\gamma_{\sigma\chi}\psi \\
&-\frac{3}{8}\sigma^{-1}\bar\psi_{[\mu}\gamma_{\nu\rho]}\psi\bar\psi^\mu\gamma^\nu\psi^\rho + \frac{1}{8}\sigma^{-1}\bar\psi_\mu\gamma^{\mu\rho\sigma\chi}\psi\bar\psi_\rho\gamma_\sigma\psi_\chi + \frac{1}{4}\sigma^{-2}\bar\psi_\mu\gamma^{\mu\rho\sigma\chi}\psi\bar\psi_\rho\gamma_{\sigma\chi}\psi \\
&-\frac{3}{8}\sigma^{-2}\bar\psi_{[\mu}\gamma_{\nu\rho]}\psi\bar\psi^\mu\gamma^{\nu\rho}\psi - \frac{1}{2}\sigma^{-2}\bar\psi\gamma^{\mu\nu}\psi_\mu\bar\psi_\nu\psi - \frac{1}{2}\sigma^{-2}\bar\psi\psi^\nu\bar\psi_\nu\psi \\
&+\frac{1}{8}\sigma^{-2}\bar\psi\gamma^{\rho\sigma\chi}\psi\bar\psi_\rho\gamma_\sigma\psi_\chi + \frac{1}{4}\sigma^{-3}\bar\psi\gamma^{\rho\sigma\chi}\psi\bar\psi_\rho\gamma_{\sigma\chi}\psi\,. \qquad (81)
\end{aligned}$$

4.3 The Total Gauged Supergravity Action

We shall review a map between the Yang-Mills supermultiplet and a set of fields in the alternative Poincaré multiplet. In the following, we shall need the full supersymmetry transformation rules only for the fields $(e^a_\mu, \psi_\mu, \mathcal{V}^{ij}_\mu, B_{\mu\nu})$, and the Yang-Mills multiplet fields $(W^I_\mu, \Omega^I, Y^{ijI})$, where I labels the adjoint representation of the Yang-Mills gauge group. We only want to establish a map between the Poincaré multiplet and the Yang-Mills multiplet and propose an R^2-invariant based on the action for the Yang-Mills multiplet. Both actions are invariant under the SU(2) R-symmetry. To prove the validity of this map, we need the full nonlinear supersymmetry transformation rules. After we construct the action we can still impose the gauge $L^{ij} = \frac{1}{\sqrt{2}}L\delta^{ij}$. This will not affect the R^2-invariant. It modifies the supersymmetry transformation rules with SU(2) compensating transformations, which leave the action separately invariant. The full version of the supersymmetry transformations is given by

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, \\
\delta\psi_\mu{}^i &= \partial_\mu\epsilon^i + \frac{1}{4}\widehat\omega_{+\mu}{}^{ab}\gamma_{ab}\epsilon^i + \mathcal{V}_\mu{}^i{}_j\epsilon^j \equiv D_\mu(\widehat\omega_+)\epsilon^i + \mathcal{V}'_\mu{}^i{}_j\epsilon^j\,, \\
\delta\mathcal{V}_\mu{}^{ij} &= -\frac{1}{2}\bar\epsilon^{(i}\gamma^\lambda\widehat R_{\lambda\mu}{}^{j)}(Q) + \frac{1}{12}\bar\epsilon^{(i}\gamma\cdot\widehat F(B)\psi_\mu{}^{j)}\,, \\
\delta B_{\mu\nu} &= -\bar\epsilon\gamma_{[\mu}\psi_{\nu]}\,, \qquad (82)
\end{aligned}$$

where the fermionic torsion and the different supercovariant objects are defined as

$$
\begin{aligned}
\widehat{\omega}_{\mu\pm}{}^{ab} &= \widehat{\omega}_\mu{}^{ab} \pm \frac{1}{2}\widehat{F}_\mu{}^{ab}(B)\,, \\
\widehat{\omega}_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\rho[a}e^{b]\sigma}e_\mu{}^c\partial_\rho e_{\sigma c} + K_\mu{}^{ab}\,, \\
K_\mu{}^{ab} &= \frac{1}{4}\big(2\bar{\psi}_\mu\gamma^{[a}\psi^{b]} + \bar{\psi}^a\gamma_\mu\psi^b\big)\,, \\
\widehat{F}_{\mu\nu\rho}(B) &= 3\partial_{[\mu}B_{\nu\rho]} + \frac{3}{2}\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]}\,, \\
\widehat{R}_{\mu\nu}{}^i(Q) &= 2\left(\partial_{[\mu} + \frac{1}{4}\widehat{\omega}_{+[\mu}{}^{ab}\gamma_{ab}\right)\psi_{\nu]}{}^i + 2\mathcal{V}_{[\mu}{}^i{}_j\psi_{\nu]}{}^j\,.
\end{aligned}
\tag{83}
$$

Next, we consider the following transformations

$$
\begin{aligned}
\delta\widehat{\omega}_{-\mu}{}^{ab} &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu\widehat{R}^{ab}(Q)\,, \\
\delta\widehat{R}^{abi}(Q) &= \frac{1}{4}\gamma^{cd}\epsilon^i\widehat{R}_{cd}{}^{ab}(\widehat{\omega}_-) - \widehat{F}^{abij}(\mathcal{V})\epsilon_j\,, \\
\delta\widehat{F}^{abij}(\mathcal{V}) &= -\frac{1}{2}\bar{\epsilon}^{(i}\gamma^\mu\widehat{D}_\mu\widehat{R}^{abj)}(Q) + \frac{1}{48}\bar{\epsilon}^{(i}\gamma\cdot\widehat{F}(B)\widehat{R}^{abj)}(Q)\,,
\end{aligned}
\tag{84}
$$

where $\widehat{F}_{\mu\nu}{}^{ij}(\mathcal{V})$ and $\widehat{R}_{\mu\nu}{}^{ab}(\widehat{\omega}_-)$ are the supercovariant curvatures of $\mathcal{V}_\mu{}^{ij}$ and $\widehat{\omega}_{-\,\mu}{}^{ab}$, respectively:

$$
\begin{aligned}
\widehat{F}_{\mu\nu}{}^{ij}(\mathcal{V}) &= F_{\mu\nu}{}^{ij}(\mathcal{V}) - \bar{\psi}_{[\mu}{}^{(i}\gamma^\rho\widehat{R}_{\nu]\rho}{}^{j)}(Q) - \frac{1}{12}\bar{\psi}_{[\mu}{}^{(i}\gamma\cdot\widehat{F}(B)\psi_{\nu]}{}^{j)}\,, \\
\widehat{R}_{\mu\nu}{}^{ab}(\widehat{\omega}^-) &= R_{\mu\nu}{}^{ab}(\widehat{\omega}^-) + \bar{\psi}_{[\mu}\gamma_{\nu]}\widehat{R}^{ab}(Q)\,, \\
\widehat{D}_\mu\widehat{R}^{abi}(Q) &= \partial_\mu\widehat{R}^{abi}(Q) + \frac{1}{4}\widehat{\omega}_\mu{}^{cd}\gamma_{cd}\widehat{R}^{abi}(Q) + \mathcal{V}_\mu{}^i{}_j\widehat{R}^{abj}(Q) \\
&\quad -\frac{1}{4}\gamma^{cd}\psi_\mu{}^i\widehat{R}_{cd}{}^{ab}(\widehat{\omega}_-) + \widehat{F}^{abij}(\mathcal{V})\psi_{\mu j} + 2\widehat{\omega}_{-\mu}{}^{[ac}\widehat{R}_c{}^{b]i}(Q)\,.
\end{aligned}
\tag{85}
$$

We now compare the above transformation rules with the $\mathcal{N}=(1,0)$, $D=6$ vector multiplet

$$
\begin{aligned}
\delta W_\mu{}^I &= -\bar{\epsilon}\gamma_\mu\Omega^I\,, \\
\delta\Omega^{Ii} &= \frac{1}{8}\gamma\cdot\widehat{F}^I(W)\epsilon^i - \frac{1}{2}Y^{Iij}\epsilon_j\,, \\
\delta Y^{Iij} &= -\bar{\epsilon}^{(i}\gamma^\mu\widehat{D}_\mu\Omega^{j)I} + \frac{1}{24}\bar{\epsilon}^{(i}\gamma\cdot\widehat{F}(B)\Omega^{j)I}\,,
\end{aligned}
\tag{86}
$$

where

$$
\begin{aligned}
\widehat{F}_{\mu\nu}{}^I(W) &= F_{\mu\nu}{}^I(W) + 2\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega^I\,, \\
\widehat{D}_\mu\Omega^{Ii} &= \partial_\mu\Omega^{Ii} + \frac{1}{4}\widehat{\omega}_\mu{}^{ab}\gamma_{ab}\Omega^{Ii} + V_\mu{}^i{}_j\Omega^{Ij} \\
&\quad -\frac{1}{8}\gamma\cdot\widehat{F}^I(W)\psi_\mu{}^i + \frac{1}{2}Y^{I\,ij}\psi_{\mu j} - f_{KL}{}^I W_\mu{}^K\Omega^{Li}\,.
\end{aligned}
\tag{87}
$$

We observe that the transformation rules (84) and (86) become identical by making the following identifications:

$$
\Big(-2\widehat{\omega}_{-\mu}{}^{ab}, -\widehat{R}^{abi}(Q), -2\widehat{F}^{abij}(\mathcal{V})\Big) \longrightarrow \Big(W_\mu{}^I, \Omega^{Ii}, Y^{Iij}\Big)\,. \tag{88}
$$

Using this observation we can now easily write down a supersymmetric R^2-action using the superconformal invariant exact action formula for the Yang-Mills multiplet. In the gauge (33) and up to quartic fermions, the Lagrangian becomes

$$e^{-1}\mathcal{L}_{\rm YM}\big|_{\sigma=1} \;=\; -\frac{1}{4}F_{\mu\nu}{}^{I}(W)F^{\mu\nu I}(W) - 2\bar{\Omega}^{I}\gamma^{\mu}D'_{\mu}(\omega)\Omega^{I} + Y^{Iij}Y^{I}_{ij} + \frac{1}{12}F_{\mu\nu\rho}(B)\bar{\Omega}^{I}\gamma^{\mu\nu\rho}\Omega^{I}$$
$$-\frac{1}{16}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}F^{I}_{\rho\sigma}(W)F^{I}_{\lambda\tau}(W) + \frac{1}{2}F_{\nu\rho}{}^{I}\bar{\Omega}^{I}\gamma^{\mu}\gamma^{\nu\rho}\psi_{\mu}\,. \tag{89}$$

Using the map (88) in this formula produces the result for the supersymmetrized Riemann tensor squared action. In presenting the results up to quartic fermion terms, it is useful to note the following simplification in the torsionful spin connection

$$\widehat{\omega}_{\mu-}{}^{ab} = \omega_{\mu+}{}^{ab} + \frac{1}{2}\bar{\psi}^{a}\gamma_{\mu}\psi^{b}\ , \qquad \omega_{\mu\pm}{}^{ab} \equiv \omega_{\mu}{}^{ab} \pm \frac{1}{2}F_{\mu}{}^{ab}(B)\ , \tag{90}$$

where $\omega_{\mu}{}^{ab}$ is the standard torsion-free connection. The map (88) applied to the action formula (89) then yields the fermionic term (supersymmetric R^2-action) in the total gauged supergravity action. The total off-shell action of minimal Poincaré supergravity is

$$\begin{aligned}
S_T = \int d^6x \Big[& \frac{1}{2}LR + \frac{1}{\sqrt{2}}gL\delta^{ij}Y_{ij} + Y^{ij}Y_{ij} + \frac{1}{2}L^{-1}\partial_{\mu}L\partial^{\mu}L - \frac{1}{24}LF_{\mu\nu\rho}(B)F^{\mu\nu\rho}(B) \\
& + 2LZ_{\mu}Z^{*\mu} - \frac{1}{4}L^{-1}E_{\mu}E^{\mu} + \frac{1}{\sqrt{2}}E^{\mu}\big(\mathcal{V}_{\mu} + \frac{1}{\sqrt{2}}gW_{\mu}\big) - \frac{1}{4}F_{\mu\nu}(W)F^{\mu\nu}(W) \\
& - \frac{1}{16}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}F_{\rho\sigma}(W)F_{\lambda\tau}(W) - \frac{1}{8M^2}\Big[R_{\mu\nu}{}^{ab}(\omega_-)R^{\mu\nu}{}_{ab}(\omega_-) - 2F^{\mu\nu}(\mathcal{V})F_{\mu\nu}(\mathcal{V}) \\
& - 8F^{\mu\nu}(Z)F^{*}_{\mu\nu}(Z) + \frac{1}{4}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}R_{\rho\sigma}{}^{ab}(\omega_-)R_{\lambda\tau ab}(\omega_-)\Big] + R_{\mu\nu}{}^{ab}(\omega_-)R^{\mu\nu}{}_{ab}(\omega_-) \\
& - 2F^{ab}(\mathcal{V})F_{ab}(\mathcal{V}) - 4F'^{abij}(\mathcal{V})F'_{abij}(\mathcal{V}) + \frac{1}{4}e^{-1}\varepsilon^{\mu\nu\rho\sigma\lambda\tau}B_{\mu\nu}R_{\rho\sigma}{}^{ab}(\omega_-)R_{\lambda\tau ab}(\omega_-) \\
& + 2\bar{R}_{+ab}(Q)\gamma^{\mu}D_{\mu}(\omega,\omega_-)R^{ab}_{+}(Q) - R_{\nu\rho}{}^{ab}(\omega_-)\bar{R}_{+ab}(Q)\gamma^{\mu}\gamma^{\nu\rho}\psi_{\mu} \\
& - 8F'_{\mu\nu}{}^{ij}(\mathcal{V})\left(\bar{\psi}^{\mu}_{i}\gamma_{\lambda}R^{\lambda\nu}_{+j}(Q) + \frac{1}{6}\bar{\psi}^{\mu}_{i}\gamma\cdot F(B)\psi^{\nu}_{j}\right) - \frac{1}{12}\bar{R}^{ab}_{+}(Q)\gamma\cdot F(B)R_{+ab}(Q) \\
& - \frac{1}{2}\Big[D_{\mu}(\omega_-,\Gamma_+)R^{\mu\rho ab}(\omega_-) - 2F_{\mu\nu}{}^{\rho}(B)R^{\mu\nu ab}(\omega_-)\Big]\bar{\psi}_{a}\gamma_{\rho}\psi_{b}\Big]\,,
\end{aligned} \tag{91}$$

where we have defined the complex vector fields and field strengths

$$Z_{\mu} \equiv \mathcal{V}'^{11}_{\mu} + \mathrm{i}\mathcal{V}'^{12}_{\mu}\ , \qquad Z^{*}_{\mu} = \mathcal{V}'_{\mu 11} - \mathrm{i}\mathcal{V}'_{\mu 12} = -\mathcal{V}'^{11}_{\mu} + \mathrm{i}\mathcal{V}'^{12}_{\mu}\ , \tag{92}$$
$$F_{\mu\nu}(\mathcal{V}) = 2\partial_{[\mu}\mathcal{V}_{\nu]} - 4\mathrm{i}Z_{[\mu}Z^{*}_{\nu]}\ , \qquad F_{\mu\nu}(Z) = 2\partial_{[\mu}Z_{\nu]} - 2\mathrm{i}\mathcal{V}_{[\mu}Z_{\nu]}\,. \tag{93}$$

and

$$\begin{aligned}
D_{\mu}(\omega,\omega_-)R^{abi}_{+}(Q) \;&=\; (\partial_{\mu} + \frac{1}{4}\omega_{\mu}{}^{cd}\gamma_{cd})R^{abi}_{+}(Q) - 2\omega_{\mu-}{}^{c[a}R_{+c}{}^{b]i}(Q) + \mathcal{V}_{\mu}{}^{i}{}_{j}R^{ab\,j}_{+}(Q)\ , \\
R_{+\mu\nu}{}^{i}(Q) \;&=\; 2D_{[\mu}(\omega_+)\psi^{i}_{\nu]} - 2\mathcal{V}'_{[\mu}{}^{ij}\psi_{\nu]j}\ ,
\end{aligned} \tag{94}$$

The torsionful modification of the Christoffel symbol $\Gamma^{\rho}_{\mu\nu\pm}$ is defined as

$$\Gamma^{\rho}_{\mu\nu\pm} \equiv \Gamma^{\rho}_{\mu\nu} \pm \frac{1}{2}F_{\mu\nu}{}^{\rho}(B)\,. \tag{95}$$

This completes the construction of the supersymmetric R^2-invariant.

We constructed the full off-shell action of minimal Poincaré supergravity in six dimensions. We obtained this action by using the methods of superconformal tensor calculus. We constructed a superconformal action by coupling a linear compensator multiplet (which is an off-shell multiplet) to the Weyl 1 multiplet in a density formula. By fixing the redundant symmetries (D, K and S) we obtained the Poincaré action. Our main goal has been the study of the R-symmetry gauging in the presence of higher derivative corrections to Poincaré supergravity. To this end, we first studied the gauging of the U(1) R-symmetry of $\mathcal{N} = (1, 0)$, $D = 6$ supergravity in the off-shell formulation. The off-shell Poincaré supergravity theory already has a local $\Upsilon(1)_R$ symmetry but it is gauged by an auxiliary vector field which is not dynamical. Expressing the action in terms of fields of the Weyl 2 multiplet led to an action that contains kinetic terms for the matter fields and has consistent field equations.

5 $N = 2$, $D = 6$ Supergravity

The $D = 6$, $N = 2$ supergravity coupled to matter was first constructed by Nishino and Sezgin and further generalized to include new couplings and more than one tensor multiplet. In the general case, the standard construction method is to start with a general ansatz for the supersymmetry variations and equations of motion involving a set of undetermined coefficients and then fix all coefficients through the requirement of closure of the supersymmetry algebra on the fields. This approach, although rather technical, has the merit that it is valid for arbitrary n_T where a Lagrangian formulation of the theory is not possible due to the self-duality conditions; in the $n_T = 1$ case, the invariant Lagrangian can be derived by integrating the equations of motion. On the other hand, since we are mostly interested in the case $n_T = 1$, it is more natural to follow the somewhat simpler Noether procedure to derive the full supergravity Lagrangian, up to (Fermi)^4 terms.

Our starting point is the the locally supersymmetric Lagrangian describing the gravity and tensor multiplets, the set of fields $(g_{\mu\nu}, B_{\mu\nu}, \psi_\mu, \phi, \chi)$. This can be derived quite easily by standard methods and we do not find it necessary to repeat the discussion here. The result is the Lagrangian

$$\begin{aligned} e^{-1}\mathcal{L}_{GT} &= \frac{1}{4}R - \frac{1}{12}e^{2\phi}G_{\mu\nu\rho}G^{\mu\nu\rho} - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{2}\bar{\chi}\Gamma^\mu D_\mu\chi \\ &- \frac{1}{24}e^\phi\left(\bar{\psi}^\lambda\Gamma_{[\lambda}\Gamma^{\mu\nu\rho}\Gamma_{\sigma]}\bar{\psi}^\sigma - 2\bar{\chi}\Gamma^\lambda\Gamma^{\mu\nu\rho}\psi_\lambda - \bar{\chi}\Gamma^{\mu\nu\rho}\chi\right)G_{\mu\nu\rho} \\ &- \frac{1}{2}\bar{\chi}\Gamma^\nu\Gamma^\mu\psi_\nu\partial_\mu\phi + (\text{Fermi})^4, \end{aligned} \tag{96}$$

which is invariant under the set of supersymmetry transformations

$$\begin{aligned} \delta e_\mu^a &= \bar{\epsilon}\Gamma^a\psi_\mu, \\ \delta B_{\mu\nu} &= e^{-\phi}\left(-\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar{\epsilon}\Gamma_{\mu\nu}\chi\right) \\ \delta\phi &= \bar{\epsilon}\chi, \\ \delta\psi_\mu &= D_\mu\epsilon + \frac{1}{24}e^\phi\Gamma^{\nu\rho\sigma}\Gamma_\mu G_{\nu\rho\sigma}\epsilon, \\ \delta\chi &= \frac{1}{2}\Gamma^\mu\partial_\mu\phi\epsilon - \frac{1}{12}e^\phi\Gamma^{\mu\nu\rho}G_{\mu\nu\rho}\epsilon. \end{aligned} \tag{97}$$

The next step in our construction is to couple the theory to vector multiplets in a way consistent with local supersymmetry. For definiteness, we take the gauge group $\mathcal{G}$ to be the full holonomy group of the scalar manifold, $\mathcal{G} = H \times \mathrm{USp}(2)$, and we let $\hat{I}$ be the adjoint index for $\mathcal{G}$ which is decomposed into the adjoint indices I and i of H and USp(2) respectively. The restriction to a subgroup of $H \times \mathrm{USp}(2)$ as well as the addition of abelian factors is a trivial matter. Our starting point is the well-known globally supersymmetric Lagrangian for vector multiplets,

$$e^{-1}\mathcal{L}_V^{(0)} = -\frac{1}{4}e^{-\phi}v_X(F^{\hat{I}}_{\mu\nu}F^{\mu\nu}_{\hat{I}})_X - \frac{1}{2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^\mu D_\mu\lambda_{\hat{I}})_X, \tag{98}$$

where the summation index X labels the simple factors of the gauge group and v_X are some numerical constants that will be determined by anomaly considerations. The Lagrangian (??) is invariant under the rigid supersymmetry transformations

$$\delta A^{\hat{I}}_\mu = \frac{1}{\sqrt{2}}e^{\phi/2}\bar{\epsilon}\Gamma_\mu\lambda^{\hat{I}}, \qquad \delta\lambda^{\hat{I}} = -\frac{1}{2\sqrt{2}}e^{-\phi/2}\Gamma^{\mu\nu}\epsilon F^{\hat{I}}_{\mu\nu}. \tag{99}$$

Our first step towards obtaining a locally supersymmetric theory is to introduce the usual Noether coupling of the gravitino to the supercurrent of the multiplet. The required term is

$$\mathcal{L}_V^{(1)} = -\frac{1}{2\sqrt{2}}ee^{-\phi/2}v_X(\bar{\psi}_\mu\Gamma^{\nu\rho}\Gamma^\mu\lambda^{\hat{I}}F_{\hat{I}\nu\rho})_X. \tag{100}$$

Next, we must cancel the $\bar{\lambda}\Gamma F\partial\phi\epsilon$ variation of $\mathcal{L}_V^{(0)}$. This variation is found to be

$$\Delta_V^{(1)} = \frac{1}{4\sqrt{2}}ee^{-\phi/2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu}\Gamma^\rho\epsilon_i F_{\hat{I}\mu\nu}\partial_\rho\phi)_X, \tag{101}$$

and can be cancelled by the $\delta\chi \sim \Gamma\partial\phi\epsilon$ variation of the additional term

$$\mathcal{L}_V^{(2)} = -\frac{1}{2\sqrt{2}}ee^{-\phi/2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu}\chi F_{\hat{I}\mu\nu})_X. \tag{102}$$

The introduction of these new interactions results in additional uncancelled terms of the form $\bar{\lambda}\Gamma FG\epsilon$ coming from the $\delta\psi$ and $\delta\chi$ variations of $\mathcal{L}_V^{(1)}$ and $\mathcal{L}_V^{(2)}$ respectively. The first one vanishes by the 6D identity $\Gamma^\mu\Gamma^{\nu\rho\sigma}\Gamma_\mu = 0$, while the second one is given by

$$\Delta_V^{(2)} = \frac{1}{24\sqrt{2}}ee^{\phi/2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu}\Gamma^{\rho\sigma\tau}\epsilon G_{\rho\sigma\tau}F_{\hat{I}\mu\nu})_X. \tag{103}$$

This can be cancelled by introducing the additional interaction

$$\mathcal{L}_V^{(3)} = \frac{1}{24}ee^{\phi}v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu\rho}\lambda_{\hat{I}}G_{\mu\nu\rho})_X. \tag{104}$$

What remains is to cancel the $\bar{\psi}\Gamma F^2\epsilon$ and $\bar{\chi}\Gamma F^2\epsilon$ terms coming from the $\delta\lambda$ variations of $\mathcal{L}_V^{(1)}$ and $\mathcal{L}_V^{(2)}$. These terms are given by

$$\Delta_V^{(3)} = \frac{1}{8}ee^{-\phi}v_X(\bar{\psi}_\mu\Gamma^{\mu\nu\rho\sigma\tau}\epsilon F^{\hat{I}}_{\nu\rho}F_{\hat{I}\sigma\tau})_X, \tag{105}$$

and

$$\Delta_V^{(4)} = \frac{1}{8}ee^{-\phi}v_X(\bar{\chi}\Gamma^{\mu\nu\rho\sigma}\epsilon F^{\hat{I}}_{\mu\nu}F_{\hat{I}\rho\sigma})_X. \tag{106}$$

To cancel these terms, we use standard spinor identities plus the gamma-matrix duality relation to write their sum in the form

$$\Delta_V^{(3)} + \Delta_V^{(4)} = -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma\tau\upsilon}\left[e^{-\phi}\left(-\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar{\epsilon}\Gamma_{\mu\nu}\chi\right)\right]v_X(F^{\hat{I}}_{\rho\sigma}F_{\hat{I}\tau\upsilon})_X. \tag{107}$$

Noting that the term inside brackets is exactly the supersymmetry variation of $B_{\mu\nu}$ in (97), we see that one can cancel this variation by introducing the interaction

$$\mathcal{L}_V^{(4)} = \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\tau\upsilon} B_{\mu\nu} v_X (F_{\rho\sigma}^{\hat{I}} F_{\hat{I}\tau\upsilon})_X. \tag{108}$$

This interaction, immediately recognized as a Green-Schwarz term, plays an important role in the context of anomaly cancellation. Here, we note that, in contrast to the 10D case, the 6D Green-Schwarz term is not a higher-derivative correction to the action of the theory but it is present in the low-energy action in the first place. Our final step in constructing the $D = 6$, $N = 2$ supergravity action is to couple the theory to hypermultiplets. In what follows, we shall derive the action describing the hypermultiplets and their couplings to the other multiplets of the theory.

Let us first examine the parameterization of the scalar manifold. According to the guidelines of S, we consider a coset representative given by a matrix L whose Maurer-Cartan form decomposes as

$$L^{-1}\partial_\alpha L = \mathcal{A}_\alpha{}^I T_I + \mathcal{A}_\alpha{}^i T_i + \mathcal{V}_\alpha{}^{aA} T_{aA}, \tag{109}$$

T_I, T_i and T_{aA} are the generators of H, USp(2) and the coset, $\mathcal{A}_\alpha{}^I$ and $\mathcal{A}_\alpha{}^i$ are the USp($2n_H$) and USp(2) connections and $\mathcal{V}_\alpha{}^{aA}$ is the coset vielbein. The pullback of this Maurer-Cartan form is given by

$$L^{-1}\partial_\mu L = \mathcal{Q}_\alpha{}^I T_I + \mathcal{Q}_\mu{}^i T_i + \mathcal{P}_\alpha{}^{aA} T_{aA}, \tag{110}$$

with

$$\mathcal{Q}_\mu{}^I = \partial_\mu\varphi^\alpha\, \mathcal{A}_\alpha{}^I, \qquad \mathcal{Q}_\mu{}^i = \partial_\mu\varphi^\alpha\, \mathcal{A}_\alpha{}^i, \qquad \mathcal{P}_\mu{}^{aA} = \partial_\mu\varphi^\alpha\, \mathcal{V}_\alpha{}^{aA}\,. \tag{111}$$

The covariant derivatives of fields carrying H and USp(2) indices must be modified by couplings to the composite connections $\mathcal{Q}_\mu$. Explicitly, for the supersymmetry spinor ϵ (one USp(2) index), the hyperino ψ (one H index) and the gauginos $\lambda^{\hat{I}}$ (one $H\times$USp(2) adjoint index and one USp(2) fundamental index) we have

$$\begin{aligned} \mathcal{D}_\mu\, \epsilon &= D_\mu\epsilon + \partial_\mu\varphi^\alpha\, \mathcal{A}_\alpha{}^i T_i\epsilon, \\ \mathcal{D}_\mu\, \psi &= D_\mu\psi + \partial_\mu\varphi^\alpha\, \mathcal{A}_\alpha{}^I T_I\psi, \\ \mathcal{D}_\mu\, \lambda^{\hat{I}} &= D_\mu\lambda^{\hat{I}} + \partial_\mu\varphi^\alpha\, \mathcal{A}_\alpha{}^i T_i\lambda^{\hat{I}}. \end{aligned} \tag{112}$$

The reason for the absence of the composite H–connection in the covariant derivative of the gauginos $\lambda^{\hat{I}}$ is a technical one and originates from the requirement of supersymmetry of the action. Given these covariant derivatives, we can define the H and USp(2) curvatures, $\mathcal{F}_{\alpha\beta}{}^I$ and $\mathcal{F}_{\alpha\beta}{}^i$ through the commutators

$$\begin{aligned} [\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon &= \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon + \partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta\, \mathcal{F}_{\alpha\beta}{}^i T_i\epsilon, \\ [\mathcal{D}_\mu, \mathcal{D}_\nu]\psi &= \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\psi + \partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta\, \mathcal{F}_{\alpha\beta}{}^I T_I\psi. \end{aligned} \tag{113}$$

Another useful geometrical quantity is the triplet of complex structures

$$J_{\alpha\beta}{}^i = (\mathcal{V}_{\alpha a}{}^A\, \mathcal{V}_\beta{}^{aB} - \mathcal{V}_{\alpha a}{}^B\, \mathcal{V}_\beta{}^{aA})(T^i)_{AB}, \tag{114}$$

which satisfy the USp(2) algebra. This will be shown to be proportional to the curvature $\mathcal{F}_{\alpha\beta}{}^i$, as a result of supersymmetry.

The supersymmetric theory of the hypermultiplets is described by the sigma-model Lagrangian

$$e^{-1}\mathcal{L}_H^{(0)} = -\frac{1}{2}g_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta - \frac{1}{2}\bar{\psi}^a\Gamma^\mu\, \mathcal{D}_\mu\, \psi_a, \tag{115}$$

which is invariant under the transformations

$$\delta\varphi^\alpha = \mathcal{V}^\alpha{}_{aA}\,\bar{\psi}^a\epsilon^A, \qquad\qquad \delta\psi^a = \Gamma^\mu\,\mathcal{P}_\mu{}^{aA}\,\epsilon_A. \tag{116}$$

Let us now construct the supersymmetric theory. In the standard way, we introduce the appropriate interaction of the gravitino with the hypermultiplet supercurrent,

$$\mathcal{L}_H^{(1)} = e\bar{\psi}_\mu^A\Gamma^\nu\Gamma^\mu\psi^a\,\mathcal{P}_{\nu aA}\,. \tag{117}$$

whose variation cancels the $\bar{\psi}_a\Gamma\,\mathcal{V}\,\partial\phi\,\mathcal{D}\,\epsilon$ variation of $\mathcal{L}_H^{(0)}$ in the usual manner. The variation of $\mathcal{L}_H^{(1)}$ under $\delta\psi_\mu \sim \Gamma G\epsilon$ yields the uncancelled term

$$\Delta_H^{(1)} = -\frac{1}{12}ee^\phi\bar{\epsilon}^A\Gamma^\mu\Gamma^{\nu\rho\sigma}\psi^a\,\mathcal{P}_{\mu aA}\,G_{\nu\rho\sigma}, \tag{118}$$

which cancels if we introduce the interaction

$$\mathcal{L}_H^{(2)} = -\frac{1}{24}ee^\phi\bar{\psi}^a\Gamma^{\mu\nu\rho}\psi_a G_{\mu\nu\rho}. \tag{119}$$

Another uncancelled term arises from the $\delta\psi_a$ variation of $\mathcal{L}_H^{(1)}$. It is given by

$$\Delta_H^{(2)} = -e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\partial_\nu\varphi^\alpha\partial_\rho\varphi^\beta J_{\alpha\beta}{}^i T_i\epsilon. \tag{120}$$

To cancel this term, we note that the modified commutator (113) implies that the gravitino kinetic term in $\mathcal{L}_{GT}$ acquires the extra term

$$\Delta_H^{(3)} = \frac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\partial_\nu\varphi^\alpha\partial_\rho\varphi^\beta\,\mathcal{F}_{\alpha\beta}{}^i\,T_i\epsilon, \tag{121}$$

This is exactly equal and opposite to $\Delta_H^{(2)}$, provided that the USp(2) curvature $\mathcal{F}_{\alpha\beta}{}^i$ is related to the complex structure $J_{\alpha\beta}{}^i$ according to

$$\mathcal{F}_{\alpha\beta}{}^i = 2J_{\alpha\beta}{}^i. \tag{122}$$

This tells us that now the holonomy of $\mathcal{M}_{\mathcal{N}=1}$ must be contained in USp($2n_H$) $\times$ USp(2) with nonzero USp(2) curvature, that is, local supersymmetry requires $\mathcal{M}_{\mathcal{N}=1}$ to be a quaternionic manifold. The complete locally supersymmetric Lagrangian for neutral hypermultiplets is given by

$$e^{-1}\mathcal{L}_{\mathcal{H}} = -\frac{1}{2}g_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta - \frac{1}{2}\bar{\psi}^a\Gamma^\mu\,\mathcal{D}_\mu\,\psi_a + \bar{\psi}_\mu^A\Gamma^\nu\Gamma^\mu\,\mathcal{P}_{\nu aA}\,\psi^a - \frac{1}{24}e^\phi\bar{\psi}^a\Gamma^{\mu\nu\rho}\psi_a G_{\mu\nu\rho}. \tag{123}$$

To gauge the theory, we consider an USp($2n_H$) $\times$ USp(2) isometry transformation on the scalar manifold. Under this transformation, the hyperscalars transform as

$$\delta\varphi^\alpha = \Lambda^{\hat{I}}\xi_{\hat{I}}^\alpha, \tag{124}$$

where $\xi_{\hat{I}}^\alpha$ are USp($2n_H$) $\times$ USp(2) Killing vectors. A convenient basis for these vectors is

$$\xi_{\hat{I}}^\alpha = (T_{\hat{I}}\varphi)^\alpha. \tag{125}$$

To promote this isometry to a local symmetry, we introduce the USp($2n_H$) $\times$ USp(2) gauge fields $A_\mu^{\hat{I}} = (A_\mu^I, A_\mu^i)$, transforming as

$$\delta A_\mu^{\hat{I}} = D_\mu\Lambda^{\hat{I}}, \tag{126}$$

whose field strengths, defined in the usual way, are denoted by $F^{\hat I}_{\mu\nu} = (F^I_{\mu\nu}, F^i_{\mu\nu})$. We then replace the ordinary derivative acting on the hyperscalars by the covariant derivative

$$\mathcal{D}_\mu\,\varphi^\alpha = \partial_\mu\varphi^\alpha - A^{\hat I}_\mu \tilde\xi^\alpha_{\hat I}, \tag{127}$$

where $\tilde\xi^\alpha_{\hat I} = (g\xi^\alpha_I, g'\xi^\alpha_i)$ with g and g' being the USp$(2n_H)$ and USp(2) gauge couplings. Appropriately, the Maurer-Cartan form (110) is replaced by the gauged version

$$L^{-1}\,\mathcal{D}_\mu\,L = \mathcal{Q}_\mu{}^I\,T_I + \mathcal{Q}_\mu{}^i\,T_i + \mathcal{P}_\mu{}^{aA}\,T_{aA}, \tag{128}$$

with

$$\mathcal{Q}_\mu{}^I = \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^I, \qquad \mathcal{Q}_\mu{}^i = \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^i, \qquad \mathcal{P}_\mu{}^{aA} = \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{V}_\alpha{}^{aA}, \tag{129}$$

and the covariant derivatives (112) are appropriately modified by using the composite connections of the gauged theory and adding gauge couplings. Explicitly, we have

$$\begin{aligned}
\mathcal{D}_\mu\,\epsilon &= D_\mu\epsilon + \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^i\,T_i\epsilon + g'A^i_\mu T_i\epsilon,\\
\mathcal{D}_\mu\,\psi &= D_\mu\psi + \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^I\,T_I\psi + gA^I_\mu T_I\psi,\\
\mathcal{D}_\mu\,\lambda^I &= D_\mu\lambda^I + \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^i\,T_i\lambda^I + g'A^i_\mu T_i\lambda^I + gf^I{}_{JK}A^J_\mu\lambda^K,\\
\mathcal{D}_\mu\,\lambda^i &= D_\mu\lambda^i + \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{A}_\alpha{}^j\,T_j\lambda^i + g'A^j_\mu T_j\lambda^i + g'\epsilon^i{}_{jk}A^j_\mu\lambda^k.
\end{aligned} \tag{130}$$

where $f^I{}_{JK}$ are the USp$(2n_H)$ structure constants. Another important building block of the gauge theory is the "prepotential" $C^{i\hat I}$, given by

$$C^{iI} = g\,\mathcal{A}_\alpha{}^i\,\xi^{\alpha I}, \qquad C^{ij} = g'(\mathcal{A}_\alpha{}^i\,\xi^{\alpha j} - \delta^{ij}), \tag{131}$$

and satisfying the identity

$$\mathcal{D}_\mu\,C^{i\hat I} = \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{D}_\alpha\,C^{i\hat I} = 2(\mathcal{D}_\mu\,\varphi^\alpha)J_{\alpha\beta}{}^i\tilde\xi^{\beta\hat I}. \tag{132}$$

This quantity appears in the generalization of the second commutator in (113) for the gauged theory, namely

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon + \mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{D}_\nu\,\varphi^\beta\,\mathcal{F}_{\alpha\beta}{}^i\,T_i\epsilon - F_{\hat I\mu\nu}C^{i\hat I}T_i\epsilon. \tag{133}$$

Having introduced the necessary formalism, we are ready to derive the Lagrangian for the gauged theory. Our first step is to replace all derivatives in the Lagrangians (96), (154) and (115) and transformation rules (97) and (116) by gauge-covariant ones. Regarding the hypermultiplets, we obtain the Lagrangian

$$\begin{aligned}
e^{-1}\mathcal{L}_H &= -\frac{1}{2}g_{\alpha\beta}(\varphi)\,\mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{D}^\mu\,\varphi^\beta - \frac{1}{2}\bar\psi^a\Gamma^\mu\,\mathcal{D}_\mu\,\psi_a + \bar\psi^A_\mu\Gamma^\nu\Gamma^\mu\,\mathcal{P}_{\nu aA}\,\psi^a\\
&\quad -\frac{1}{24}e^\phi\bar\psi^a\Gamma^{\mu\nu\rho}\psi_a G_{\mu\nu\rho},
\end{aligned} \tag{134}$$

and the supersymmetry transformations

$$\delta\varphi^\alpha = \mathcal{V}^\alpha{}_{aA}\,\bar\psi^a\epsilon^A, \qquad\qquad \delta\psi^a = \Gamma^\mu\,\mathcal{P}_\mu{}^{aA}\,\epsilon_A. \tag{135}$$

The combination $\mathcal{L}_{GT} + \mathcal{L}_V + \mathcal{L}_H$ obtained this way is not locally supersymmetric. The reason is that $\mathcal{D}_\mu\,\epsilon$ and $\mathcal{D}_\mu\,\varphi^\alpha$ now include extra contributions involving the USp(2) gauge fields; these do not affect the supersymmetry variations of the various interaction terms, but they do modify the variations of the gravitino and hyperino kinetic terms. For the gravitino kinetic term, the modified commutator (133) implies that its supersymmetry variation acquires the extra terms

$$\Delta_G^{(1)} = \frac{1}{2} e\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu \varphi^\alpha \mathcal{D}_\rho \varphi^\beta \mathcal{F}_{\alpha\beta}{}^i T_i \epsilon, \tag{136}$$

and

$$\Delta_G^{(2)} = \frac{1}{2} e C^{i\hat{I}} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} F_{\hat{I}\nu\rho} T_i \epsilon. \tag{137}$$

By exactly the same considerations, it is easy to see that the hyperino kinetic term also gives rise to the extra variation

$$\Delta_G^{(3)} = -\frac{1}{2} e\bar{\psi}_a \Gamma^{\mu\nu} F^{\hat{I}}_{\mu\nu} \mathcal{V}_\alpha{}^{aA} \tilde{\xi}^\alpha_{\hat{I}} \epsilon. \tag{138}$$

We immediately see that $\Delta_G^{(1)}$ exactly cancels the covariant version of the variation $\Delta_H^{(2)}$. On the other hand, the variations $\Delta_G^{(2)}$ and $\Delta_G^{(3)}$ require additional terms for their cancellation. Starting from $\Delta_G^{(2)}$, the possible sources for its cancellation may be the modification of the gaugino supersymmetry transformation law by the extra term

$$\delta' \lambda^{\hat{I}} = a e^{\phi/2} v_X^{-1} (C^{i\hat{I}})_X T_i \epsilon, \tag{139}$$

or an additional interaction of the form

$$\mathcal{L}_G^{(1)} = b e e^{\phi/2} \bar{\psi}_\mu \Gamma^\mu T_i \lambda_{\hat{I}} C^{i\hat{I}}. \tag{140}$$

Here, a and b are two coefficients, which can be determined by considering the $\bar{\psi}\Gamma F \epsilon$ terms. The variations of this type arising from the $\delta'\lambda$ variation of $\mathcal{L}_V$ is

$$\Delta_G^{(4)} = \frac{a}{2\sqrt{2}} e \left(\bar{\psi}_\mu \Gamma^{\mu\nu\rho} T_i \epsilon F_{\hat{I}\nu\rho} C^{i\hat{I}} - 2\bar{\psi}^\mu \Gamma^\nu T_i \epsilon F_{\hat{I}\mu\nu} C^{i\hat{I}} \right), \tag{141}$$

while the $\delta\lambda$ variation of $\mathcal{L}_G^{(1)}$ gives

$$\Delta_G^{(5)} = -\frac{b}{2\sqrt{2}} e \left(\bar{\psi}_\mu \Gamma^{\mu\nu\rho} T_i \epsilon F_{\hat{I}\nu\rho} C^{i\hat{I}} + 2\bar{\psi}^\mu \Gamma^\nu T_i \epsilon F_{\hat{I}\mu\nu} C^{i\hat{I}} \right). \tag{142}$$

We observe that the requirement for cancellation of the terms proportional to $\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\epsilon$ and $\bar{\psi}^\mu \Gamma^\nu \epsilon$ fixes the coefficients. Next, let us consider the $\delta'\lambda$ variation of the gaugino kinetic term in $\mathcal{L}_V$. Doing an integration by parts and using (132), we obtain

$$\Delta_G^{(6)} = \frac{1}{2\sqrt{2}} e e^{\phi/2} \left(2\bar{\lambda}_{\hat{I}} \Gamma^\mu T_i \mathcal{D}_\mu \epsilon C^{i\hat{I}} + 2\bar{\lambda}^{\hat{I}} \Gamma^\mu \mathcal{F}_{\alpha\beta}{}^i \mathcal{D}_\mu \varphi^\alpha \xi^\beta_{\hat{I}} T_i \epsilon + \bar{\lambda}_{\hat{I}} \Gamma^\mu T_i \epsilon \partial_\mu \phi C^{i\hat{I}} \right). \tag{143}$$

On the other hand, using $\Gamma^\mu \Gamma^{\nu\rho\sigma} \Gamma_\mu = 0$, we find that the $\delta\psi_\mu$ variation of $\mathcal{L}_G^{(1)}$ is given by

$$\Delta_G^{(7)} = -\frac{1}{\sqrt{2}} e e^{\phi/2} \bar{\lambda}_{\hat{I}} \Gamma^\mu T_i \mathcal{D}_\mu \epsilon C^{i\hat{I}}, \tag{144}$$

and it cancels the first term of $\Delta_G^{(6)}$. The second term can cancel by the $\delta\psi^a$ variation of the new term

$$\mathcal{L}_G^{(2)} = -\sqrt{2} e e^{\phi/2} \bar{\lambda}^{\hat{I}}_A \psi_a \mathcal{V}_\alpha{}^{aA} \tilde{\xi}^\alpha_{\hat{I}}, \tag{145}$$

whose $\delta\lambda$ variation cancels $\Delta_G^{(3)}$. Two other uncancelled terms are the $\delta'\lambda$ variations of the $\bar{\lambda}\Gamma\chi F$ and $\bar{\lambda}\Gamma\lambda G$ terms of $\mathcal{L}_V$, given by

$$\Delta_G^{(8)} = \frac{1}{4} e \bar{\chi} \Gamma^{\mu\nu} T_i \epsilon F_{\hat{I}\mu\nu} C^{i\hat{I}}, \tag{146}$$

and

$$\Delta_G^{(9)} = -\frac{1}{12\sqrt{2}} e e^{3\phi/2} \bar{\lambda}_{\hat{I}} \Gamma^{\mu\nu\rho} T_i \epsilon G_{\mu\nu\rho} C^{i\hat{I}}, \tag{147}$$

respectively. $\Delta_G^{(8)}$ is cancelled by the $\delta\lambda$ variation of yet another new term

$$\mathcal{L}_G^{(3)} = \frac{1}{\sqrt{2}} e e^{\phi/2} \bar{\chi} T_i \lambda_{\hat{I}} C^{i\hat{I}}, \tag{148}$$

whose $\delta\chi$ variation is given by

$$\Delta_G^{(10)} = -\frac{1}{2\sqrt{2}} e e^{\phi/2} \bar{\lambda}_{\hat{I}} \Gamma^{\mu} T_i \epsilon \partial_\mu \phi C^{i\hat{I}} + \frac{1}{12\sqrt{2}} e e^{3\phi/2} \bar{\lambda}_{\hat{I}} \Gamma^{\mu\nu\rho} T_i \epsilon G_{\mu\nu\rho} C^{i\hat{I}}, \tag{149}$$

so that its first part cancels the third term of $\Delta_G^{(6)}$ and its second part cancels $\Delta_G^{(9)}$. What remains to be cancelled are the $\delta'\lambda$ variations of $\mathcal{L}_G^{(1)}$ and $\mathcal{L}_G^{(3)}$, given by

$$\Delta_G^{(11)} = \frac{1}{8} e e^{\phi} v_X^{-1} \bar{\psi}_\mu \Gamma^\mu \epsilon (C^{i\hat{I}} C_{i\hat{I}})_X, \qquad \Delta_G^{(12)} = \frac{1}{8} e e^{\phi} v_X^{-1} \bar{\chi} \epsilon (C^{i\hat{I}} C_{i\hat{I}})_X, \tag{150}$$

respectively. To cancel them, we introduce the term

$$\mathcal{L}_G^{(4)} = -\frac{1}{8} e e^{\phi} v_X^{-1} (C^{i\hat{I}} C_{i\hat{I}})_X, \tag{151}$$

whose δe and $\delta\phi$ variation is given by

$$\Delta_G^{(13)} = -\frac{1}{8} e e^{\phi} v_X^{-1} \bar{\psi}_\mu \Gamma^\mu \epsilon (C^{i\hat{I}} C_{i\hat{I}})_X - \frac{1}{8} e e^{\phi} v_X^{-1} \bar{\chi} \epsilon (C^{i\hat{I}} C_{i\hat{I}})_X, \tag{152}$$

and indeed yields the desired cancellation.

To summarize, the terms that should be added to $\mathcal{L}_{GT} + \mathcal{L}_V + \mathcal{L}_{GS} + \mathcal{L}_H$ in order to restore local supersymmetry in case USp(2) is gauged are the following

$$e^{-1}\mathcal{L}_G = \frac{1}{\sqrt{2}} e^{\phi/2} \left(\bar{\psi}_\mu \Gamma^\mu T_i \lambda_{\hat{I}} C^{i\hat{I}} + \bar{\chi} T_i \lambda_{\hat{I}} C^{i\hat{I}} \right) - \sqrt{2} e^{\phi/2} \bar{\lambda}^{\hat{I}}_A \psi_a V_\alpha{}^{aA} \tilde{\xi}^\alpha_{\hat{I}} - \frac{1}{8} e^{\phi} v_X^{-1} (C^{i\hat{I}} C_{i\hat{I}})_X. \tag{153}$$

The important fact emerging from the above rather technical discussion is that the theory includes a scalar potential given by the last term of (153). The appearance of this potential has interesting consequences for the allowed compactifications of the theory.

Combining the above terms, we finally find that the locally supersymmetric Lagrangian describing the vector multiplets and their couplings to the gravity and tensor multiplet is

$$\begin{aligned} e^{-1}\mathcal{L}_V \;\; = \;\; & -\frac{1}{4} e^{-\phi} v_X (F^{\hat{I}}_{\mu\nu} F^{\mu\nu}_{\hat{I}})_X - \frac{1}{2} v_X (\bar{\lambda}^{\hat{I}} \Gamma^\mu D_\mu \lambda_{\hat{I}})_X - \frac{1}{2\sqrt{2}} e e^{-\phi/2} v_X (\bar{\psi}_\mu \Gamma^{\nu\rho} \Gamma^\mu \lambda^{\hat{I}} F_{\hat{I}\nu\rho})_X \\ & -\frac{1}{2\sqrt{2}} e e^{-\phi/2} v_X (\bar{\lambda}^{\hat{I}} \Gamma^{\mu\nu} \chi F_{\hat{I}\mu\nu})_X + \frac{1}{24} e e^{\phi} v_X (\bar{\lambda}^{\hat{I}} \Gamma^{\mu\nu\rho} \lambda_{\hat{I}} G_{\mu\nu\rho})_X \\ & +\frac{1}{8} \epsilon^{\mu\nu\rho\sigma\tau\upsilon} B_{\mu\nu} v_X (F^{\hat{I}}_{\rho\sigma} F_{\hat{I}\tau\upsilon})_X. \end{aligned} \tag{154}$$

We are now in a position to summarize the results. By combining (96), (154), (134) and

(153), we obtain the full Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L} &= \frac{1}{4}R - \frac{1}{12}e^{2\phi}G_{\mu\nu\rho}G^{\mu\nu\rho} - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\,\mathcal{D}_\nu\,\psi_\rho - \frac{1}{2}\bar{\chi}\Gamma^\mu\,\mathcal{D}_\mu\,\chi \\
&\quad -\frac{1}{4}e^{-\phi}v_X(F^{\hat{I}}_{\mu\nu}F^{\mu\nu}_{\hat{I}})_X - \frac{1}{2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^\mu\,\mathcal{D}_\mu\,\lambda_{\hat{I}})_X - \frac{1}{2}g_{\alpha\beta}(\varphi)\,\mathcal{D}_\mu\,\varphi^\alpha\,\mathcal{D}^\mu\,\varphi^\beta - \frac{1}{2}\bar{\psi}^a\Gamma^\mu\,\mathcal{D}_\mu\,\psi_a \\
&\quad -\frac{1}{24}e^\phi\left[\bar{\psi}^\lambda\Gamma_{[\lambda}\Gamma^{\mu\nu\rho}\Gamma_{\sigma]}\bar{\psi}^\sigma - 2\bar{\chi}\Gamma^\lambda\Gamma^{\mu\nu\rho}\psi_\lambda - \bar{\chi}\Gamma^{\mu\nu\rho}\chi - v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu\rho}\lambda_{\hat{I}})_X + \bar{\psi}^a\Gamma^{\mu\nu\rho}\psi_a\right]G_{\mu\nu\rho} \\
&\quad -\frac{1}{2\sqrt{2}}ee^{-\phi/2}v_X(\bar{\psi}_\mu\Gamma^{\nu\rho}\Gamma^\mu\lambda^{\hat{I}}F_{\hat{I}\nu\rho})_X - \frac{1}{2\sqrt{2}}ee^{-\phi/2}v_X(\bar{\lambda}^{\hat{I}}\Gamma^{\mu\nu}\chi F_{\hat{I}\mu\nu})_X \\
&\quad -\frac{1}{2}\bar{\chi}\Gamma^\nu\Gamma^\mu\psi_\nu\partial_\mu\phi + \bar{\psi}^A_\mu\Gamma^\nu\Gamma^\mu\,\mathcal{P}_{\nu aA}\,\psi^a + \frac{1}{8}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau\upsilon}B_{\mu\nu}v_X(F^{\hat{I}}_{\rho\sigma}F_{\hat{I}\tau\upsilon})_X \\
&\quad +\frac{1}{\sqrt{2}}e^{\phi/2}\bar{\psi}_\mu\Gamma^\mu T_i\lambda_{\hat{I}}C^{i\hat{I}} + \frac{1}{\sqrt{2}}e^{\phi/2}\bar{\chi}T_i\lambda_{\hat{I}}C^{i\hat{I}} - \sqrt{2}e^{\phi/2}\bar{\lambda}^{\hat{I}}_A\psi_a V_\alpha{}^{aA}\tilde{\xi}^\alpha_{\hat{I}} \\
&\quad -\frac{1}{8}e^\phi v_X^{-1}(C^{i\hat{I}}C_{i\hat{I}})_X + (\text{Fermi})^4. \qquad (155)
\end{aligned}
$$

which is invariant under the local supersymmetry transformations

$$
\begin{aligned}
\delta e^a_\mu &= \bar{\epsilon}\Gamma^a\psi_\mu, \\
\delta B_{\mu\nu} &= e^{-\phi}\left(-\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar{\epsilon}\Gamma_{\mu\nu}\chi\right), \\
\delta\phi &= \bar{\epsilon}\chi, \\
\delta\psi_\mu &= \mathcal{D}_\mu\,\epsilon + \frac{1}{24}e^\phi\Gamma^{\nu\rho\sigma}\Gamma_\mu G_{\nu\rho\sigma}\epsilon, \\
\delta\chi &= \frac{1}{2}\Gamma^\mu\partial_\mu\phi\epsilon - \frac{1}{12}e^\phi\Gamma^{\mu\nu\rho}G_{\mu\nu\rho}\epsilon, \\
\delta A^{\hat{I}}_\mu &= \frac{1}{\sqrt{2}}e^{\phi/2}\bar{\epsilon}\Gamma_\mu\lambda^{\hat{I}}, \\
\delta\lambda^{\hat{I}} &= -\frac{1}{2\sqrt{2}}e^{-\phi/2}\Gamma^{\mu\nu}\epsilon F^{\hat{I}}_{\mu\nu} - \frac{1}{\sqrt{2}}e^{\phi/2}v_X^{-1}(C^{i\hat{I}})_X T_i\epsilon, \\
\delta\varphi^\alpha &= \mathcal{V}^\alpha{}_{aA}\,\bar{\psi}^a\epsilon^A, \\
\delta\psi^a &= \Gamma^\mu\,\mathcal{P}_\mu{}^{aA}\,\epsilon_A. \qquad (156)
\end{aligned}
$$

The Lagrangian (155) correspond to one of the two possible formulations of $D = 6$, $N = 2$ supergravity. The Lagrangian derived in the original papers is the dual to the one considered here. The main differences are (i) the reversed dilaton signs in the kinetic terms of the vector fields, (ii) the absence of the Green-Schwarz term and (iii) a modification of the $B_{\mu\nu}$ supersymmetry transformation law and to its field strength.

6 Bulk Branes in Supergravity

We present a new anomaly-free gauged $N = 1$ supergravity model in six dimensions. We construct a consistent supersymmetric action for brane chiral and vector multiplets in a six-dimensional chiral gauged supergravity. When the brane chiral multiplet is charged under the bulk $U(1)_R$, we obtain a nontrivial coupling to the extra component of the $U(1)_R$ gauge field strength and a singular scalar self-interaction term. The six-dimensional Salam-Sezgin supergravity consists of gravity coupled to a dilaton field ϕ, a Kalb-Ramond(KR) field B_{MN}, along with the SUSY fermionic partners, the gravitino ψ_M, the dilatino χ. Moreover, it also contains a bulk $U(1)_R$

vector multiplet (A_M, λ) that gauges the R-symmetry of six-dimensional supergravity. All the bulk fermions are 6D Weyl. In order to do this analysis, we need the spinor part of the action and in particular the part that is quadratic in fermionic terms. This is given by

$$
\begin{aligned}
e^{-1}\mathcal{L}_f &= \bar{\psi}_M\Gamma^{MNP}\mathcal{D}_N\psi_P + \bar{\chi}\Gamma^M\mathcal{D}_M\chi + \bar{\lambda}\Gamma^M\mathcal{D}_M\lambda + \bar{\lambda}\Gamma^M\mathcal{D}_M\lambda \\
&+\frac{1}{4}(\partial_M\phi)(\bar{\psi}_N\Gamma^M\Gamma^N\chi + \text{h.c.}) + \sqrt{2}ge^{-\frac{1}{4}\phi}(i\bar{\psi}_M\Gamma^M\lambda - i\bar{\chi}\lambda + \text{h.c.}) \\
&-\frac{1}{4\sqrt{2}}e^{\frac{1}{4}\phi}\Big\{F_{MN}(\bar{\psi}_Q\Gamma^{MN}\Gamma^Q\lambda + \bar{\chi}\Gamma^{MN}\lambda) + F^I_{MN}(\bar{\psi}_Q\Gamma^{MN}\Gamma^Q\lambda^I + \bar{\chi}\Gamma^{MN}\lambda^I) + \text{h.c.}\Big\}.
\end{aligned}
\tag{157}
$$

The complete bulk Langrangian up to four fermion terms is

$$
\begin{aligned}
e_6^{-1}\mathcal{L}_{\text{bulk}} &= R - \frac{1}{4}(\partial_M\phi)^2 - \frac{1}{12}e^{\phi}G_{MNP}G^{MNP} - \frac{1}{4}e^{\frac{1}{2}\phi}F_{MN}F^{MN} - 8g^2e^{-\frac{1}{2}\phi} \\
&+\bar{\psi}_M\Gamma^{MNP}\mathcal{D}_N\psi_P + \bar{\chi}\Gamma^M\mathcal{D}_M\chi + \bar{\lambda}\Gamma^M\mathcal{D}_M\lambda + \frac{1}{4}(\partial_M\phi)(\bar{\psi}_N\Gamma^M\Gamma^N\chi + \bar{\chi}\Gamma^N\Gamma^M\psi_N) \\
&+\frac{1}{24}e^{\frac{1}{2}\phi}G_{MNP}(\bar{\psi}^R\Gamma_{[R}\Gamma^{MNP}\Gamma_{S]}\psi^S + \bar{\psi}_R\Gamma^{MNP}\Gamma^R\chi - \bar{\chi}\Gamma^R\Gamma^{MNP}\psi_R - \bar{\chi}\Gamma^{MNP}\chi \\
&+\bar{\lambda}\Gamma^{MNP}\lambda) - \frac{1}{4\sqrt{2}}e^{\frac{1}{4}\phi}F_{MN}(\bar{\psi}_Q\Gamma^{MN}\Gamma^Q\lambda + \bar{\lambda}\Gamma^Q\Gamma^{MN}\psi_Q + \bar{\chi}\Gamma^{MN}\lambda - \bar{\lambda}\Gamma^{MN}\chi) \\
&+i\sqrt{2}ge^{-\frac{1}{4}\phi}(\bar{\psi}_M\Gamma^M\lambda + \bar{\lambda}\Gamma^M\psi_M - \bar{\chi}\lambda + \bar{\lambda}\chi).
\end{aligned}
\tag{158}
$$

The field strengths of the gauge and Kalb-Ramond(KR) fields are defined as

$$
F_{MN} = \partial_M A_N - \partial_N A_N, \qquad G_{MNP} = 3\partial_{[M}B_{NP]} + \frac{3}{2}F_{[MN}A_{P]}, \tag{159}
$$

and satisfy the Bianchi identities

$$
\partial_{[Q}F_{MN]} = 0, \qquad \partial_{[Q}G_{MNP]} = \frac{3}{4}F_{[MN}F_{QP]}. \tag{160}
$$

For $\delta_\Lambda A_M = \partial_M\Lambda$ under the $U(1)_R$, the field strength for the KR field is made gauge invariant by allowing for B_{MN} to transform as

$$
\delta_\Lambda B_{MN} = -\frac{1}{2}\Lambda F_{MN}. \tag{161}
$$

All the spinors have the same R charge $+1$, so the covariant derivative of the gravitino, for instance, is given by

$$
\mathcal{D}_M\psi_N = (\partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB} - igA_M)\psi_N. \tag{162}
$$

The local $\mathcal{N} = 2$ SUSY transformations are (up to trilinear fermion terms):

$$
\begin{aligned}
\delta e^A_M &= -\frac{1}{4}\bar{\varepsilon}\Gamma^A\psi_M + \text{h.c.}, \qquad \delta\phi = \frac{1}{2}\bar{\varepsilon}\chi + \text{h.c.}, \\
\delta B_{MN} &= A_{[M}\delta A_{N]} + \frac{1}{4}e^{-\frac{1}{2}\phi}(\bar{\varepsilon}\Gamma_M\psi_N - \bar{\varepsilon}\Gamma_N\psi_M + \bar{\varepsilon}\Gamma_{MN}\chi + \text{h.c.}), \\
\delta\chi &= -\frac{1}{4}(\partial_M\phi)\Gamma^M\varepsilon + \frac{1}{24}e^{\frac{1}{2}\phi}G_{MNP}\Gamma^{MNP}\varepsilon, \\
\delta\psi_M &= \mathcal{D}_M\varepsilon + \frac{1}{48}e^{\frac{1}{2}\phi}G_{PQR}\Gamma^{PQR}\Gamma_M\varepsilon, \\
\delta A_M &= \frac{1}{2\sqrt{2}}e^{-\frac{1}{4}\phi}(\bar{\varepsilon}\Gamma_M\lambda + \text{h.c.}), \qquad \delta\lambda = \frac{1}{4\sqrt{2}}e^{\frac{1}{4}\phi}F_{MN}\Gamma^{MN}\varepsilon - i\sqrt{2}g\, e^{-\frac{1}{4}\phi}\varepsilon.
\end{aligned}
\tag{163}
$$

The above spinors are chiral with handednesses

$$\mathcal{G}^7\psi_M = +\psi_M, \quad \mathcal{G}^7\chi = -\chi, \quad \mathcal{G}^7\langle = +\langle, \quad \mathcal{G}^7\varepsilon = +\varepsilon. \tag{164}$$

Taking into account that $\mathcal{G}^7 = \sigma^3 \otimes \mathbf{1}$, the 6D (8-component) spinors can be decomposed to 6D Weyl (4-component) spinors as

$$\psi_M = (\tilde{\psi}_M, 0)^T, \quad \chi = (0, \tilde{\chi})^T, \quad \lambda = (\tilde{\langle}, 0)^T, \quad \varepsilon = (\tilde{\varepsilon}, 0)^T. \tag{165}$$

For later use, we decompose the 6D Weyl spinor $\tilde{\psi}$ to $\tilde{\psi} = (\tilde{\psi}_L, \tilde{\psi}_R)^T$, satisfying $\gamma^5(\tilde{\psi}_L, 0)^T = +(\tilde{\psi}_L, 0)^T$ and $\gamma^5(0, \tilde{\psi}_R)^T = -(0, \tilde{\psi}_R)^T$. Henceforth we drop the tildes for simplicity.

We can show that the action for the Lagrangian (158) is invariant under the above SUSY transformations up to the trilinear fermion terms and the Bianchi identities as follows,

$$\begin{aligned}\delta\mathcal{L}_{\rm bulk} &= e_6\Big[-\frac{1}{24}e^{\frac{1}{2}\phi}\Big(\partial_S G_{MNP} - \frac{3}{4}F_{MN}F_{SP}\Big)\Big(\bar{\psi}_R\Gamma^{RMNPS}\varepsilon - \bar{\chi}\Gamma^{SMNP}\varepsilon + {\rm h.c.}\Big) \\ &\quad + \frac{1}{4\sqrt{2}}e^{\frac{1}{4}\phi}\Big(\partial_Q F_{MN}\bar{\lambda}\Gamma^{QMN}\varepsilon + {\rm h.c.}\Big)\Big].\end{aligned} \tag{166}$$

The SUSY variation of the brane action can be cancelled with the bulk variation (166) by modifying the Bianchi identities.

We consider a nonzero brane tension as well as brane matter multiplets: a brane chiral multiplet (Q, ψ_Q), the superfield of which has an R charge $-r$, and a brane vector multiplet (W_μ, Λ). The 4D chirality of the fermion in the brane chiral multiplet is taken to be right-handed in contrast to the Z_2-even gravitino and the Z_2-even gaugino and the brane gaugino. So, the conventional chiral superfield containing a left-handed fermion, $(Q^*, (\psi_Q)^c)$, should have an opposite R charge, namely, r for Q^* and $r-1$ for $(\psi_Q)^c$. Then, by employing the Noether method for the local SUSY, we find that the supersymmetric action for the bulk-brane system up to four fermion terms is composed of the original bulk action (158) with the field strength tensors G_{MNP} and F_{MN} being replaced by the modified ones $\hat{G}_{MNP}$ and $\hat{F}_{MN}$, respectively, and the brane action as follows,

$$\mathcal{L} = \mathcal{L}_{\rm bulk}(G \to \hat{G}, F \to \hat{F}) + \delta^2(y)\mathcal{L}_{\rm brane} \tag{167}$$

with the special type brane lagrangian

$$\begin{aligned}\mathcal{L}_{\rm brane} = e_4\Big[&e^{\frac{1}{2}\phi}\Big(-(D^\mu Q)^\dagger D_\mu Q + \frac{1}{2}\bar{\psi}_Q\gamma^\mu D_\mu\psi_Q + {\rm h.c.}\Big) + \sqrt{2}irge^{\frac{1}{4}\phi}\bar{\psi}_Q\lambda_+ Q + {\rm h.c.} \\ &-4rg^2|Q|^2 - T + e^{\frac{1}{2}\phi}\Big(\frac{1}{2}\bar{\psi}_{\mu+}\gamma^\nu\gamma^\mu\psi_Q(D_\nu Q)^\dagger + \frac{1}{2}\bar{\psi}_Q\gamma^\mu\chi_+ D_\mu Q + {\rm h.c.}\Big) \\ &-\frac{1}{4}W_{\mu\nu}W^{\mu\nu} + \frac{1}{2}\bar{\Lambda}\gamma^\mu D_\mu\Lambda + {\rm h.c.} - ie\sqrt{2}e^{\frac{1}{2}\phi}Q\bar{\psi}_Q\Lambda + {\rm h.c.} - \frac{1}{2}e^2|Q|^4 e^\phi \\ &-\frac{1}{4\sqrt{2}}\bar{\Lambda}\gamma^\mu\gamma^{\nu\rho}\psi_{\mu+}W_{\nu\rho} - \frac{i}{2\sqrt{2}}e|Q|^2 e^{\frac{1}{2}\phi}\bar{\Lambda}\gamma^\mu\psi_{\mu+} + {\rm h.c.} \\ &-\frac{i}{\sqrt{2}}e|Q|^2 e^{\frac{1}{2}\phi}\bar{\chi}_+\Lambda + {\rm h.c.}\Big].\end{aligned} \tag{168}$$

The SUSY transformations of the brane chiral multiplet are

$$\delta Q = \frac{1}{2}\bar{\varepsilon}_+\psi_Q\,, \qquad \delta\psi_Q = -\frac{1}{2}\gamma^\mu\varepsilon_+ D_\mu Q$$

.

On the other hand, the SUSY transformations of the brane vector multiplet are

$$\delta W_\mu = \frac{1}{2\sqrt{2}}\bar{\varepsilon}_+\gamma_\mu\Lambda + \text{h.c.}, \qquad \delta\Lambda = \frac{1}{4\sqrt{2}}\gamma^{\mu\nu}\varepsilon_+ W_{\mu\nu} + \frac{i}{2\sqrt{2}}e|Q|^2 e^{\frac{1}{2}\phi}\varepsilon_+. \tag{170}$$

Here the brane gauge field strength is $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ and the covariant derivatives of the brane multiplets are

$$D_\mu Q = (\partial_\mu + irgA_\mu - ieW_\mu)Q, \tag{171}$$

$$D_\mu\psi_Q = (\partial_\mu + i(r-1)gA_\mu - ieW_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta})\psi_Q, \tag{172}$$

$$D_\mu\Lambda = (\partial_\mu - igA_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta})\Lambda. \tag{173}$$

We note that the R charges of the component fields in the brane chiral multiplet are different by $+1$ as known to be the case in 4D local SUSY. The gaugino of a brane vector multiplet also has the same R charge $+1$ as the bulk gravitino.

The modified field strength tensors are

$$\hat{G}_{\mu mn} = G_{\mu mn} + \Big(J_\mu - \xi A_\mu\Big)\epsilon_{mn}\frac{\delta^2(y)}{e_2}, \tag{174}$$

$$\hat{G}_{\tau\rho\sigma} = G_{\tau\rho\sigma} + J_{\tau\rho\sigma}\frac{\delta^2(y)}{e_2}, \tag{175}$$

$$\hat{F}_{mn} = F_{mn} - (rg|Q|^2 + \xi)\epsilon_{mn}\frac{\delta^2(y)}{e_2} \tag{176}$$

where $\xi = \frac{T}{4g}$ is the localized FI term, ϵ_{mn} is the 2D volume form and

$$J_\mu = \frac{1}{2}i\Big[Q^\dagger D_\mu Q - (D_\mu Q)^\dagger Q + \frac{1}{2}\bar{\psi}_Q\gamma_\mu\psi_Q - \frac{1}{2}e^{-\frac{1}{2}\phi}\bar{\Lambda}\gamma_\mu\Lambda\Big], \tag{177}$$

$$J_{\tau\rho\sigma} = -\frac{1}{4}\bar{\psi}_Q\gamma_{\tau\rho\sigma}\psi_Q - \frac{1}{8}e^{-\frac{1}{2}\phi}\bar{\Lambda}\gamma_{\tau\rho\sigma}\Lambda. \tag{178}$$

Here in order to cancel the variation of the brane tension action, we needed to modify the gauge field strength with the localized FI term proportional to the brane tension. Moreover, the modified field strength for the KR field contains a gauge non-invariant piece proportional to the localized FI term so the gauge transformation of the KR field needs to be modified to

$$\delta_\Lambda B_{mn} = \Lambda\Big(-\frac{1}{2}F_{mn} + \xi\epsilon_{mn}\frac{\delta^2(y)}{e_2}\Big). \tag{179}$$

The SUSY transformations of the bulk fields are the same as (163) and (164) with G_{MNP} and F_{MN} being replaced by $\hat{G}_{MNP}$ and $\hat{F}_{MN}$, and the gauge field A_M being kept the same as in the no-brane case, with an exception that the SUSY transformation of the extra components of the KR field has an additional term as

$$\delta B_{mn} = \frac{1}{4}i\bar{\psi}_Q\varepsilon_+ Q\epsilon_{mn}\frac{\delta^2(y)}{e_2} + \text{h.c.}. \tag{180}$$

Furthermore, for the modified field strength tensors, we obtain the Bianchi identities as follows,

$$\partial_{[\mu}\hat{G}_{\nu mn]} = \frac{3}{4}\hat{F}_{[\mu\nu}\hat{F}_{mn]} + \Big[\frac{i}{2}(D_{[\mu}Q)^\dagger(D_{\nu]}Q) + \frac{1}{4}e|Q|^2 W_{\mu\nu}\Big]\epsilon_{mn}\frac{\delta^2(y)}{e_2}, \tag{181}$$

$$\partial_{[\mu}\hat{F}_{mn]} = -\frac{1}{3}rg\partial_\mu|Q|^2\epsilon_{mn}\frac{\delta^2(y)}{e_2}. \tag{182}$$

Then, by using (166) with the modified Bianchi identities (181) and (182), we are able to cancel all the remaining variations of the brane action given in (168). We can extend the result to the more general case with multiple branes. When all the branes preserve the same 4D $\mathcal{N}=1$ SUSY, we only have to replace the delta terms appearing in the action and the SUSY/gauge transformations: $T\delta^2(y)$ with $\sum_i T_i\delta^2(y-y_i)$, and $f(Q)\delta^2(y)$ with $\sum_i f(Q_i)\delta^2(y-y_i)$.

7 IIA Supergravity

The type IIA supergravity in ten dimensions describes the low energy limit of type IIA superstrings. This theory is non-chiral and therefore it has no anomalies. We start by reviewing bosonic part of the standard IIA supergravity action. The 10D NSNS fields are the dilaton ϕ, 2-form potential $\hat{B}_2$ and string frame metric $\hat{g}_{\hat{M}\hat{N}}$, where $\hat{M},\hat{N}=0,\dots 9$. The 10D RR fieldstrenghts are the 4-form G_4, 2-form G_2 and 0-form G_0.

We measure all dimensionful fields in units of 11D Planck length l_p and set $k_{11}=\pi$, so

$$\begin{aligned} S^{(10)}_{bos} &= \frac{1}{2\pi}\int_{X_{10}} e^{-2\phi}\left(\sqrt{g_{10}}(\hat{g}) + 4d\phi\wedge\hat{*}d\phi + \tfrac{1}{2}\hat{H}_3\wedge\hat{*}\hat{H}_3\right) \\ &+\frac{1}{4\pi}\int_{X_{10}}\left(\widetilde{G}_4{\wedge}\hat{*}\widetilde{G}_4 + i\hat{B}{\wedge}\widetilde{G}_4{\wedge}\widetilde{G}_4 + \widetilde{G}_2{\wedge}\hat{*}\widetilde{G}_2 + \sqrt{g_{10}}G_0^2\right) \end{aligned} \tag{183}$$

where $\hat{*}$ stands for the 10D Hodge duality operator. The fields in the previous action are defined as

$$\tilde{G}_2 = G_2 + \hat{B}_2 G_0, \quad \tilde{G}_4 = G_4 + \hat{B}_2 G_2 + \tfrac{1}{2}\hat{B}_2\hat{B}_2 G_0, \quad \hat{H}_3 = d\hat{B}_2. \tag{184}$$

The next step is the part of the $10D$ IIA supergravity action quadratic in fermions. We work in the string frame.

$$\begin{aligned} S^{(10)}_{ferm} &= \int\sqrt{-g_{10}}e^{-2\phi}\left[\tfrac{1}{2}\bar{\hat{\psi}}_{\hat{A}}\hat{\Gamma}^{\hat{A}\hat{N}\hat{B}}D_{\hat{N}}\hat{\psi}_{\hat{B}} + \tfrac{1}{2}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{N}}D_{\hat{N}}\hat{\Lambda} - \frac{1}{\sqrt{2}}(\pi_{\hat{N}}\phi)\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}}\hat{\Gamma}^{\hat{N}}\hat{\psi}_{\hat{A}}\right] \\ &+\frac{1}{16}\int\sqrt{-g_{10}}e^{-\phi}\widetilde{G}_{\hat{A}\hat{C}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}_{\hat{F}]}\hat{\Gamma}^{11}\hat{\psi}^{\hat{F}} + \frac{3}{\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}^{11}\hat{\psi}_{\hat{E}} + \frac{5}{4}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}\hat{C}}\hat{\Gamma}^{11}\hat{\Lambda}\right] \\ &+\int\sqrt{-g_{10}}e^{-\phi}G_0\left[\frac{1}{8}\bar{\hat{\psi}}_{\hat{A}}\hat{\Gamma}^{\hat{A}\hat{B}}\hat{\psi}_{\hat{B}} + \frac{5}{8\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}}\hat{\psi}_{\hat{A}} - \frac{21}{32}\bar{\hat{\Lambda}}\hat{\Lambda}\right] \\ &+\frac{1}{192}\int\sqrt{-g_{10}}e^{-\phi}\widetilde{G}_{\hat{A}\hat{B}\hat{C}\hat{D}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\Gamma}_{\hat{F}]}\hat{\psi}^{\hat{F}} + \frac{1}{\sqrt{2}}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\psi}_{\hat{E}} + \frac{3}{4}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}\hat{D}}\hat{\Lambda}\right] \\ &+\frac{1}{48}\int\sqrt{-g_{10}}e^{-2\phi}H_{\hat{A}\hat{B}\hat{C}}\left[\bar{\hat{\psi}}^{\hat{E}}\hat{\Gamma}_{[\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}}\hat{\Gamma}_{\hat{F}]}\hat{\Gamma}^{11}\hat{\psi}^{\hat{F}} + \sqrt{2}\bar{\hat{\Lambda}}\hat{\Gamma}^{\hat{E}}\hat{\Gamma}^{\hat{A}\hat{B}\hat{C}}\hat{\Gamma}^{11}\hat{\psi}_{\hat{E}}\right] \end{aligned} \tag{185}$$

where $\hat{\Lambda}$ and $\hat{\psi}^{\hat{A}}$ are the Majorana dilatino and gravitino and covariant derivatives act on them

$$D_{\hat{N}}\hat{\psi}^{\hat{A}} = \pi_{\hat{N}}\hat{\psi}^{\hat{A}} + \omega_{\hat{N}}{}^{\hat{A}}{}_{\hat{B}}\hat{\psi}^{\hat{B}} + \frac{1}{4}\omega_{\hat{N}\hat{B}\hat{C}}\Gamma^{\hat{B}\hat{C}}\hat{\psi}^{\hat{A}}, \qquad D_{\hat{N}}\hat{\Lambda} = \pi_{\hat{N}}\hat{\Lambda} + \frac{1}{4}\omega_{\hat{N}\hat{B}\hat{C}}\Gamma^{\hat{B}\hat{C}}\hat{\Lambda} \tag{186}$$

There are also terms quartic in fermions in the action. It turns out that it is important to take them into account to check the T-duality invariance of partition sum. We collect 4-fermionic terms in D=10 IIA supergravity action which are obtained from reduction of the D=11 action

$$
\begin{aligned}
S^{(10)}_{4-ferm} = \frac{\pi}{2}\int\sqrt{-g_{10}}e^{-2\phi}\Big\{ & -\frac{1}{64}\Big[\bar{\chi}_{\mathbf{E}}\hat{\Gamma}^{\mathbf{ABCDEF}}\chi_{\mathbf{F}} + 12\bar{\chi}^{[\mathbf{A}}\hat{\Gamma}^{\mathbf{BC}}\chi^{\mathbf{D}]}\Big]\bar{\chi}_{[\mathbf{A}}\hat{\Gamma}_{\mathbf{BC}}\chi_{\mathbf{D}]} \\
& + \frac{1}{32}\left(\bar{\chi}_{\mathbf{E}}\hat{\Gamma}^{\mathbf{ABCEF}}\chi_{\mathbf{F}}\right)\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right) + \frac{1}{4}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}^{\mathbf{A}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}^{\mathbf{B}}\chi^{\mathbf{C}}\right) \\
& - \frac{1}{8}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}^{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}_{\mathbf{B}}\hat{\Gamma}^{\mathbf{A}}\chi^{\mathbf{C}}\right) - \frac{1}{16}\left(\bar{\chi}_{\mathbf{A}}\hat{\Gamma}_{\mathbf{B}}\chi_{\mathbf{C}}\right)\left(\bar{\chi}^{\mathbf{A}}\hat{\Gamma}^{\mathbf{B}}\chi^{\mathbf{C}}\right)\Big\}
\end{aligned} \tag{187}
$$

where

$$
\chi_{\hat{A}} = \left[\hat{\psi}_{\hat{A}} + \frac{1}{6\sqrt{2}}\hat{\Gamma}_{\hat{A}}\hat{\Lambda}\right], \qquad \chi_{11} = -\frac{2\sqrt{2}}{3}\hat{\Gamma}^{11}\hat{\Lambda} \tag{188}
$$

Recall that the graviton $\mathbf{E}^{\mathbf{A}}_{\mathbf{M}}$ and the gravitino $\psi^{(11)}_{\mathbf{A}}$ of 11D supergravity are related to 10D fields as

$$
\mathbf{E}^{\hat{\mathcal{A}}}_{\hat{M}} = e^{-\frac{\phi}{3}}\hat{E}^{\hat{\mathcal{A}}}_{\hat{M}}, \quad \mathbf{E}^{11}_{11} = e^{\frac{2\phi}{3}}, \quad \mathbf{E}^{11}_{\hat{M}} = e^{\frac{2\phi}{3}}C_{\hat{M}}
$$

$$
\psi^{(11)}_{\mathbf{A}} = \frac{1}{\sqrt{2\pi}}e^{\frac{\phi}{6}}\chi_{\mathbf{A}}
$$

We are in the position to read off the values of the sources and constants introduced in the dualization procedure and sum the different contributions to obtain the full action of the bulk and brane system in the Einstein frame,

$$
\begin{aligned}
S^E \;=\; & -\frac{1}{2\kappa_4^2}\int\Big[R*1 + \frac{1}{2}\mathrm{d}\phi\wedge*\mathrm{d}\phi + 2G_{KL}\mathrm{d}q^K\wedge*\mathrm{d}q^L + 2(G_{ab} + \frac{9}{4}\frac{\mathcal{K}_{ab}}{2})\mathrm{d}v^a\wedge*\mathrm{d}v^b \\
& + 2e^{-\phi}G_{ab}\mathrm{d}b^a\wedge*\mathrm{d}b^b + \frac{3}{4}e^{-\frac{\phi}{4}}\mathcal{H}^{IJ}\left(\mathrm{d}\Phi_I\wedge*\mathrm{d}\Phi_J + M_I^L M_J^K\mathrm{d}\Phi_L\wedge*\mathrm{d}\Phi_K\right. \\
& \left. - 2M_I^L\mathrm{d}\Phi_L\wedge*\mathrm{d}a_J + \mathrm{d}a_I\wedge*\mathrm{d}a_J\right) + \frac{3}{-}e^{\frac{\phi}{2}}A_{\hat{K}\hat{L}}\left[\nabla\xi^{\hat{K}} + 2J^{\hat{K}}\right]\wedge*\left[\nabla\xi^{\hat{L}} + 2J^{\hat{L}}\right] \\
& + \frac{}{3}e^{\frac{\phi}{2}}G_{\alpha\beta}\mathrm{d}V^\alpha\wedge*\mathrm{d}V^\beta + \frac{1}{2}{}_{\alpha\beta a}\mathcal{B}^a\mathrm{d}V^\alpha\wedge\mathrm{d}V^\beta \\
& + \frac{}{3}e^{\frac{\phi}{2}}G_{\alpha\beta}\tilde{\mathcal{M}}^{I\beta}\Phi_I\mathrm{d}V^\alpha\wedge*F + \frac{1}{2}\left({}_{\alpha\beta a}\tilde{\mathcal{M}}^{\beta I}\Phi_I\mathcal{B}^a - \tilde{\mathcal{L}}^I_\alpha a_I\right)\mathrm{d}V^\alpha\wedge F \\
& + \frac{1}{8}e^{-\frac{\phi}{4}}V_{\Pi_+}F\wedge*F + \frac{}{12}e^{\frac{\phi}{2}}G_{\alpha\beta}\tilde{\mathcal{M}}^{I\alpha}\tilde{\mathcal{M}}^{J\beta}\Phi_J\Phi_I F\wedge*F \\
& + \frac{1}{8}\left({}_{\alpha\beta a}\tilde{\mathcal{M}}^{\beta I}\Phi_I\mathcal{B}^a - \tilde{\mathcal{L}}^I_\alpha a_I\right)\tilde{\mathcal{M}}^{\alpha J}\Phi_J F\wedge F - \frac{1}{8}l_s^3\pi_{0\hat{K}}\xi^{\hat{K}}F\wedge F \\
& + \frac{1}{4l_s^2}\left(\frac{6}{-}\right)^3 e^{-\frac{\phi}{2}}\left(\frac{1}{2}\mathcal{M}^{Ia}\Phi_{Iabc}\mathcal{B}^b\mathcal{B}^c - \mathcal{L}^I_a a_I\mathcal{B}^a - \eta_a\mathcal{B}^a + \lambda\right)^2 *_4 1 \\
& + \frac{1}{16l_s^2}\left(\frac{6}{-}\right)^3 e^{\frac{\phi}{2}}G^{ab}\left(\mathcal{L}^I_a a_I -_{acd}\mathcal{M}^{Ic}\mathcal{B}^d\Phi_I -_{acd}\nu^c\mathcal{B}^d + \eta_a\right) \\
& \times\left(\mathcal{L}^I_b a_I -_{bef}\mathcal{M}^{Je}\mathcal{B}^f\Phi_J -_{bef}\nu^e\mathcal{B}^f + \eta_b\right) *_4 1 \\
& + \frac{6}{l_s^2}e^{\frac{3}{2}\phi}G_{ab}\left(\mathcal{M}^{Ia}\Phi_I + \nu^a\right)\left(\mathcal{M}^{Jb}\Phi_J + \nu^b\right)\Big].
\end{aligned} \tag{189}
$$

The abbreviations used in the above are

$$\begin{aligned}
V_{\Pi_+} &= \frac{1}{l_s^3}\int_{\Pi_+} \mathrm{dvol}|^E_{\Pi_+}\,, \qquad \mathcal{H}^{IJ} = \frac{1}{l_s^3}\int_{\Pi_+} A^I \wedge *_{\Pi_+} A^J\,,\\
\mathcal{M}^{Ia} &= \frac{1}{l_s^3}\int_{\Pi_+} \iota^*(i_{s^I}\tilde{\omega}^a), \qquad \tilde{\mathcal{M}}^{I\alpha} = \frac{1}{l_s^3}\int_{\Pi_-} \iota^*(i_{s^I}\tilde{\omega}^\alpha)\,,\\
\mathcal{L}^I_a &= \frac{1}{l_s^3}\int_{\Pi_+} (\iota^*\omega_a)\wedge A^I, \qquad \tilde{\mathcal{L}}^I_\alpha = \frac{1}{l_s^3}\int_{\Pi_-} (\iota^*\omega_\alpha)\wedge A^I\,.
\end{aligned} \tag{190}$$

The source current $J^{\hat K}$ arising from the dualization of the two-form ρ is given by

$$J^{\hat K} = -\frac{1}{8}\mathcal{Q}^{IJ\hat K}\left[\Phi_I \mathrm{d}(a_J - M^L_J\Phi_L) - (a_J - M^L_J\Phi_L)\mathrm{d}\Phi_I\right]. \tag{191}$$

λ, η_a and ν^a are constants arising, by the spelled-out procedure, from elimination of the original three-forms $\lambda \leftrightarrow c_{(3)}$, $\eta_a \leftrightarrow \tilde{c}^a_{(3)}$, $\nu^a \leftrightarrow d_{(3)_a}$.

In the action (189) the kinetic terms of the axionic scalars $\xi^{\hat K}$ contain the gauge covariant derivatives

$$\nabla\xi^{\hat K} = \mathrm{d}\xi^{\hat K} + \frac{1}{2l_s}\pi_0^{\hat K} A. \tag{192}$$

The appearance of this covariant derivative is a manifestation of the Green-Schwarz mechanism for anomaly cancellation. It arises after dualization from the Green-Schwarz term in the Chern-Simons action which describes a linear coupling of the $U(1)$ field strength to the spacetime two-form ρ. The bulk action has a set of shift symmetries

$$\xi^{\hat K} \to \xi^{\hat K} + c^{\hat K}, \tag{193}$$

which are gauged upon inclusion of a D6-brane, resulting in the covariant derivatives given above. The non-gauge-invariant part of the Lagrangian obtained by gauging a shift symmetry of a Ramond-Ramond scalar fields appearing in the gauge kinetic functions as is the case for the $\xi^{\hat K}$ exhibits exactly the right gauge transformation behavior to cancel the chiral anomalies.

In the sequel we will assume that this is the case so that the part of the Lagrangian including the kinetic terms of the scalar fields encoded in (189) may be written as

$$\begin{aligned}
\mathcal{L}_{sc} \;=\;& 2e^{-\phi}G_{ab}\mathrm{d}t^a \wedge *\mathrm{d}\bar{t}^b + 4e^{2D}A_{\hat K\hat L}\mathrm{d}l^{\hat K}\wedge *\mathrm{d}l^{\hat L}\\
&+4e^{2D}A_{\hat K\hat L}\left[\nabla\xi^{\hat K} + \frac{i}{8}\mathcal{Q}^{IJ\hat K}\left[\bar{u}_I\mathrm{d}u_J - u_I\mathrm{d}\bar{u}_J\right]\right]\\
&\wedge *\left[\nabla\xi^{\hat L} + \frac{i}{8}\mathcal{Q}^{IJ\hat L}\left[\bar{u}_I\mathrm{d}u_J - u_I\mathrm{d}\bar{u}_J\right]\right] + \frac{3}{4}e^{-\frac{\phi}{4}}\mathcal{H}^{IJ}\mathrm{d}u_I\wedge *\mathrm{d}\bar{u}_J.
\end{aligned} \tag{194}$$

To proceed with the evaluation of the Kähler potential requires more information about the integrals in (**??**) and (**??**). It turns out that to find a Kähler potential reproducing the kinetic terms in the action we need to assume that

$$\mathcal{Q}^{IJ\hat K} = \frac{4}{l_s^3}\delta^{\hat K\hat L}\frac{\pi_{0\hat L}}{\|\pi_0\|^2}\tilde{\mathcal{L}}^I_\alpha\tilde{\mathcal{M}}^{J\alpha}. \tag{195}$$

It is now straightforward but tedious to use the chain rule and the various equalities presented above to check that the kinetic terms of (194) may be written as

$$\begin{aligned}
\mathcal{L}_{sc} &= 2\partial_{\bar M}\partial_N K \mathrm{d}N\wedge *\mathrm{d}\bar M\\
&= 2K_{\bar t^a t^b}\mathrm{d}t^a\wedge *\mathrm{d}\bar t^b + 2K_{\bar N^{\hat K}N^{\hat L}}\nabla N^{\hat L}\wedge *\nabla\bar N^{\hat K}\\
&\quad +4\mathrm{Re}(K_{\bar N^{\hat K}u_I}\mathrm{d}u_I\wedge *\nabla\bar N^{\hat K}) + 2K_{\bar u_J u_I}\mathrm{d}u_I\wedge *\mathrm{d}\bar u_J
\end{aligned} \tag{196}$$

Let us now take a closer look at the scalar potential evaluated for the F-flat configuration in which we performed our dimensional reduction. From what we said above it reduces to

$$
\begin{aligned}
V_{sc} &= \frac{1}{16l_s^2}(\frac{6}{-})^3 e^{\frac{\phi}{2}} G^{ab}(\eta_a -_{acd}\nu^c\mathcal{B}^d)(\eta_b -_{bef}\nu^e\mathcal{B}^f) *_4 1 \\
&+\frac{1}{4l_s^2}(\frac{6}{-})^3 e^{-\frac{\phi}{2}}(\lambda-\eta_a\mathcal{B}^a)^2 *_4 1 + \frac{6}{l_s^2}e^{\frac{3}{2}\phi}G_{ab}\nu^a\nu^b *_4 1
\end{aligned} \tag{197}
$$

and depends on the Kähler structure moduli t^a as well as the dualization constants λ, ν^a, η_a. First we stress again that the Kähler moduli should be viewed as fixed at the F-flat value consistent with the calibration condition and are thus not to be treated dynamically in this potential.

7.1 Massive IIA Supergravity

Our starting point is the supersymmetric bosonic action of massive type IIA supergravity. In the language of differential forms, it is given by

$$
\begin{aligned}
S^{10}_{IIA} &= \int \Big[\hat{R}\,\hat{*}1\!\!1 - \tfrac{1}{2}\hat{*}d\hat{\phi}\wedge d\hat{\phi} - \tfrac{1}{2}e^{\frac{3}{2}\hat{\phi}}\,\hat{*}\hat{F}_{(2)}\wedge\hat{F}_{(2)} - \tfrac{1}{2}e^{-\hat{\phi}}\,\hat{*}\hat{F}_{(3)}\wedge\hat{F}_{(3)} - \tfrac{1}{2}e^{\frac{1}{2}\hat{\phi}}\,\hat{*}\hat{F}_{(4)}\wedge\hat{F}_{(4)} \\
&-\tfrac{1}{2}d\hat{A}_{(3)}\wedge d\hat{A}_{(3)}\wedge\hat{A}_{(2)} - \tfrac{1}{6}m\,d\hat{A}_{(3)}\wedge(\hat{A}_{(2)})^3 - \tfrac{1}{40}m^2\,(\hat{A}_{(2)})^5 - \tfrac{1}{2}m^2\,e^{\frac{5}{2}\hat{\phi}}\,\hat{*}1\!\!1\Big],
\end{aligned} \tag{198}
$$

where the field strengths are given in terms of potentials by

$$
\begin{aligned}
\hat{F}_{(2)} &= d\hat{A}_{(1)} + m\,\hat{A}_{(2)}\,, \qquad \hat{F}_{(3)} = d\hat{A}_{(2)}\,, \\
\hat{F}_{(4)} &= d\hat{A}_{(3)} + \hat{A}_{(1)}\wedge d\hat{A}_{(2)} + \tfrac{1}{2}m\,\hat{A}_{(2)}\wedge\hat{A}_{(2)}\,.
\end{aligned} \tag{199}
$$

For the form fields and dilaton, together with the Einstein equation

$$
\begin{aligned}
\hat{R}_{AB} &= \tfrac{1}{2}\partial_A\hat{\phi}\,\partial_B\hat{\phi} + \tfrac{1}{16}m^2\,e^{\frac{5}{2}\hat{\phi}}\,\eta_{AB} + \tfrac{1}{12}e^{\frac{1}{2}\hat{\phi}}\left(\hat{F}^2_{(4)AB} - \tfrac{3}{32}\hat{F}^2_{(4)}\,\eta_{AB}\right) \\
&+\tfrac{1}{4}e^{-\hat{\phi}}\left(\hat{F}^2_{(3)AB} - \tfrac{1}{12}\hat{F}^2_{(3)}\,\eta_{AB}\right) + \tfrac{1}{2}e^{\frac{3}{2}\hat{\phi}}\left(\hat{F}^2_{(2)AB} - \tfrac{1}{16}\hat{F}^2_{(2)}\,\eta_{AB}\right).
\end{aligned} \tag{200}
$$

It is useful also to present the expressions for the ten-dimensional Hodge duals of the form fields given above, and for $d\hat{\phi}$. We find that they are given by

$$
\begin{aligned}
e^{\frac{1}{2}\hat{\phi}}\,\hat{*}\hat{F}_{(4)} &= -\tfrac{\sqrt{2}}{3}g\,U\,\epsilon_{(6)} + 4\sqrt{2}g^{-1}\,s\,c\,X^{-1}*dX\wedge d\xi \\
&-\tfrac{\sqrt{2}}{4}g^{-3}\,c^4\,\Delta^{-1}\,X\,F_{(3)}\wedge\epsilon_{(3)} + \tfrac{1}{2\sqrt{2}}g^{-4}\,s\,c^3\,\Delta^{-1}\,X^{-3}\,F_{(2)}\wedge d\xi\wedge\epsilon_{(3)} \\
&-\tfrac{1}{4\sqrt{2}}g^{-2}\,c^2\,X^{-2}*F^i_{(2)}\wedge h^j\wedge h^k\,\epsilon_{ijk} + \tfrac{1}{\sqrt{2}}g^{-2}\,s\,c\,X^{-2}*F^i_{(2)}\wedge h^i\wedge d\xi\,, \\
e^{-\hat{\phi}}\,\hat{*}\hat{F}_{(3)} &= \tfrac{1}{2}g^{-4}\,s^{5/3}\,c^3\,\Delta^{-1}\,X*F_{(3)}\wedge d\xi\wedge\epsilon_{(3)} - \tfrac{1}{4}g^{-3}\,s^{2/3}\,c^4\,\Delta^{-1}\,X^{-1}*F_{(2)}\wedge\epsilon_{(3)}\,, \\
e^{\frac{3}{2}\hat{\phi}}\,\hat{*}\hat{F}_{(2)} &= \tfrac{1}{2\sqrt{2}}g^{-4}\,s^{-1/3}\,c^3\,X^{-2}*F_{(2)}\wedge d\xi\wedge\epsilon_{(3)}\,, \\
\hat{*}d\hat{\phi} &= -\tfrac{1}{2}g^{-4}\,s^{1/3}\,c^3\,(X\,c^2 + 2X^{-3}\,s^2)\,\Delta^{-1}\,X^{-1}*dX\wedge d\xi\wedge\epsilon_{(3)} \\
&+\tfrac{1}{16}g^{-2}\,s^{1/3}\,c^3\,(\Delta^{-1}\,\partial_\xi\Delta - \tfrac{10}{3}\cot\xi)X^{-2}\,\epsilon_{(6)}\wedge\epsilon_{(3)}\,,
\end{aligned} \tag{201}
$$

where $\epsilon_{(6)}$ is the volume form of the metric ds_6^2.

The action for massive type IIA supergravity in the Einstein frame reads

$$
\begin{aligned}
S^{10}_{IIA} \;=\; & \int \Big(\tfrac{1}{2}\hat{R} \star \mathbb{1} - \frac{1}{4} d\hat{\phi} \wedge \star d\hat{\phi} - \frac{1}{4} e^{-\hat{\phi}} \hat{F}_3 \wedge \star \hat{F}_3 - \frac{1}{4} e^{\frac{1}{2}\hat{\phi}} \hat{F}_4 \wedge \star \hat{F}_4 \\
- \; & m^2 e^{\frac{3}{2}\hat{\phi}} \hat{B}_2 \wedge \star \hat{B}_2 - m^2 e^{\frac{5}{2}\hat{\phi}} \star \mathbb{1} \\
+ \; & \frac{1}{4} d\hat{C}_3 \wedge d\hat{C}_3 \wedge \hat{B}_2 + \frac{1}{6} m d\hat{C}_3 \wedge \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 + \frac{1}{20} m^2 \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 \Big) \\
+ \; & \int \sqrt{-\hat{g}}\, d^{10}X \Big[- \bar{\hat{\Psi}}_M \Gamma^{MNP} D_N \hat{\Psi}_P - \tfrac{1}{2} \bar{\hat{\langle}} \Gamma^M D_M \hat{\langle} - \tfrac{1}{2} (d\hat{\phi})_N \bar{\hat{\langle}} \Gamma^M \Gamma^N \hat{\Psi}_M \\
- \; & \frac{1}{96} e^{\frac{1}{4}\hat{\phi}} (\hat{F}_4)_{PRST} \Big(\bar{\hat{\Psi}}^M \Gamma_{[M} \Gamma^{PRST} \Gamma_{N]} \hat{\Psi}^N + \frac{1}{2} \bar{\hat{\langle}} \Gamma^M \Gamma^{PRST} \hat{\Psi}_M + \frac{3}{8} \bar{\hat{\langle}} \Gamma^{PRST} \hat{\langle} \Big) \\
+ \; & \frac{1}{24} e^{-\frac{1}{2}\hat{\phi}} (\hat{F}_3)_{PRS} \Big(\bar{\hat{\Psi}}^M \Gamma_{[M} \Gamma^{PRS} \Gamma_{N]} \Gamma_{11} \hat{\Psi}^N + \bar{\hat{\langle}} \Gamma^M \Gamma^{PRS} \Gamma_{11} \hat{\Psi}_M \Big) \\
+ \; & \frac{1}{4} m e^{\frac{3}{4}\hat{\phi}} \hat{B}_{PR} \Big(\bar{\hat{\Psi}}^M \Gamma_{[M} \Gamma^{PR} \Gamma_{N]} \Gamma_{11} \hat{\Psi}^N + \frac{3}{4} \bar{\hat{\langle}} \Gamma^M \Gamma^{PR} \Gamma_{11} \hat{\Psi}_M + \frac{5}{8} \bar{\hat{\langle}} \Gamma^{PR} \Gamma_{11} \hat{\langle} \Big) \\
- \; & \tfrac{1}{2} m e^{\frac{5}{4}\hat{\phi}} \bar{\hat{\Psi}}_M \Gamma^{MN} \hat{\Psi}_N - \frac{5}{4} m e^{\frac{5}{4}\hat{\phi}} \bar{\hat{\langle}} \Gamma^M \hat{\Psi}_M + \frac{21}{16} m e^{\frac{5}{4}\hat{\phi}} \bar{\hat{\langle}} \hat{\langle} \Big] \; .
\end{aligned}
\tag{202}
$$

This action is a generalisation of the type IIA supergravity that is obtained from the low-energy limit of type IIA string theory, although some care must be taken and he massless limit is $m \to 0$.

We now turn to notation and field content. The indices $M, N \ldots$ run from 0 to 9, and the ten dimensional space-time coordinates are X^M. In the Neveau-Schwarz-Neveau-Schwarz (NS-NS) sector the action contains the bosonic fields $\hat{\phi}, \hat{B}_2, \hat{g}$, which are the ten-dimensional dilaton, a massive two-form and the metric, together with the fermionic fields $\hat{\Psi}, \hat{\langle}$, which are the gravitino and dilatino. The Ramond-Ramond (RR) sector contains the three-form $\hat{C}_3$ and a one-form $\hat{A}^0$ which is eliminated by a gauge transformation of $\hat{B}_2$. The field strengths in the action are given by

$$
\hat{F}_4 := d\hat{C}_3 + m\hat{B}_2 \wedge \hat{B}_2 \hat{F}_3 := d\hat{B}_2 \; . \tag{203}
$$

The terms in the ten-dimensional action (202) will contribute to the gravitino masses by

$$
\begin{aligned}
S^{10}_{\text{mass}} = & \int \sqrt{-\hat{g}}\, d^{10}X \Big[- \bar{\hat{\Psi}}_\mu \Gamma^{\mu n \nu} D_n \hat{\Psi}_\nu - \frac{1}{96} e^{\frac{1}{4}\hat{\phi}} (\hat{F}_4)_{prst} \bar{\hat{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{prst} \Gamma_{\nu]} \hat{\Psi}^\nu \\
& - \frac{1}{96} e^{\frac{1}{4}\hat{\phi}} (\hat{F}_4)_{\rho\sigma\delta\epsilon} \bar{\hat{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{\rho\sigma\delta\epsilon} \Gamma_{\nu]} \hat{\Psi}^\nu + \frac{1}{24} e^{-\frac{1}{2}\hat{\phi}} (\hat{F}_3)_{prs} \bar{\hat{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{prs} \Gamma_{\nu]} \Gamma_{11} \hat{\Psi}^\nu \\
& + \frac{1}{4} m e^{\frac{3}{4}\hat{\phi}} \hat{B}_{pr} \bar{\hat{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{pr} \Gamma_{\nu]} \Gamma_{11} \hat{\Psi}^\nu - \tfrac{1}{2} m e^{\frac{5}{4}\hat{\phi}} \bar{\hat{\Psi}}_\mu \Gamma^{\mu\nu} \hat{\Psi}_\nu \Big] \; .
\end{aligned}
\tag{204}
$$

8 IIB Supergravity

One of the most striking features of perturbative superstring theory in ten dimensions is the absence of anomalies. In the type-IIB theory this is realized by miraculous cancellations between various contributions, while in the type-I and heterotic theories the Green-Schwarz mechanism generates anomalous couplings that exactly cancel the contributions of fermion loops, once one restricts the gauge group to be $SO(32)$ for the type I theory and $SO(32)$ or $E_8 \times E_8$ for the heterotic theory. All these $N = 1$ theories are very interesting, since they can be naturally compactified to rich spectra of $N=1$ theories in four dimensions. In this context, an interesting intermediate step is the study of $(1,0)$ vacua in six dimensions, since in these compactifications the absence of anomalies is a strong restriction on the low-energy physics. The equations of motion of type IIB supergravity theory can not be obtained from a covariant action because of the presence of a four-form gauge field with the self-dual field strength in the spectrum. This gauge field couples to a self-dual three-brane which can give rise to string solution in $D \leq 8$. But, we are not going to consider this type of string solution and set the corresponding field-strength F_5 to zero. There are also magnetically charged string solution for type II theory in $D \leq 6$, but since we are not restricting ourselves to any particular dimensionality we will not consider those kinds of solutions also. Now as we set $F_5 = 0$, the type IIB equations of motion can be derived from the following covariant action

$$\begin{aligned}\tilde{S}^{\rm IIB}_{10} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\Big[& e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\right) \\ & -\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{12}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\Big] \,.\end{aligned} \tag{205}$$

Here $\mu, \nu, \ldots = 0, 1, \ldots, D-1$ are the space-time indices and $m, n, \ldots = D, \ldots, 9$ are the internal indices. The metric $G_{\mu\nu}$, the dilaton ϕ and the antisymmetric tensor $B^{(1)}_{\mu\nu}$ (with $H^{(1)} = dB^{(1)}$) represent the massless modes in the NS-NS sector of type IIB theory. Also the scalar χ and $B^{(2)}_{\mu\nu}$ (with $H^{(2)} = dB^{(2)}$) represent the massless modes in the R-R sector. The reduced action takes the form

$$\begin{aligned}S_D = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\Big[& e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}G_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} + \frac{1}{4}\partial_\mu G_{mn}\partial^\mu G^{mn} \\ & -\frac{1}{4}G^{mp}G^{nq}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq} - \frac{1}{4}G^{mp}H^{(1)}_{\mu\nu\,m}H^{(1)\,\mu\nu}_{\;\;p} - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\Big) \\ & -\frac{1}{2}\Delta\partial_\mu\chi\partial^\mu\chi - \frac{1}{4}\Delta G^{mp}G^{nq}\left(\partial_\mu B^{(2)}_{mn} + \chi\partial_\mu B^{(1)}_{mn}\right)\left(\partial^\mu B^{(2)}_{pq} + \chi\partial^\mu B^{(1)}_{pq}\right) \\ & -\frac{1}{4}\Delta G^{mp}\left(H^{(2)}_{\mu\nu\,m} + \chi H^{(1)}_{\mu\nu\,m}\right)\left(H^{(2)\,\mu\nu}_{\;\;p} + \chi H^{(1)\,\mu\nu}_{\;\;p}\right) \\ & -\frac{1}{12}\Delta\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\Big]\end{aligned} \tag{206}$$

The corresponding field-strengths are given below

$$H^{(i)}_{\mu mn} = \eta H^{(i)}_{\mu mn} \;=\; \partial_\mu B^{(i)}_{mn}, \qquad\qquad H^{(i)}_{\mu\nu m} = F^{(i)}_{\mu\nu m} - B^{(i)}_{mn}F^{(3)\,n}_{\mu\nu} \tag{207}$$

where $F^{(i)}_{\mu\nu\,m} = \partial_\mu A^{(i)}_{\nu\,m} - \partial_\nu A^{(i)}_{\mu\,m}$ and $F^{(3)\,m}_{\mu\nu} = \partial_\mu A^{(3)\,m}_\nu - \partial_\nu A^{(3)\,m}_\mu$ and finally

$$H^{(i)}_{\mu\nu\lambda} \;=\; \partial_\mu B^{(i)}_{\nu\lambda} - F^{(3)\,m}_{\mu\nu}A^{(i)}_{\lambda\,m} + \text{cyc. in } \;\mu\nu\lambda \tag{208}$$

The reduced action (205) have an $SL(2,R)$ invariance which can be better understood by rewriting the action in the Einstein frame. The metric in the Einstein frame is related with the string metric as given in the second section. The action (205) in the Einstein frame takes the following form

$$
\begin{aligned}
S_D = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\Big[&R - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{1}{2}e^{2\tilde{\phi}}\partial_\mu\chi\partial^\mu\chi + \frac{1}{8}\partial_\mu\log\beta\Delta\partial^\mu\log\beta\Delta \\
&+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} - \frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} - \frac{1}{4}(\beta\Delta)^{1/2}g^{mp}g^{nq}e^{-\tilde{\phi}}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq} \\
&-\frac{1}{4}(\beta\Delta)^{1/2}g^{mp}g^{nq}e^{\tilde{\phi}}\left(\partial_\mu B^{(2)}_{mn} + \chi\partial_\mu B^{(1)}_{mn}\right)\left(\partial^\mu B^{(2)}_{pq} + \chi\partial^\mu B^{(1)}_{pq}\right) \\
&-\frac{1}{4}(\beta\Delta)^{1/2}g^{mp}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu m}H^{(1)\,\mu\nu}_{\;\;p} + e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\,m} + \chi H^{(1)}_{\mu\nu\,m}\right)\left(H^{(2)\,\mu\nu}_{\;\;p} + \chi H^{(1)\,\mu\nu}_{\;\;p}\right)\right\} \\
&-\frac{1}{12}(\beta\Delta)^{1/2}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda} + e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right\}\Big]
\end{aligned}
\tag{209}
$$

where we have defined $\tilde{\phi} = \phi + \frac{1}{2}\log\Delta$. Also, $G_{mn} = e^{\frac{4}{D-2}\phi}g_{mn}$ and $\Delta = e^{2\frac{(10-D)}{(D-2)}\phi}\beta\Delta$ with $(\beta\Delta)^2 = (\det g_{mn})$. If we define the following $SL(2,R)$ matrix then the action (90) can be expressed in the manifestly $SL(2,R)$ invariant form as

$$
\begin{aligned}
S_D = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\Big[&R + \frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1} + \frac{1}{8}\partial_\mu\log\beta\Delta\partial^\mu\log\beta\Delta + \frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} \\
&-\frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} - \frac{1}{4}(\beta\Delta)^{1/2}g^{mp}g^{nq}\partial_\mu\mathcal{B}^T_{mn}\mathcal{M}_D\partial^\mu\mathcal{B}_{pq} \\
&-\frac{1}{4}(\beta\Delta)^{1/2}g^{mp}\mathcal{H}^T_{\mu\nu\,m}\mathcal{M}_D\mathcal{H}^{\mu\nu}_{\;\;p} - \frac{1}{12}(\beta\Delta)^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\Big]
\end{aligned}
\tag{210}
$$

Here we have defined $\mathcal{B}_{mn} \equiv \begin{pmatrix} B^{(1)}_{mn} \\ B^{(2)}_{mn}\end{pmatrix}$, $\mathcal{H}_{\mu\nu\,m} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\,m} \\ H^{(2)}_{\mu\nu\,m}\end{pmatrix}$, $\mathcal{H}_{\mu\nu\lambda} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda}\end{pmatrix}$. The action (91) is invariant under the following global $SL(2,R)$ transformation

$$
\begin{aligned}
\mathcal{M}_D &\to \Lambda\mathcal{M}_D\Lambda^T, & \mathcal{B}_{mn} &\to (\Lambda^{-1})^T\mathcal{B}_{mn} \\
\begin{pmatrix} A^{(1)}_{\mu\,m} \\ A^{(2)}_{\mu\,m}\end{pmatrix} \equiv \mathcal{A}_{\mu\,m} &\to (\Lambda^{-1})^T\mathcal{A}_{\mu\,m}, & \begin{pmatrix} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu}\end{pmatrix} \equiv \mathcal{B}_{\mu\nu} &\to (\Lambda^{-1})^T\mathcal{B}_{\mu\nu}
\end{aligned}
\tag{211}
$$

where $\Lambda = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is the $SL(2,R)$ transformation matrix and a, b, c, d are constants satisfying $ad - bc = 1$. If we set $G_{mn} = \delta_{mn}$, $\Delta = 1$, $A^{(3)\,m}_\mu = A^{(i)}_{\mu\,n} = B^{(i)}_{mn} = 0$, then the action (91) reduces

$$
\begin{aligned}
S_D = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\Big[&R + \frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1} \\
&+\frac{1}{8}\partial_\mu\log\beta\Delta\partial^\mu\log\beta\Delta + \frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} - \frac{1}{12}(\beta\Delta)^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\Big]
\end{aligned}
\tag{213}
$$

This action is $SL(2,R)$ invariant under the transformation (211). Note that both g_{mn} and $\beta\Delta$ are $SL(2,R)$ invariant. Also, $\mathcal{M}_D$ in (92) is as given in (90) with $\tilde{\phi}$ replaced by ϕ, the D-dimensional dilaton as $\Delta = 1$ in this case. Note also that although we have set $G_{mn} = \delta_{mn}$ and $\Delta = 1$, but as they are not $SL(2,R)$ invariant, non-trivial values of G_{mn} and Δ will be generated through $SL(2,R)$ transformation. It can be easily checked that the $SL(2,R)$ invariant action (212) gets precisely converted to the effective action considered by Dabholkar by setting the R-R fields to zero. Thus, we note that the action in the Einstein frame is a special case of the more general type II action (212) and so the solution is a particular case of a general solution that we are going to construct. The 10-dimensional effective action is invariant under general coordinate transformations as well as the gauge transformations associated with the two antisymmetric tensor fields. When we examine the local symmetries of the theory in D-dimensions after dimensional reduction, we find that there is general coordinate transformation invariance in D-dimensions. The Abelian gauge transformation, associated with $\mathcal{A}^{\alpha}_{\mu}$, has its origin in 10-dimensional general coordinate transformations. The field strength $H^{(i)}_{\mu\nu\alpha}$ is invariant under a suitable gauge transformation once we define the gauge transformation for $F^{(i)}_{\mu\nu\alpha}$ since $\mathcal{F}^{\alpha}_{\mu\nu}$ is gauge invariant under the gauge transformation of $\mathcal{A}$-gauge fields. Finally, the tensor field strength $H^{(i)}_{\mu\nu\rho}$, defined above, can be shown to be gauge invariant by defining appropriate gauge transformations for $B^{(i)}_{\mu\nu}$:

$$\delta B^{(i)}_{\mu\nu} = \partial_\mu \xi^{(i)}_\nu - \partial_\nu \xi^{(i)}_\mu . \tag{214}$$

The D-dimensional effective action takes the following form

$$\begin{aligned} S_E \;\; = \;\; & \int d^D x \sqrt{-g}\sqrt{\mathcal{G}} \Big\{ R + \frac{1}{4}[\partial_\mu \mathcal{G}_{\alpha\beta}\partial^\mu \mathcal{G}^{\alpha\beta} + g^{\mu\nu}\partial_\mu \log\mathcal{G}\partial_\nu \log\mathcal{G} - g^{\mu\lambda}g^{\nu\rho}\mathcal{G}_{\alpha\beta}\mathcal{F}^{\alpha}_{\mu\nu}\mathcal{F}^{\beta}_{\lambda\rho}] \\ & -\frac{1}{4}\mathcal{G}^{\alpha\beta}\mathcal{G}^{\gamma\delta}\partial_\mu B^{(i)}_{\alpha\gamma}\mathcal{M}_{ij}\partial^\mu B^{(j)}_{\beta\delta} - \frac{1}{4}\mathcal{G}^{\alpha\beta}g^{\mu\lambda}g^{\nu\rho}H^{(i)}_{\mu\nu\alpha}\mathcal{M}_{ij}H^{(j)}_{\lambda\rho\beta} \\ & -\frac{1}{12}H^{(i)}_{\mu\nu\rho}\mathcal{M}_{ij}H^{(j)\mu\nu\rho} + \frac{1}{4}Tr(\partial_\mu \mathcal{M}\Sigma\partial^\mu \mathcal{M}\Sigma) \Big\} \end{aligned} \tag{215}$$

The above action is expressed in the Einstein frame, $\mathcal{G}$ being determinant of $\mathcal{G}_{\alpha\beta}$. If we demand $SL(2,R)$ invariance of the above action, then the backgrounds are required to satisfy following transformation properties:

$$\mathcal{M} \to \Lambda\mathcal{M}\Lambda^T, \quad H^{(i)}_{\mu\nu\rho} \to (\Lambda^T)^{-1}{}_{ij}H^{(j)}_{\mu\nu\rho} A^{(i)}_{\mu\alpha} \to (\Lambda^T)^{-1}{}_{ij}A^{(j)}_{\mu\alpha}, \quad B^{(i)}_{\alpha\beta} \to (\Lambda^T)^{-1}{}_{ij}B^{(j)}_{\alpha\beta} \tag{216}$$

It is evident from the D-dimensional action that dilaton and axion interact with antisymmetric tensor fields, gauge fields and the scalars due to the presence of $\mathcal{M}$ matrix in various terms and these interaction terms respect the $SL(2,R)$ symmetry. It is important know what type of dilatonic potential is admissible in the above action which respects the S-duality symmetry. The only permissible interaction terms preserving the symmetry are of the form $Tr[\mathcal{M}\Sigma]^n$. It is easy to check using the properties of Σ and $\mathcal{M}$ matrices, such as $Tr(\mathcal{M}\Sigma) = 0$ and $Tr(\mathcal{M}\Sigma\mathcal{M}\Sigma) = 2$, that

$$Tr[\mathcal{M}\Sigma]^n = 0, \qquad Tr[\mathcal{M}\Sigma]^n = 2 \tag{217}$$

For odd $n \in Z$ and even $n \in Z$ respectively. Therefore, we reach a surprizing conclusion that the presence of interaction terms of the form only adds constant term which amounts to adding cosmological constant term to the reduced action. Note that the Einstein metric is $SL(2,R)$ invariant and one can add terms involving higher powers of curvature (higher derivatives of metric) to the action and maintain the symmetry. However, we are considering the case when the gravitational part of the action has the Einstein-Hilbert term only.

9 $D = 11$ Supergravity

The dimension D = 11 is the maximal dimension for which one can realize supersymmetry in terms of an ordinary supergravity theory. The conjectured 11-dimensional M-theory is required to have 11-dimensional supergravity as a low energy limit. The low-energy approximation of M-theory is given by the eleven-dimensional supergravity which describes the dynamics of the $N = 1$ supermultiplet in eleven dimensions. This contains the metric $\hat{g}_{MN}$ and an antisymmetric tensor field $\hat{A}_{MNP}$ as bosonic components and the gravitino $\hat{\Psi}_M$, which is a Majorana spinor in eleven dimensions, as their fermionic superpartner. We use D for the spinor covariant derivative. The indices $M, N \ldots = 0, 1, \ldots, 10$ denote curved eleven-dimensional indices. The supergravity action can be written as

$$\begin{aligned} S &= \frac{1}{2}\int_{M_{11}} \sqrt{-\hat{g}}\, d^{11}x \left[\hat{R} - \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNP} D_N \hat{\Psi}_P - \frac{1}{2}\hat{F}_4 \wedge *\hat{F}^4\right] \\ & -\frac{1}{192}\int_{M_{11}} \sqrt{-\hat{g}}\, d^{11}x \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS}\hat{\Psi}_N (\hat{F}_4)_{PQRS} - \frac{1}{2}\int_{M_{11}} \hat{F}_4 \wedge *\hat{C}_4 \\ & -\frac{1}{12}\int_{M_{11}} \hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3 \, , \end{aligned} \tag{218}$$

where we have set the eleven-dimensional Newton's constant to unity and denoted

$$(\hat{C}_4)_{MNPQ} = 3\bar{\hat{\Psi}}_{[M}\hat{\Gamma}_{NP}\hat{\Psi}_{Q]} \, . \tag{219}$$

The spinor conjugation is defined to be $\bar{\hat{\Psi}}_M = \hat{\Psi}^\dagger_M \Gamma^0$. We have ignored the four-fermionic terms, which play no role in our analysis, and kept only bilinear terms in the gravitino field $\hat{\Psi}_M$. The action (218) is invariant under the usual supersymmetry transformations. For the gravitino this takes the form

$$\delta\Psi_M = D_M\epsilon + \frac{1}{288}\left(\hat{\Gamma}_M{}^{NPQR} - 8\delta^N_M \hat{\Gamma}^{NPQ}\right)(\hat{F}_4)_{NPQR}\,\epsilon \, , \tag{220}$$

where ϵ is the Majorana spinor parameterising the supersymmetry transformation.

Inserting the decomposition $\hat{\Psi}_M = (\hat{\psi}_M + \hat{\psi}^*_M) \otimes \eta$ in the gravitino kinetic term from (218) and keeping only the terms which are relevant for the gravitino mass term one finds

$$\begin{aligned} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNP} D_N \hat{\Psi}_P &= \bar{\hat{\Psi}}_\mu \hat{\Gamma}^{\mu n \nu} D_n \hat{\Psi}_\nu + \text{ terms not contributing to gravitino mass} \\ &= -(\bar{\hat{\psi}}_\mu + \overline{\hat{\psi}^*}_\mu)\hat{\gamma}^{\mu\nu}\gamma(\hat{\psi}_\nu + \hat{\psi}^*_\nu)\eta^T \gamma^n D_n \eta + \ldots \end{aligned} \tag{221}$$

We denote the internal fluxes by $\hat{F}_4$ and do not discuss their origin here. The relevant term can be written

$$\begin{aligned} \bar{\hat{\Psi}}_M \hat{\Gamma}^{MNPQRS}\hat{\Psi}_N (\hat{F}_4)_{PQRS} &= \bar{\hat{\Psi}}_\mu \Gamma^{\mu\nu pqrs}\hat{\Psi}_\nu (\hat{F}_4)_{pqrs} + \ldots \\ &= (\bar{\hat{\psi}}_\mu + \overline{\hat{\psi}^*}_\mu)\hat{\gamma}^{\mu\nu}(\hat{\psi}_\nu + \hat{\psi}^*_\nu)\eta^T \gamma^{pqrs}\eta (\hat{F}_4)_{pqrs} + \ldots \end{aligned} \tag{222}$$

The total action is

$$\begin{aligned} S_T = \frac{2}{\kappa^2}\int_{\mathcal{M}} d^{11}x\sqrt{g}\,\Big(&-\frac{1}{2}R(\Omega) - \frac{1}{2}\bar{\psi}_I \Gamma^{IJK} D_J(\Omega^*)\psi_K - \frac{1}{48}G_{IJKL}G^{IJKL} \\ &- \frac{\sqrt{2}}{192}\left(\bar{\psi}_I \Gamma^{IJKLMP}\psi_P + 12\bar{\psi}^J \Gamma^{KL}\psi^M\right) G^*_{JKLM} - \frac{\sqrt{2}}{10!}\epsilon^{I_1 \ldots I_{11}}(C \wedge G \wedge G)_{I_1 \ldots I_{11}}\Big) \\ &+ \frac{2}{\kappa^2}\int_{\partial\mathcal{M}} d^{11}x\sqrt{g}\left(K \mp \frac{1}{4}\bar{\psi}_A \Gamma^{AB}\psi_B + \frac{1}{2}\bar{\psi}_A \Gamma^A \psi_N\right) - \frac{2\epsilon}{\kappa^2}\int_{\partial\mathcal{M}} d^{11}x\sqrt{g}\,\Big(\frac{1}{4}F^a{}_{AB}F^{aAB} \\ &+ \frac{1}{2}\bar{\chi}^a \Gamma^A D_A(\hat{\Omega})\chi^a + \frac{1}{4}\bar{\psi}_A \Gamma^{BC}\Gamma^A F^{a*}{}_{BC}\chi^a + \frac{1}{192}\bar{\chi}\Gamma_{ABC}\chi\bar{\psi}_D \Gamma^{ABCDE}\psi_E\Big) \end{aligned} \tag{223}$$

The version of equation which describes the supersymmetric variation of the action and includes the four fermi terms is

$$\delta S = \frac{2}{\kappa^2}\int_{\partial\mathcal{M}} d^{11}x\sqrt{g}\,\Big(\bar\eta D_A(\hat\Omega)\theta^A + \frac{1}{2}(K^{AB}-Kg^{AB})\bar\eta\Gamma_A\psi_B - \frac{1}{2}\bar\eta D_A(\hat\Omega)J^A$$
$$-\frac{\sqrt{2}}{96}\bar\eta\Gamma^{ABCDE}\psi_A\hat G_{BCDE} + \frac{\sqrt{2}}{8}\bar\eta\Gamma^{AB}\psi^C\hat G_{ABCN} + \epsilon\bar\eta\Gamma^{AB}(D_A(\hat\Omega)\Gamma)\psi_B$$
$$-\epsilon\bar\eta[\Gamma,\Gamma^{AB}]P_- D_A(\hat\Omega)\psi_B + \delta g_{AB}\,p^{AB} + \delta r_{AB}q^{AB} + \delta C_{ABC}p^{ABC}$$
$$+\frac{1}{2}\bar\eta D_A(\hat\Omega)J^A - \epsilon\delta\bar\chi\Gamma^A\Gamma^{BC}\hat F_{BC}\psi_A + \frac{1}{4}\epsilon\bar\eta\Gamma_B\chi\, D_A(\hat\omega)(\bar\psi_C\Gamma^{ABC}\chi)$$
$$-\frac{1}{4}\epsilon\bar\eta\Gamma^{ABC}\chi\, D_A(\hat\Omega)(\bar\psi_B\Gamma_C\chi) + \frac{1}{2}\epsilon\bar\eta\Gamma_A\chi\, D_B(\hat{\mathcal{K}})\hat F^{AB}\Big). \tag{224}$$

where $D(\hat{\mathcal{K}}) = D(\hat\Omega) - D(\omega)$.

The coefficients appearing here are the field equations and boundary conditions, given explicitly by

$$p^{AB} = -\frac{1}{2}\left(\hat K^{AB} - \hat K g^{AB}\right) + \frac{1}{4}\kappa^2\hat T^{AB} \tag{225}$$

$$p^{ABC} = \hat G^{ABCN} + \frac{\sqrt{2}}{4}\bar\psi_D\Gamma^{DEABC}P_+\psi_E - \frac{\sqrt{2}}{3\times 7!}\epsilon^{ABCD_1\ldots D_7}(C\wedge G)_{D_1\ldots D_7} \tag{226}$$

$$\Xi = \epsilon\left(-\Gamma^A D_A(\hat\Omega)\chi - \frac{1}{4}\Gamma^A\Gamma^{BC}\hat F_{BC}\psi_A\right) \tag{227}$$

$$Y^A = \epsilon\left(-D_B(\omega)\hat F^{AB} + \frac{1}{2}D_B(\omega)(\bar\psi_C\Gamma^{BAC}\chi) + \frac{1}{8}\bar\psi_D\Gamma^{ABCDE}\psi_E F_{BC}\right) \tag{228}$$

The variation of the metric is the most complicated. This has been simplified by assuming that $\hat T^{AB}$ is in supercovariant form,

$$\frac{1}{4}\kappa^2\hat T^{AB} = \epsilon\Big(\frac{1}{2}\hat F^{aAC}\hat F^{aB}{}_C - \frac{1}{8}g^{AB}\hat F^{aCD}\hat F^a{}_{CD} + \frac{1}{4}\bar\chi^a\Gamma^A D^B(\hat\Omega)\chi^a - \frac{1}{4}g^{AB}\bar\chi^a\Gamma^C D_C(\hat\Omega)\chi^a$$
$$+\frac{1}{16}\bar\chi^a\Gamma^A\Gamma^{CD}\psi^B\hat F^a{}_{CD} - \frac{1}{16}\bar\chi^a\Gamma^C\Gamma^{DE}\psi_C\hat F^a{}_{DE}g^{AB}\Big) \tag{229}$$

We will require explicit expressions for two of the boundary terms,

$$\theta^A = -\Gamma^{AB}P_+\psi_B \tag{230}$$

$$p^{AB} = -\frac{1}{2}\left(K^{AB} - Kg^{AB}\right) + 2\bar\psi^{(A}\Gamma^{B)C}P_+\psi_C - g^{AB}\bar\psi_C\Gamma^{CD}P_+\psi_D \tag{231}$$

In the new theory, the boundary conditions

$$\hat G_{ABCD} = -3\sqrt{2}\epsilon\hat F^a{}_{[AB}\hat F^a{}_{CD]} + \sqrt{2}\epsilon\bar\chi^a\Gamma_{[ABC}D_{D]}(\hat\Omega)\chi^a + \frac{\sqrt{2}}{4}\epsilon\bar\chi^a\Gamma_{[ABC}\Gamma^{EF}\psi_{D]}\hat F^a{}_{EF} \tag{232}$$

lead to a similar effect.

Now we are ready to examine the variation of the three terms in c_{ABC} under supersymmetry transformations. Firstly, using the transformation (??), equations (??), (??) and gamma-matrix identities

$$\delta C_{ABC} = -\frac{\sqrt{2}}{96}\epsilon\bar\eta\Gamma_{ABC}{}^{DE}\chi\hat F_{DE} + \frac{\sqrt{2}}{16}\epsilon\bar\eta\Gamma_{[AB}{}^D\chi\hat F_{C]D} - \frac{3\sqrt{2}}{16}\epsilon\bar\eta\Gamma_{[A}\chi\hat F_{BC]} + \frac{\sqrt{2}}{8}\epsilon\bar\eta\{\Gamma,\Gamma_{[AB}\}P_-\psi_{C]} \tag{233}$$

9.1 Hořava-Witten Theory

Hořava-Witten theory can be formulated as an expansion in the 11-dimensional gravitational coupling κ. To lowest order in this expansion, Hořava-Witten theory is 11-dimensional supergravity which is of order κ^{-2}, with the fields restricted under the $\mathbb{Z}_2$ action. In the upstairs picture, the action is

$$S_{\rm SG} = -\frac{1}{\kappa^2}\int_{\mathbf{M}^{11}} d^{11}x\sqrt{g}\Bigg[\frac{1}{2}R + \frac{1}{2}\overline{\psi}_I\Gamma^{IJK}D_J(\Omega)\psi_K + \frac{1}{48}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{192}\left(\overline{\psi}_I\Gamma^{IJKLMN}\psi_N + 12\overline{\psi}^J\Gamma^{KL}\psi^M\right)G_{JKLM} + \frac{\sqrt{2}}{3456}\epsilon^{I_1\dots I_{11}}C_{I_1I_2I_3}G_{I_4\dots I_7}G_{I_8\dots I_{11}} + (\text{Fermi})^4\Bigg]. \quad (234)$$

The terms which are quartic in the gravitino can be absorbed into the definition of supercovariant objects. The condition (**??**) means that the gravitino is chiral from a 10-dimensional perspective, and so the theory has a gravitational anomaly localized on the fixed planes. $S_{\rm SG}$ is invariant under the local supersymmetry transformations

$$\delta e_I^m = \frac{1}{2}\bar{\eta}\Gamma^m\psi_I \quad (235)$$

$$\delta C_{IJK} = -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[IJ}\psi_{K]} \quad (236)$$

$$\delta\psi_I = D_I(\Omega)\eta + \frac{\sqrt{2}}{288}(\Gamma_{IJKLM} - 8g_{IJ}\Gamma_{KLM})G^{JKLM}\eta + (\text{Fermi})^2 \quad (237)$$

whose infinitesimal spacetime dependent Grassmann parameter η satisfies the orbifold condition

$$\eta(x^{11}) = \Gamma_{11}\eta(-x^{11}). \quad (238)$$

This condition means that the theory has 32 supersymmetries in the bulk, but only 16 chiral supersymmetries on the orbifold fixed planes. At this order in κ, the 4-form field strength G_{IJKL} satisfies the boundary conditions

$$G_{\bar{I}\bar{J}\bar{K}\bar{L}}|_{x^{11}=0} = 0 \quad (239)$$

$$G_{\bar{I}\bar{J}\bar{K}\bar{L}}|_{x^{11}=\pi\rho} = 0, \quad (240)$$

the equation of motion

$$D_I(\Omega)G^{IJKL} = 0, \quad (241)$$

and the Bianchi identity

$$(dG)_{IJKLM} = 0. \quad (242)$$

Cancellation of the gravitational anomaly requires the introduction of one E_8 Yang-Mills supermultiplet $(A_{\bar{I}}^{(i)a}, \chi^{(i)a})$ on each orbifold fixed plane $\mathbf{M}^{10}_{(i)}$ $(i = 1, 2)$. The minimal Yang-Mills action is

$$S_{\rm YM} = -\frac{1}{\lambda^2}\sum_{i=1}^{2}\int_{\mathbf{M}^{10}_{(i)}} d^{10}x\sqrt{g}\ \text{tr}\left(\frac{1}{4}F^{(i)}_{\bar{I}\bar{J}}F^{(i)\bar{I}\bar{J}} + \frac{1}{2}\bar{\chi}^{(i)}\Gamma^{\bar{I}}D_{\bar{J}}(\Omega)\chi^{(i)}\right) \quad (243)$$

where λ is the 10-dimensional gauge coupling. This action is invariant under the global supersymmetry transformations

$$\delta A^{(i)a}_{\bar{I}} = \frac{1}{2}\bar{\eta}\Gamma_{\bar{I}}\chi^{(i)a} \tag{244}$$

$$\delta\chi^{(i)a} = -\frac{1}{4}\Gamma^{\bar{I}\bar{J}}F^{(i)a}_{\bar{I}\bar{J}}\eta. \tag{245}$$

The challenge is then to add interactions and modify the supersymmetry transformation laws so that $S = S_{\rm SG} + S_{\rm YM} + \cdots$ is locally supersymmetric. This involves coupling the gravitino to the Yang-Mills supercurrent. However, since the gravitino lives in the 11-dimensional bulk, while the Yang-Mills supermultiplets live on the 10-dimensional fixed planes, a locally supersymmetric theory cannot be achieved simply by adding interactions on the fixed planes. To achieve local supersymmetry, the Bianchi identity must be modified to read

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = 8\pi^2\sqrt{2}\,\frac{\kappa^2}{\lambda^2}\Bigg\{J^{(0)}\delta(x^{11}) + J^{(N+1)}\delta(x^{11}-\pi\rho) + \frac{1}{2}\sum_{n=1}^{N} J^{(n)}\left[\delta(x^{11}-x_n) + \delta(x^{11}+x_n)\right]\Bigg\}_{\bar{I}\bar{J}\bar{K}\bar{L}} \tag{246}$$

where

$$J^{(0)} = -\frac{1}{16\pi^2}\left[\mathrm{tr}\left(F^{(1)}\wedge F^{(1)}\right) - \frac{1}{2}\,\mathrm{tr}\left(R\wedge R\right)\right]_{x^{11}=0} \tag{247}$$

$$J^{(N+1)} = -\frac{1}{16\pi^2}\left[\mathrm{tr}\left(F^{(2)}\wedge F^{(2)}\right) - \frac{1}{2}\,\mathrm{tr}\left(R\wedge R\right)\right]_{x^{11}=\pi\rho} \tag{248}$$

are the sources on the fixed planes at $x^{11} = x_0 \equiv 0$ and $x^{11} = x_{N+1} \equiv \pi\rho$, respectively, and $J^{(n)}$ $(n = 1, \ldots, N)$ are the M5-brane sources located at $x^{11} = x_1, \ldots, x_N$ $(0 \leq x_1 \leq \ldots \leq x_N \leq \pi\rho)$. Each M5-brane at $x = x_n$ is paired with a mirror M5-brane at $x = -x_n$ with the same source since the Bianchi identity must be even under the $\mathbb{Z}_2$ action.

With the modified Bianchi identity (246), $S = S_{\rm SG} + S_{\rm YM} + \ldots$ can be made locally supersymmetric. However, having gained supersymmetry, Yang Mills gauge invariance has been lost. The modified Bianchi identity implies that $G_{11\bar{I}\bar{J}\bar{K}}$ is invariant under the infinitesimal gauge transformations

$$\delta A^{(i)a}_{\bar{I}} = D_{\bar{I}}(\Omega)\epsilon^{(i)a} \tag{249}$$

if $C_{11\bar{I}\bar{J}}$ transforms as

$$\delta C_{11\bar{I}\bar{J}} = -\frac{\kappa^2}{6\sqrt{2}\lambda^2}\left[\delta(x^{11})\ \mathrm{tr}\left(\epsilon^{(1)}F^{(1)}_{\bar{I}\bar{J}}\right) + \delta(x^{11}-\pi\rho)\ \mathrm{tr}\left(\epsilon^{(2)}F^{(2)}_{\bar{I}\bar{J}}\right)\right]. \tag{250}$$

The quantum theory is anomaly free. Gauge, gravitational and mixed anomalies are cancelled with a refinement of the standard Green-Schwarz mechanism.

To order $\kappa^{-2+(2/3)}$, the Hořava-Witten action is

$$
\begin{aligned}
S = -\frac{1}{\kappa^2}\int_{\mathbf{M}^{11}} d^{11}x\sqrt{g}\Bigg[&\frac{1}{2}R + \frac{1}{2}\overline{\psi}_I\Gamma^{IJK}D_J\left(\tfrac{1}{2}(\Omega+\hat{\Omega})\right)\psi_K \\
&+\frac{\sqrt{2}}{384}\left(\overline{\psi}_I\Gamma^{IJKLMN}\psi_N + 12\overline{\psi}^J\Gamma^{KL}\psi^M\right)\left(G_{JKLM}+\hat{G}_{JKLM}\right) \\
&+\frac{1}{48}G_{IJKL}G^{IJKL} + \frac{\sqrt{2}}{3456}\epsilon^{I_1\ldots I_{11}}C_{I_1I_2I_3}G_{I_4\ldots I_7}G_{I_8\ldots I_{11}}\Bigg] \\
-\frac{1}{2\pi\kappa^2}\left(\frac{\kappa}{4\pi}\right)^{2/3}\sum_{i=1}^{2}\int_{\mathbf{M}^{10}_{(i)}} d^{10}x\sqrt{g}\Bigg[&\frac{1}{4}F^{(i)a}_{\bar{I}\bar{J}}F^{(i)a\bar{I}\bar{J}} + \frac{1}{2}\bar{\chi}^{(i)a}\Gamma^{\bar{I}}D_{\bar{J}}(\hat{\Omega})\chi^{(i)a} \\
&+\frac{1}{8}\bar{\psi}_I\Gamma^{\bar{J}\bar{K}}\Gamma^{\bar{I}}\left(F^{(i)a}_{\bar{J}\bar{K}} + \hat{F}^{(i)a}_{\bar{J}\bar{K}}\right)\chi^{(i)a} - \frac{\sqrt{2}}{48}\bar{\chi}^{(i)a}\Gamma^{\bar{I}\bar{J}\bar{K}}\chi^{(i)a}\hat{G}_{\bar{I}\bar{J}\bar{K}11}\Bigg]. \qquad (251)
\end{aligned}
$$

where the quartic Fermi terms are absorbed into the supercovariant objects

$$\hat{\Omega}_{IJK} = \Omega_{IJK} + \frac{1}{8}\bar{\psi}^L\Gamma_{LIJKM}\psi^M \qquad (252)$$

$$\hat{G}_{IJKL} = G_{IJKL} + \frac{3\sqrt{2}}{4}\bar{\psi}_{[I}\Gamma_{JK}\psi_{L]} \qquad (253)$$

$$\hat{F}^{(i)a}_{\bar{I}\bar{J}} = F^{(i)a}_{\bar{I}\bar{J}} - \bar{\psi}_{[\bar{I}}\Gamma_{\bar{J}]}\chi^{(i)a}. \qquad (254)$$

This action is invariant under the local supersymmetry transformations

$$\delta e^m_I = \frac{1}{2}\bar{\eta}\Gamma^m\psi_I \qquad (255)$$

$$\delta\psi_{\bar{I}} = D_{\bar{I}}(\hat{\Omega})\eta + \frac{\sqrt{2}}{288}(\Gamma_{\bar{I}JKLM} - 8g_{\bar{I}J}\Gamma_{KLM})\hat{G}^{JKLM}\eta + \delta'\psi_{\bar{I}} \qquad (256)$$

$$\delta\psi_{11} = D_{11}(\hat{\Omega})\eta + \frac{\sqrt{2}}{288}(\Gamma_{11JKLM} - 8g_{11J}\Gamma_{KLM})\hat{G}^{JKLM}\eta + \delta'\psi_{11} \qquad (257)$$

$$\delta A^{(i)a}_{\bar{I}} = \frac{1}{2}\bar{\eta}\Gamma_{\bar{I}}\chi^{(i)a} \qquad (258)$$

$$\delta\chi^{(i)a} = -\frac{1}{4}\Gamma^{\bar{I}\bar{J}}F^{(i)a}_{\bar{I}\bar{J}}\eta + \delta'\chi^{(i)a} \qquad (259)$$

where the supersymmetry transformation law corrections are

$$\delta' C_{11\bar{J}\bar{K}} = \frac{1}{6\sqrt{2}}\frac{1}{2\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\sum_{i=1}^{2}\mathrm{tr}\left(A^{(i)}_{\bar{J}}\delta A^{(i)}_{\bar{K}} - A^{(i)}_{\bar{K}}\delta A^{(i)}_{\bar{J}}\right) \tag{260}$$

$$\begin{aligned}\delta'\psi_{\bar{I}} = -\frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\Big[&\delta(x^{11})\,\bar{\chi}^{(1)a}\Gamma^{\bar{J}\bar{K}\bar{L}}\chi^{(1)a}\\ &+\delta(x^{11}-\pi\rho)\,\bar{\chi}^{(2)a}\Gamma^{\bar{J}\bar{K}\bar{L}}\chi^{(2)a}\Big]\left(\Gamma_{\bar{J}\bar{K}\bar{L}} - 6g_{\bar{I}\bar{J}}\Gamma_{\bar{K}\bar{L}}\right)\eta\end{aligned} \tag{261}$$

$$\begin{aligned}\delta'\psi_{11} = +\frac{1}{576\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\Big[&\delta(x^{11})\,\bar{\chi}^{(1)a}\Gamma^{JKL}\chi^{(1)a}\\ &+\delta(x^{11}-\pi\rho)\,\bar{\chi}^{(2)a}\Gamma^{JKL}\chi^{(2)a}\Big]\Gamma_{JKL}\,\eta\end{aligned} \tag{262}$$

$$\delta'\chi^{(i)a} = \frac{1}{4}\bar{\psi}_{\bar{I}}\Gamma_{\bar{J}}\chi^{(i)a}\Gamma^{\bar{I}\bar{J}}\eta. \tag{263}$$

At this order in κ, the 4-form field strength G_{IJKL} satisfies the boundary conditions

$$G_{\bar{I}\bar{J}\bar{K}\bar{L}}|_{x^{11}=0} = -\frac{3}{\sqrt{2}}\frac{1}{2\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\left(F^{(1)a}_{[\bar{I}\bar{J}}F^{(1)a}_{\bar{K}\bar{L}]} - \frac{1}{2}\,\mathrm{tr}\,R_{[\bar{I}\bar{J}}R_{\bar{K}\bar{L}]}\right) \tag{264}$$

$$G_{\bar{I}\bar{J}\bar{K}\bar{L}}|_{x^{11}=\pi\rho} = +\frac{3}{\sqrt{2}}\frac{1}{2\pi}\left(\frac{\kappa}{4\pi}\right)^{2/3}\left(F^{(2)a}_{[\bar{I}\bar{J}}F^{(2)a}_{\bar{K}\bar{L}]} - \frac{1}{2}\,\mathrm{tr}\,R_{[\bar{I}\bar{J}}R_{\bar{K}\bar{L}]}\right), \tag{265}$$

the equation of motion

$$D_I(\Omega)G^{IJKL} = 0, \tag{266}$$

and the Bianchi identity

$$\begin{aligned}(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = 4\sqrt{2}\pi\left(\frac{\kappa}{4\pi}\right)^{2/3}\Bigg\{&J^{(0)}\delta(x^{11}) + J^{(N+1)}\delta(x^{11}-\pi\rho)\\ &+\frac{1}{2}\sum_{n=1}^{N}J^{(n)}\left[\delta(x^{11}-x_n)+\delta(x^{11}+x_n)\right]\Bigg\}_{\bar{I}\bar{J}\bar{K}\bar{L}}.\end{aligned} \tag{267}$$

Note that the $\mathrm{tr}\,(R\wedge R)$ terms appearing in the above boundary conditions and Bianchi identity are not required by the low energy theory, but (since they are needed for anomaly cancellation, given the structure of the one-loop chiral anomalies) must be present in the full quantum M-theory.

9.2 Duality–symmetric D=11 Supergravity

The duality–symmetric action for $D=11$ supergravity is

$$\begin{aligned}S = &\int_{\mathcal{M}^{11}}\left[\hat{R}^{\hat{a}_1\hat{a}_2}\wedge\hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!}\hat{\bar{\Psi}}\wedge\mathcal{D}[\frac{1}{2}(\hat{\omega}+\hat{\tilde{\omega}})]\hat{\Psi}\mathcal{G}^{\hat{a}_1\hat{a}_2\hat{a}_3}\wedge\hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3}\right]\\ &-\int_{\mathcal{M}^{11}}\left[\frac{1}{2}(\hat{C}^{(7)}+\hat{*}\hat{C}^{(4)})\wedge(\hat{F}^{(4)}+(\hat{F}^{(4)}-\hat{C}^{(4)})) - \frac{1}{2}\hat{F}^{(4)}\wedge\hat{*}\hat{F}^{(4)} + \frac{1}{3}\hat{A}^{(3)}\wedge\hat{F}^{(4)}\wedge\hat{F}^{(4)}\right]\\ &+\int_{\mathcal{M}^{11}}\frac{1}{2}i_{\hat{v}}\hat{\mathcal{F}}^{(4)}\wedge\hat{*}i_{\hat{v}}\hat{\mathcal{F}}^{(4)},\end{aligned} \tag{268}$$

or in a more symmetric form

$$S = \int_{\mathcal{M}^{11}} \Big[\hat{R}^{\hat{a}_1\hat{a}_2} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!}\hat{\bar{\Psi}} \wedge \mathcal{D}[\frac{1}{2}(\hat{\omega} + \hat{\tilde{\omega}})]\hat{\Psi}\mathcal{G}^{\hat{a}_1\hat{a}_2\hat{a}_3} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3}$$

$$-\frac{1}{2}(\hat{C}^{(7)} + \hat{*}\hat{C}^{(4)}) \wedge (\hat{F}^{(4)} - \frac{1}{2}\hat{C}^{(4)}) - \frac{1}{2}(\hat{C}^{(4)} + \hat{*}\hat{C}^{(7)}) \wedge (\hat{F}^{(7)} + \frac{1}{2}\hat{C}^{(7)}) \Big]$$

$$+ \int_{\mathcal{M}^{11}} \Big[\frac{1}{4}\hat{F}^{(4)} \wedge \hat{*}\hat{F}^{(4)} - \frac{1}{4}\hat{F}^{(7)} \wedge \hat{*}\hat{F}^{(7)} \tag{269}$$

$$+\frac{1}{4} i_v\hat{\mathcal{F}}^{(4)} \wedge \hat{*} i_v\hat{\mathcal{F}}^{(4)} - \frac{1}{4} i_v\hat{\mathcal{F}}^{(7)} \wedge \hat{*} i_v\hat{\mathcal{F}}^{(7)} + \frac{1}{6}\hat{F}^{(7)} \wedge \hat{F}^{(4)} \Big].$$

where $\hat{*}$ is the Hodge operator in $D = 11$.

Modulo the last term the action (268) is the conventional $D = 11$ supergravity action written in the same notation as in the original paper except for the coefficient in front of the Einstein-Hilbert term, the first term in (268) and (269), and the coefficient in the definition of the spin connection. To write the Einstein–Hilbert term and the gravitino kinetic term of the action in the differential form notation it is convenient to introduce a form dual to the wedge product of the vielbeine

$$\hat{\Sigma}_{\hat{a}_1\ldots\hat{a}_n} = \frac{1}{(11-n)!}\epsilon_{\hat{a}_1\hat{a}_2\ldots\hat{a}_{11}}\hat{E}^{\hat{a}_{n+1}} \wedge \cdots \wedge \hat{E}^{\hat{a}_{11}}. \tag{270}$$

Other building blocks of the action are the covariant derivative of the gravitino field

$$\hat{\Psi}^{\hat{\alpha}} = dX^{\hat{m}}\hat{\Psi}^{\hat{\alpha}}_{\hat{m}} , \tag{271}$$

$$\mathcal{D}\hat{\Psi}^{\hat{\alpha}} = d\hat{\Psi}^{\hat{\alpha}} - \hat{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} \wedge \hat{\Psi}^{\hat{\beta}}, \qquad \hat{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} = \frac{1}{4}\hat{\omega}_{\hat{a}\hat{b}}(\mathcal{G}^{\hat{a}\hat{b}})^{\hat{\alpha}}{}_{\hat{\beta}} , \tag{272}$$

the bilinear fermionic terms

$$\hat{C}^{(4)} = -\frac{1}{4}\hat{\bar{\Psi}} \wedge \hat{\mathcal{G}}^{(2)} \wedge \hat{\Psi}, \qquad \hat{C}^{(7)} = \frac{i}{4}\hat{\bar{\Psi}} \wedge \hat{\mathcal{G}}^{(5)} \wedge \hat{\Psi} , \tag{273}$$

the supercovariant connection $\hat{\omega}$ determined by $d\hat{E}^{\hat{a}} - \hat{E}^{\hat{b}} \wedge \hat{\omega}_{\hat{b}}{}^{\hat{a}} = \frac{i}{4}\hat{\bar{\Psi}}\mathcal{G}^{\hat{a}} \wedge \hat{\Psi}$,

$$\hat{\tilde{\omega}}_{\hat{m}\hat{a}\hat{b}} = \hat{\omega}_{\hat{m}\hat{a}\hat{b}} + \frac{i}{8}\hat{\bar{\Psi}}^{\hat{n}}(\mathcal{G}_{\hat{m}\hat{a}\hat{b}\hat{n}\hat{p}})\hat{\Psi}^{\hat{p}} , \tag{274}$$

and the field strength

$$\hat{F}^{(4)} = d\hat{A}^{(3)} \tag{275}$$

of the three–form gauge field $\hat{A}^{(3)}$.

The last term of (268) and the corresponding terms in (269) encode the information on duality relations between $\hat{A}^{(3)}$ and a six–form gauge field $\hat{A}^{(6)}$, which can be derived directly from the action (268), and contains the following (anti–)dual combinations of the field strengths

$$\hat{\mathcal{F}}^{(4)} = (\hat{F}^{(4)} - \hat{C}^{(4)}) - \hat{*}(\hat{F}^{(7)} + \hat{C}^{(7)}) , \tag{276}$$

$$\hat{\mathcal{F}}^{(7)} = \hat{F}^{(7)} + \hat{C}^{(7)} - \hat{*}(\hat{F}^{(4)} - \hat{C}^{(4)}) = -\hat{*}\hat{\mathcal{F}}^{(4)} , \tag{277}$$

where

$$\hat{F}^{(7)} = d\hat{A}^{(6)} + \hat{A}^{(3)} \wedge \hat{F}^{(4)}. \tag{278}$$

This part of the actions is constructed with the use of the space–like unit vector $\hat{v}_{\hat{m}}$ composed of derivatives of the PST scalar $a(x)$ and $i_{\hat{v}}\hat{\mathcal{F}}^{(n)}$ is the inner product of $\hat{v}$ with $\hat{\mathcal{F}}^{(n)}$ $(n = 4, 7)$

$$i_{\hat{v}}\hat{\mathcal{F}}^{(n)} = \frac{1}{(n-1)!}dX^{\hat{m}_{n-1}} \wedge \cdots \wedge dX^{\hat{m}_1}\hat{v}_{\hat{n}}\hat{g}^{\hat{n}\hat{m}}\hat{\mathcal{F}}^{(n)}_{\hat{m}\hat{m}_1\cdots\hat{m}_{n-1}} \equiv \frac{1}{(n-1)!}\hat{E}^{\hat{a}_{n-1}} \wedge \cdots \wedge \hat{E}^{\hat{a}_1}\ \hat{v}^{\hat{a}}\hat{\mathcal{F}}^{(n)}_{\hat{a}\hat{a}_1\cdots\hat{a}_{n-1}} . \tag{279}$$

We should also point out that the duality–symmetric version of $D = 11$ supergravity has the following structure

$$S = S_{EH} + S_{\hat{\Psi}} + S_{\hat{A}}, \tag{280}$$

where S_{EH} is the Einstein–Hilbert term, $S_{\hat{\Psi}}$ is the fermion kinetic term and $S_{\hat{A}}$ is a specific term of the form which contains the information on the duality relations. In fact, looking ahead, we can claim that modulo quartic fermion terms which can not be included by the supercovariantization of the gauge field strengths any duality-symmetric supergravity can be presented in the form of the action containing the Einstein–Hilbert term, kinetic terms of the fermionic fields and a specific construction.

To conclude this section let us recall that in the conventional Cremmer–Julia–Scherk formulation of $D = 11$ supergravity

$$\begin{aligned} S_{CJS} = &\int \left[\hat{R}^{\hat{a}_1\hat{a}_2} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!}\hat{\bar{\Psi}} \wedge \mathcal{D}[\frac{1}{2}(\hat{\omega} + \hat{\tilde{\omega}})]\hat{\Psi}\mathcal{G}^{\hat{a}_1\hat{a}_2\hat{a}_3} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3} + \frac{1}{2}\hat{F}^{(4)} \wedge \hat{*}\hat{F}^{(4)}\right] \\ &- \int_{\mathcal{M}^{11}} \left[\frac{1}{2}(\hat{C}^{(7)} + \hat{*}\hat{C}^{(4)}) \wedge (\hat{F}^{(4)} + (\hat{F}^{(4)} - \hat{C}^{(4)})) + \frac{1}{3}\hat{A}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)}\right] \end{aligned} \tag{281}$$

the duality conditions arise as a solution of the second order equation of motion of $\hat{A}^{(3)}$

$$d\left(\hat{*}(\hat{F}^{(4)} - \hat{C}^{(4)}) - \hat{A}^{(3)} \wedge \hat{F}^{(4)} - \hat{C}^{(7)}\right) = 0 . \tag{282}$$

The previous equation implies that the differential form under the external differential is closed, hence in space–time with trivial topology its solution is an exact form $d\hat{A}^{(6)}$,

$$\hat{*}(\hat{F}^{(4)} - \hat{C}^{(4)}) - \hat{A}^{(3)} \wedge \hat{F}^{(4)} - \hat{C}^{(7)} = d\hat{A}^{(6)} . \tag{283}$$

The solution of the total action is

$$\begin{aligned} S = &\int_{\mathcal{M}^{11}} \left[\hat{R}^{\hat{a}_1\hat{a}_2} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2} + \frac{i}{3!}\hat{\bar{\Psi}} \wedge \mathcal{D}[\frac{1}{2}(\hat{\omega} + \hat{\tilde{\omega}})]\hat{\Psi}\mathcal{G}^{\hat{a}_1\hat{a}_2\hat{a}_3} \wedge \hat{\Sigma}_{\hat{a}_1\hat{a}_2\hat{a}_3}\right] \\ &- \int_{\mathcal{M}^{11}} \left[\frac{1}{2}(\hat{C}^{(7)} + \hat{*}\hat{C}^{(4)}) \wedge (\hat{F}^{(4)} + (\hat{F}^{(4)} - \hat{C}^{(4)})) - \frac{1}{2}\hat{F}^{(4)} \wedge \hat{*}\hat{F}^{(4)} + \frac{1}{3}\hat{A}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)}\right] \\ &+ \int_{\mathcal{M}^{11}} \left[\frac{1}{2}\hat{v} \wedge (\hat{F}^{(4)} - \hat{C}^{(4)}) \wedge i_{\hat{v}}\hat{\mathcal{F}}^{(7)} - \frac{1}{2}\hat{v} \wedge (\hat{F}^{(7)} + \hat{C}^{(7)}) \wedge i_{\hat{v}}\hat{\mathcal{F}}^{(4)} + \frac{1}{6}\hat{F}^{(7)} \wedge \hat{F}^{(4)}\right. \\ &\left. -\frac{1}{2}\hat{C}^{(4)} \wedge \hat{F}^{(7)} - \frac{1}{2}\hat{C}^{(7)} \wedge \hat{F}^{(4)}\right] . \end{aligned} \tag{284}$$

The general variation of the last term in the previous equation is

$$\begin{aligned} \delta S_{\hat{A}} = &\int_{\mathcal{M}^{11}} \left[\delta\hat{v} \wedge \hat{v} \wedge (i_{\hat{v}}\hat{\mathcal{F}}^{(4)} \wedge i_{\hat{v}}\hat{\mathcal{F}}^{(7)}) + (\hat{v} \wedge i_{\hat{v}}\hat{\mathcal{F}}^{(7)}) \wedge \delta(\hat{F}^{(4)} - \hat{C}^{(4)}) + \frac{1}{2}(\hat{F}^{(7)} + \hat{C}^{(7)}) \wedge \delta(\hat{F}^{(4)} - \hat{C}^{(4)})\right. \\ &\left. +(\hat{v} \wedge i_{\hat{v}}\hat{\mathcal{F}}^{(4)}) \wedge \delta(\hat{F}^{(7)} + \hat{C}^{(7)}) - \frac{1}{2}(\hat{F}^{(4)} - \hat{C}^{(4)}) \wedge \delta(\hat{F}^{(7)} + \hat{C}^{(7)}) + \frac{1}{2}\delta\hat{A}^{(3)} \wedge \hat{F}^{(4)} \wedge \hat{F}^{(4)}\right. \end{aligned} \tag{285}$$

$$-\frac{1}{2}\delta(\hat{C}^{(4)} \wedge \hat{F}^{(7)}) - \frac{1}{2}\delta(\hat{C}^{(7)} \wedge \hat{F}^{(4)})\Big] .$$

The 11-dimensional theory generated considerable excitement as the first potential candidate for the theory of everything. The action and boundary conditions provide a supersymmetric theory which is a natural candidate for the a low energy limit of M-theory. Our understanding of M-theory is still very limited, mainly due to the lack of powerful methods to probe it at the quantum level. One approach to encoding information about M-theory is through its low energy effective field theory.

10 Conclusion

The main purpose of this publication was to investigate the consistent supergravity theories. We have shown how general gauge theories with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies can be formulated in a way that is covariant with respect to electric-magnetic duality transformations. We performed our analysis first in rigid supersymmetry. Using superconformal techniques, we could then show that only one cancellation had to be checked to extend the results to supergravity. It turns out that the Chern-Simons term does not need any gravitino corrections and can thus be added as such to the matter-coupled supergravity actions. Our paper provides thus an extension to the general framework of coupled chiral and vector multiplets in $N = 1$ supergravity. We have completed the coupling of (1, 0) six-dimensional supergravity to tensor and vector multiplets. The coupling to tensor multiplets is of a more conventional nature, and parallels similar constructions in other supergravity models. Our work is here confined to the field equations, but a lagrangian formulation of the (anti)self-dual two-forms is now possible, following the proposal of Pasti, Sorokin and Tonin and indeed, results to this effect have been presented in a superspace formulation. The Yang-Mills currents are not conserved, and the consistent residual gauge anomaly is accompanied by a corresponding anomaly in the supersymmetry current. In completing these results to all orders in the fermi fields, we have come to terms with another peculiar feature of anomalies, neatly displayed by these field equations: anomalous divergences of gauge currents are typically accompanied by corresponding anomalies in current commutators. We have shown that cancellation of gravitational, gauge, and mixed anomalies gives a sufficient constraint on six-dimensional supersymmetric theories of gravity with gauge and matter fields that in some cases all models consistent with anomaly cancellation admit a realization through string theory. We have ruled out a number of infinite families of models which satisfy anomaly factorization, so that the gap is rather small between the set of known 6D models satisfying anomaly cancellation and the set of models realized through string compactification. We have conjectured that all consistent 6D supergravity theories with Lagrangian descriptions can be realized in string theory, and that this set of models can be identified from low-energy considerations. All $N = (1, 0)$ supersymmetric theories in 6D with one gravity and one tensor multiplet which are free of anomalies or other quantum inconsistencies admit a string construction. It would also be interesting to formulate the matter coupled anomaly-free supergravity theories in six dimensions such that the classically gauge invariant and supersymmetric part of the action is identified and the anomaly corrections are determined by means of the anomaly equations. We conduct a systematic search for anomaly-free six-dimensional $N = 1$ chiral supergravity theories. Under a certain set of restrictions on the allowed gauge groups and the representations of the hypermultiplets, we present possible Poincaré and gauged supergravities with one tensor multiplet satisfying the 6D anomaly cancellation criteria. In six dimensions, cancellation of gauge, gravitational, and mixed anomalies strongly constrains the set of quantum field theories which can be coupled consistently to gravity. Anomaly cancellation has turned out to be a crucial guiding principle for the identification of consistent D = 6 theories for the same reason as in the D = 10 case. The D = 6 anomaly cancellation conditions are weaker than those in D = 10, they are still very stringent, especially in the case of gauged supergravity theories. The corrections to the equations of motion determined here by anomaly considerations are only those which originate from the Green-Schwarz anomaly cancelling local counterterms. To obtain the fully consistent equations of motion at the quantum level one must also take into account the non-local corrections to the one loop effective action. This raises the question of which equations of motion are to be solved in search of special solutions of the theory. Constructing a consistent quantum theory of gravity has proven to be substantially more difficult than identifying a quantum theory describing the other forces in nature. Even if it is known that consistent superstring theories can be formulated in six dimensions and that six-dimensional supergravity can arise as their low-energy limit, this is not the only reason for investigating D = 6 supergravity. In fact, while supergravities in D = 10 and D = 11 spacetime dimensions are of direct interest as backgrounds for strings, membranes and M-theory, one frequently performs compactifications down to D = 6 to clarify relations among these theories, which are hidden in their ten or eleven-dimensional formulations. Supergravity theories in diverse dimensions play nowadays an important role as low-energy effective field theories of superstring and membrane theories. Supergravity theories have been extensively studied in four dimensions, of course because of their direct physical relevance, and in ten and eleven dimensions of their fundamental features. Explicit knowledge of this set of theories gives us a powerful tool for exploring the connection between string theory and low-energy physics. The anomalies are the key to a deeper research and understanding of gauged supergravity. We hope to have conveyed the idea that anomalies play an important role in supergravity and their cancellation has been and still is a valuable guide for constructing consistent quantum supergravity theories. The treatment of anomalies makes fascinating contacts with several branches of modern theoretical physics.

References

[1] H. Samtleben, "Lectures on Gauged Supergravity and Flux Compactifications", arXiv:hep-th/0808.4076.

[2] H. Nishino, E. Sezgin, "New Couplings of Six-Dimensional Supergravity", arXiv:hep-th/9703075.

[3] H. Lü , C.N. Pope, E. Sezgin, "Yang-Mills-Chern-Simons Supergravity", arXiv:hep-th/0305242.

[4] R. Suzuki, Y. Tachikawa, "More anomaly-free models of six-dimensional gauged supergravity", arXiv:hep-th/0512019.

[5] F. Riccioni, A. Sagnotti, "Consistent and Covariant Anomalies in Six-dimensional Supergravity", arXiv:hep-th/9806129.

[6] S. Ferrara, F. Riccioni, A. Sagnotti, "Tensor and Vector Multiplets in Six-Dimensional Supergravity", arXiv:hep-th/9711059.

[7] E. Bergshoeff, F. Coomans, E. Sezgin, A. Van Proeyen, "Higher Derivative Extension of $6D$ Chiral Gauged Supergravity", arXiv:1203.2975.

[8] F. Riccioni, "Low-energy structure of six-dimensional open-string vacua", arXiv:hep-th/0203157.

[9] G. Dall'Agata, K. Lechner, "N = 1, D = 6 Supergravity: Duality and non Minimal Couplings", arXiv:hep-th/9707236.

[10] C. Vafa, "Evidence for F-theory", arXiv:hep-th/9602022.

[11] S. Ferrara, R. Minasian, A. Sagnotti, "Low-energy analysis of M and F theories on Calabi-Yau manifolds", arXiv:hep-th/9604097.

[12] C. Scrucca, M. Serone, "Anomalies in field theories with extra dimensions", arXiv:hep-th/0403163.

[13] G. Dall'Agata, K. Lechner, M. Tonin, "Covariant actions for $N = 1$, $D = 6$ Supergravity theories with chiral bosons", arXiv:hep-th/9710127.

[14] J. Babington, J. Erdmenger, "Space-time dependent couplings in $\mathcal{N} = 1$ SUSY gauge theories: Anomalies and Central Functions", arXiv:hep-th/0502214.

[15] L. Andrianopoli, S. Ferrara, M. Lledo,â® "Axion gauge symmetries and generalized ChernâSimons terms in $N = 1$ supersymmetric theories", arXiv:hep-th/0402142.

[16] V. Kumar, W.Taylor, "A bound on 6D $\mathcal{N} = 1$ supergravities", arXiv:hep-th/0901.1586.

[17] S. Randjbar-Daemi, E.Sezgin, "Scalar Potential and Dyonic Strings in 6D Gauged Supergravity", arXiv:hep-th/0402217.

[18] R. Güven, J. Liu, C.Pope, E. Sezgin, "Fine Tuning and Six-Dimensional Gauged $N = (1, 0)$ Supergravity Vacua", arXiv:hep-th/0306201.

[19] P. Howe, E. Sezgin, "Anomaly-Free Tensor-Yang-Mills System and Its Dual Formulation", arXiv:hep-th/9806050.

[20] E. Bergshoeff, E. Sezgin, E. Sokatchev, "Couplings of Self-Dual Tensor Multiplet in Six Dimensions", arXiv:hep-th/9605087.

[21] F. Brandt, "Deformed supergravity with local R-symmetry", arXiv:hep-th/9704046.

[22] S. Nibbelink, J. van Holten, "Consistent σ-models in $N = 1$ supergravity", arXiv:hep-th/9903006.

[23] F. Brandt, "Anomaly candidates and invariants of $D = 4$, $N = 1$ supergravity theories", arXiv:hep-th/9306054.

[24] P. Anastasopoulos, M. Bianchi, E. Dudas, E. Kiritsis, "Anomalies, Anomalous U(1)'s and generalized Chern-Simons terms", arXiv:hep-th/0605225.

[25] A. V. Smilga, "Chiral anomalies in higher-derivative supersymmetric $6D$ gauge theories", arXiv:hep-th/0606139.

[26] J. Schwarz, "Anomaly-Free Supersymmetric Models in Six Dimensions", arXiv:hep-th/9512053.

[27] S. Avramis, "Anomaly-Free Supergravities in Six Dimensions", arXiv:hep-th/0611133.

[28] V. Kumar, W. Taylor, "String Universality in Six Dimensions", arXiv:hep-th/0906.0987.

[29] N. Seiberg, E. Witten, "Comments on String Dynamics in Six Dimensions", arXiv:hep-th/9603003.

[30] M. Duff, J. Liu, H. Lü, C. Pope, "Gauge Dyonic Strings and Their Global Limit", arXiv:hep-th/9711089.

[31] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen, "Brane plus Bulk Supersymmetry in Ten Dimensions", arXiv:hep-th/0105061.

[32] S. Metzger, "Supersymmetric Gauge Theories from String Theory", arXiv:hep-th/0512285.

[33] T. Asaka, W. Buchmüller, L. Covi, "Bulk and Brane Anomalies In Six Dimensions", arXiv:hep-ph/0209144.

[34] G. Cardoso, B. Ovrut, "Supersymmetric Calculation of Mixed Kähler-Gauge and Mixed Kähler-Lorentz Anomalies", arXiv:hep-th/9308066.

[35] S. Randjbar-Daemi, "Aspects of Six Dimensional Supersymmetric Theories", arXiv:hep-th/9809044.

[36] F. Riccioni, "Abelian Vectors and Self-Dual Tensors in Six-Dimensional Supergravity", arXiv:hep-th/9910246.

[37] P. Horava, E. Witten, "Eleven Dimensional Supergravity on a Manifold with Boundary", arXiv:hep-th/9603142.

[38] Y. Shamir, "Compensating fields and Anomalies in Supergravity", arXiv:hep-th/9207038.

[39] A. Bilal, "Lectures on Anomalies", arXiv:hep-th/08020634.

[40] J. Maharana, "S-Duality and Compactification of Type IIB Superstring Action", arXiv:hep-ph/0202233.

[41] J. Dixon, "The Search for Supersymmetry Anomalies–Does Supersymmetry Break Itself?", arXiv:hep-th/9308088.

[42] S. Deguchi, T. Mukai, T. Nakajima, "Anomalous Gauge Theories with Antisymmetric Tensor Fields", arXiv:hep-th/9804070.

[43] A. Bilal, "Lectures on Anomalies", arXiv:hep-th/08020634.