ON THE GAUSS CIRCLE PROBLEM

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Abstract. Using the compression method, we prove an inequality related to the Gauss circle problem. Let $\mathcal{N}_r$ denotes the number of integral points in a circle of radius $r > 0$, then we have

$$2r^2\left(1+\frac{1}{4}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r}) \leq \mathcal{N}_r \leq 8r^2\left(1+\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})$$

for all $r > 1$. This implies that the error function $E(r)$ of the counting function $\mathcal{N}_r \ll r^{1-\epsilon}$ for any $\epsilon > 0$.

1. Introduction

The Gauss circle problem is a classic question in number theory that concerns the approximation of the number of lattice points within a circle in the Euclidean plane. Specifically, the problem asks about the error term in the approximation of the number of lattice points $N(r)$ inside a circle of radius r, where the exact number of points is compared to the area of the circle πr^2. The main challenge lies in understanding the difference between the exact count of lattice points and the area, known as the error term, and establishing its asymptotic behaviour as r grows large. This problem connects to deep areas of mathematics, such as analytic number theory, geometric analysis, and the distribution of integer solutions to polynomial equations. In this work, we explore the error term in the Gauss circle problem under specific transformations of the lattice points, aiming to refine the asymptotic bounds for the error term and deepen our understanding of the distribution of lattice points in circles. By deriving precise upper and lower bounds, we seek to contribute to the ongoing exploration of the error term in lattice point counting functions. Precisely, the Gauss circle problem is a problem that seeks to counts the number of integral points in a circle centered at the origin and of radius r. It is fairly easy to see that the area of a circle of radius $r > 0$ gives a fairly good approximation for the number of such integral points in the circle, since on average each unit square in the circle contains at least an integral point. In particular, by denoting $N(r)$ to be the number of integral points in a circle of radius r, then the following elementary estimate is well-known

$$N(r) = \pi r^2 + |E(r)|$$

where $|E(r)|$ is the error term. The real and the main problem in this area is to obtain a reasonably good estimate for the error term. In fact, it is conjectured that

$$|E(r)| \ll r^{\frac{1}{2}+\epsilon}$$

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for $\epsilon > 0$. The first fundamental progress was made by Gauss [3], where it is shown that

$$|E(r)| \leq 2\pi r\sqrt{2}.$$

G.H Hardy and Edmund Landau almost independently obtained a lower bound [1] by showing that

$$|E(r)| \neq o(r^{\frac{1}{2}}(\log r)^{\frac{1}{4}}).$$

The current best upper bound (see [2]) is given by

$$|E(r)| \ll r^{\frac{131}{208}}.$$

In this paper we prove a general upper bound and lower bound for the number of integral points in a circle of radius $r > 1$. This upper bound is of the desired quality as does the Gauss circle problem, where the quest is to be obtain an error of quality as that in the following result

Theorem 1.1 (The inequality)**.** *Let $\mathcal{N}_r$ denote the number of integral points in a circle of radius r. Then*

$$2r^2\left(1+\frac{1}{4}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})\leq \mathcal{N}_r\leq 8r^2\left(1+\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}\right)+O(\frac{r}{\log r})$$

for all $r > 1$.

Now we describe the steps used to achieve these inequalities. We write them chronologically as follows:

(i) We pick a point in the plane with compression gap $2r$ and construct the compression circle. This circle has radius r by the choice of compression gap.
(ii) We first count the number of integral points on the boundary of the circle of radius r using the upper and the lower bounds of the compression gap. The error terms of the upper and the lower bound emanate from this particular analysis.
(iii) We construct a further smaller circle of compression by shrinking the radius of each successive circle by a factor of 2. This procedure has a tendency to create annular regions in the circle.
(iv) For each annular region, we construct an integer square grid that exactly covers the upper circle and count the number of points in the gird and in this annular region. The main terms in the inequalities follow by upper and lower bounds this count.

2. **Preliminary results**

Definition 2.1. By the compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$, we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is a process of rescaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus, it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by a certain scale and similarly draws points away from the origin close to the origin. Intuitively, compression induces some kind of motion on points in the Euclidean space $\mathbb{R}^n$ for $n \geq 2$.

Proposition 2.3. *A compression of scale* $1 \geq m > 0$ *with* $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. **The mass of compression.** In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.4. By the mass of a compression of scale $1 \geq m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\operatorname{Inf}(x_j) = \operatorname{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m \sum_{k=0}^{n-1} \frac{1}{\operatorname{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\operatorname{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \neq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.5. *We have*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.6. Next we prove upper and lower bounding the mass of the compression of scale $1 \geq m > 0$.

Proposition 2.7. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for each $1 \leq i \leq n$ and $x_i \neq x_j$ for $i \neq j$, then we have*

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Definition 2.8. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \ldots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \ldots, x_n - \frac{m}{x_n}\right)\right|\right|$$

3. The ball induced by compression

In this section we introduce the notion of the ball induced by a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of a given scale. We launch in a more formal way the following language.

Definition 3.1. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ and $x_i \neq 0$ for all $1 \leq i \leq n$. Then by the ball induced by $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ under compression of scale $1 \geq m > 0$, denoted by $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \ldots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \ldots, x_n)]}[(x_1, x_2, \ldots, x_n)]$ if it satisfies the inequality.

Remark 3.2. Next, we prove that smaller balls induced by points should essentially be covered by the larger balls in which they are embedded. We state and prove this statement in the following result.

In the geometry of balls induced under compression of scale $m > 0$, we assume implicitly that

$$0 < m \leq 1.$$

For simplicity, we will on occasion choose to write the ball induced by the point $\vec{x} = (x_1, x_2, \ldots, x_n)$ under compression as

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

We adopt this notation to save enough work space in many circumstances. We first prove a preparatory result in the following sequel. We find the following estimates for the compression gap useful.

Proposition 3.3. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$ for $j = 1, \ldots, n$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2, \ldots, x_n^2)] - 2mn.$$

In particular, if $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

for $\vec{x} \in \mathbb{R}^n$ with $x_i \geq 1$ for each $1 \leq i \leq n$.

Proposition 3.3 offers us an extremely useful identity. It allows us to pass from the compression gap on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin compared to points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}] < \mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

with $m := m(n) = o(1)$ as $n \longrightarrow \infty$ if and only if $||\vec{x}|| \lesssim ||\vec{y}||$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $x_i \geq 1$ for all $1 \leq i \leq n$. This important transference principle will be mostly put to use in obtaining our results. In particular, we note that in the latter case, we can write the asymptotic

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \sim \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \ldots, \frac{1}{x_n^2}\right)\right] = ||\vec{x}||^2.$$

Corollary 3.4. *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq x_i$ for $j \neq i$ and $x_i, x_j \geq 1$ for each $1 \leq i, j \leq n$. If $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \geq n\mathrm{Inf}(x_j^2) - 2mn + O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \leq n\mathrm{sup}(x_j^2) - 2mn + O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2, \ldots, x_n^2)]\right)$$

Lemma 3.5 (Compression estimate)**.** *Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_i \geq 1$ for all $1 \leq i \leq n$ with $x_i \neq x_j$ $(i \neq j)$. If $m := m(n) = o(1)$ as $n \longrightarrow \infty$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]^2 \ll n\mathrm{sup}(x_j^2) + m^2\log\left(1 + \frac{n-1}{\mathrm{Inf}(x_j)^2}\right) - 2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 \gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\sup(x_j^2)}\right)^{-1}-2mn.$$

Remark 3.6. It is important to note that the inequality in Corollary 3.4 implies the inequalities in Lemma 3.5. At any given moment, we will decide which of the versions of these inequalities to use. In fact, the inequalities in Corollary 3.4 are more applicable to various problems than those of Lemma 3.5.

Theorem 3.7. *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{R}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$ with $y_i,z_i\geq 1$ for all $1\leq i\leq n$ and $m:=m(n)=o(1)$ as $n\longrightarrow\infty$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ with $||\vec{z}||<||\vec{y}||$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

with $||\vec{y}-\vec{z}||<\epsilon$ for some $\epsilon>0$.

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{R}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$ and $z_i\geq 1$ for all $1\leq i\leq n$ such that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]>\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\lesssim||\vec{z}||$, which is absurd. In this case, we can take $\epsilon:=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 3.3 that $||\vec{z}||\lesssim||\vec{y}||$. Under the requirement $||\vec{y}-\vec{z}||<\epsilon$ for some $\epsilon>0$, we obtain the inequality

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right| &\leq \left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|+\epsilon\\ &=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]+\epsilon\end{aligned}$$

with $m=m(n)=o(1)$ as $n\longrightarrow\infty$. By choosing $\epsilon>0$ sufficiently small, we deduce that $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

In the geometry of balls under compression, we will assume that n is sufficiently large for $\mathbb{R}^n$. In this regime, we will always take the scale of compression $m:=m(n)=o(1)$ as $n\longrightarrow\infty$.

Theorem 3.8. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$ with $y_i,x_i\geq 1$ for each $1\leq i\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}||<||\vec{x}||$ for $||\vec{y}-\vec{x}||<\delta$ for $\delta>0$ sufficiently small, then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for $m:=m(n)=o(1)$ as $n\longrightarrow\infty$.

Proof. First, let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}|| < ||\vec{x}||$ for $||\vec{y}-\vec{x}|| < \delta$, then it follows from Theorem 3.7 that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}] \gtrsim \mathcal{G}\circ\mathbb{V}_m[\vec{y}]$ with $||\vec{y}-\vec{x}|| < \delta$ for $\delta > 0$ sufficiently small. Consequently the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is slightly bigger than the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ by virtue of their compression gaps and the latter does not contain the point $\vec{x}$ by construction. It is easy to see that $||\mathbb{V}_m[\vec{y}]|| > ||\mathbb{V}_m[\vec{x}]||$ and

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\mathbb{V}_m[\vec{y}]] &= \mathcal{G}\circ\mathbb{V}_m[\vec{y}]\\ &\lesssim \mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\ &= \mathcal{G}\circ\mathbb{V}_m[\mathbb{V}_m[\vec{x}]]\end{aligned}$$

with $||\mathbb{V}_m[\vec{y}] - \mathbb{V}_m[\vec{x}]|| < \epsilon$ for small $\epsilon > 0$. It implies that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and this completes the proof. □

Remark 3.9. Theorem 3.8 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

3.1. **Interior points and the limit points of balls induced under compression.** In this section we launch the notion of an interior and the limit point of balls induced under compression. We study this notion in depth and explore some connections.

Definition 3.10. Let $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with $y_i \neq y_j$ for all $1 \leq i < j \leq n$. Then a point $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ is an interior point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for most $\vec{x} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$. An interior point $\vec{z}$ is then said to be a limit point if

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}]}[\vec{z}] \subseteq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

for all $\vec{x} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$

Remark 3.11. Next we prove that there must exist an interior and limit point in any ball induced by points under compression of any scale in any dimension.

Theorem 3.12. *Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ with $y_i \geq 1$ for all $1 \leq i \leq n$. Then the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains an interior point and a limit point.*

Proof. Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \neq x_j$ for all $1 \leq i < j \leq n$ with $x_i \geq 1$ for all $1 \leq i \leq n$ and suppose on the contrary that $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contains no limit point. Then pick

$$\vec{z}_1 \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

such that $z_{1_i} \geq 1$ for all $1 \leq i \leq n$ with $||\vec{z}_1|| < ||\vec{x}||$ such that $||\vec{z}_1 - \vec{x}|| < \epsilon$ for $\epsilon > 0$ sufficiently small. Then by Theorem 3.8 and Theorem 3.7, it follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1] \subset \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Again pick $\vec{z}_2\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$ such that $z_{2_i}\geq 1$ for all $1\leq i\leq n$ with $||\vec{z}_2||<||\vec{z}_1||$ such that $||\vec{z}_2-\vec{z}_1||<\delta$ for $\delta>0$ sufficiently small. Then by employing Theorem 3.8 and Theorem 3.7, we have

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]}[\vec{z}_2]\subset\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]}[\vec{z}_1]$$

with $\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]\lesssim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]$. By continuing the argument in this manner we obtain the infinite descending sequence of the gap of compression

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_1]\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_2]\gtrsim\cdots\gtrsim\mathcal{G}\circ\mathbb{V}_m[\vec{z}_n]\gtrsim\cdots$$

thereby ending the proof of the theorem. □

Proposition 3.13. *The point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$ is the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$ for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$.*

Proof. Applying the compression $\mathbb{V}_1:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ on the point $\vec{x}=(x_1,x_2,\ldots,x_n)$ with $x_i=1$ for each $1\leq i\leq n$, we obtain $\mathbb{V}_1[\vec{x}]=(1,1,\ldots,1)$ so that $\mathcal{G}\circ\mathbb{V}_1[\vec{x}]=0$ and the corresponding ball induced under compression $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ contains only the point $\vec{x}$. It follows by Definition 3.12 the point $\vec{x}$ must be the limit point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$. It follows that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

for any $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for all $1\leq i\leq n$. For if the contrary

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

holds for some $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i>1$ for each $1\leq i\leq n$, then there must exists some point $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$. Since $\vec{x}$ is the only point in the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}]}[\vec{x}]$, it follows that

$$\vec{x}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{y}]}[\vec{y}]$$

which is inconsistent with the fact that $\vec{x}$ is the limit point of the ball. □

3.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 3.14. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 3.15. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 3.16. *Let $\vec{x} \in \mathbb{R}^n$ with $x_i \neq x_j$ $(i \neq j)$ such that $x_i, y_i \geq 1$ for all $1 \leq i \leq n$ and set $m := m(n) = o(1)$ as $n \longrightarrow \infty$. The point $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}|| < ||\vec{x}||$ such that $||\vec{y} - \vec{x}|| < \epsilon$ for $\epsilon > 0$ sufficiently small is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{y}|| < ||\vec{x}||$ such that $||\vec{y} - \vec{x}|| < \epsilon$ for $\epsilon > 0$ sufficiently small be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] \neq \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Without loss of generality, we can choose some $\vec{z} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ with $||\vec{z}|| < ||\vec{x}||$ such that

$$\vec{z} \notin \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

for $||\vec{z} - \vec{x}|| < \delta$ for $\delta > 0$ sufficiently small. Applying Theorem 3.7, we obtain the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{y}] \lesssim \mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

This already contradicts the equality $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. The latter equality of compression gaps follows from the requirement that the balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G} \circ \mathbb{V}_m[\vec{y}] = \mathcal{G} \circ \mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the two indistinguishable balls and so must satisfy the equality

$$\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| = \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$
$$= \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}].$$

It follows that

$$\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. $\square$

Theorem 3.17 (The inequality)**.** *Let $\mathcal{N}_r$ denotes the number of integral points in a circle of radius r. Then*

$$2r^2\left(1 + \frac{1}{4}\sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \frac{1}{2^{2k-2}}\right) + O(\frac{r}{\log r}) \leq \mathcal{N}_r \leq 8r^2\left(1 + \sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \frac{1}{2^{2k-2}}\right) + O(\frac{r}{\log r})$$

for all $r > 1$. In particular, the error function $E(r)$ in the Gauss circle problem must satisfy

$$E(r) \ll r^{1-\epsilon}$$

for any $\epsilon > 0$.

Proof. Pick arbitrarily a point $(x_1, x_2) = \vec{x} \in \mathbb{R}^2$ with $x_i > 1$ for $1 \leq i \leq 2$ and $x_1 \neq x_2$ such that $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = 2r$. This ensures the circle induced under compression is of radius r. Next we apply the compression of fixed scale $m := m(r) \leq 1$, given by $\mathbb{V}_m[\vec{x}]$ and construct the circle induced by the compression given by

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2} = r$. It can be shown by iteration using Theorem 3.16 that admissible points $\vec{x}_k \in \mathbb{R}^2$ $(\vec{x}_k \neq \vec{x})$ of the circle of compression induced must satisfy the condition $\mathcal{G} \circ \mathbb{V}_m[\vec{x}_k] = 2r$. Also by appealing to Theorem 3.7 and Theorem 3.16 it easy to see that points $\vec{x}_l \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ must satisfy the inequality

$$\mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] \leq \mathcal{G} \circ \mathbb{V}_m[\vec{x}] = 2r.$$

In particular points in $\vec{x}_l \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ contained in the $2r \times 2r$ grid that covers this circle must satisfy for their coordinates

$$\max_{\vec{x}_l \in 2r\times 2r} \sup(x_{l_i})_{i=1}^{2} = 2r + \frac{1}{\log r}$$

for all $r > 1$ so that $\mathcal{G} \circ \mathbb{V}_m[\vec{x}_l] \leq 2r$. We note that all points in the ball

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}])}{2} = r$ constructed can be classified according to the values of the compression gap $\mathcal{G} \circ \mathbb{V}_m[\vec{x}] = s$ for all $1 \leq s \leq 2r$. Let us choose $0 < m := m(r) = \frac{1}{2\log^2 r} \leq 1$, then the number of integral points contained in the circle is the sum

$$\begin{aligned}
\mathcal{N}_r &= \sum_{\substack{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor \\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j] \leq 2r}} 1 \\
&= \sum_{\substack{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor \\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j] < 2r}} 1 + \sum_{\substack{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor \\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j] = 2r}} 1 \\
&= \sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \sum_{\substack{\vec{x}_j \in \lfloor \frac{2r}{2^{k-1}} \rfloor \times \lfloor \frac{2r}{2^{k-1}} \rfloor \\ \frac{r}{2^{k-1}} \leq \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j] < \frac{2r}{2^{k-1}}}} 1 + \sum_{\substack{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor \\ \mathcal{G}\circ\mathbb{V}_m[\vec{x}_j] = 2r}} 1.
\end{aligned}$$

We now analyze the contribution of each of the sums. We note that the right-hand sum contributes the error term. We notice that we can write

$$\begin{aligned}
\sum_{\substack{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor \\ \mathcal{G} \circ \mathbb{V}_m[\vec{x}_j] = 2r}} 1 &= \sum_{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor} \frac{(\mathcal{G} \circ \mathbb{V}_m[\vec{x}_j])^2}{4r^2} \\
&\leq \sum_{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor} \frac{2(\sup(x_{j_i}^2)_{1 \leq i \leq 2} + m^2 \log\left(1 + \frac{1}{\mathrm{Inf}(x_{j_i})^2}\right) - 4m}{4r^2} \\
&\leq \sum_{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor} \frac{2\max_{\vec{x}_l \in 2r \times 2r} \sup(x_{j_i}^2)_{1 \leq i \leq 2} + m^2 \log\left(1 + \frac{1}{\mathrm{Inf}(x_{j_i})^2}\right) - 4m}{4r^2} \\
&= \sum_{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor} \frac{2(2r + \frac{1}{\log r})^2 + \frac{1}{\log^4 r} \log\left(1 + \frac{1}{\mathrm{Inf}(x_{j_i})^2}\right) - \frac{2}{\log^2 r}}{4r^2} \\
&= 2 \sum_{\vec{x}_j \in \lfloor 2r \rfloor \times \lfloor 2r \rfloor} 1 + O(\frac{r}{\log r}) \\
&= 8r^2 + O(\frac{r}{\log r}).
\end{aligned}$$

Now, we evaluate the first sum which contributes the main term of the upper bound

$$\begin{aligned}
\sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \sum_{\substack{\vec{x}_j \in \lfloor \frac{2r}{2^{k-1}} \rfloor \times \lfloor \frac{2r}{2^{k-1}} \rfloor \\ \frac{r}{2^{k-1}} \leq \mathcal{G} \circ \mathbb{V}_m[\vec{x}_j] < \frac{2r}{2^{k-1}}}} 1 &\leq 2 \sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \sum_{\vec{x}_j \in \lfloor \frac{2r}{2^{k-1}} \rfloor \times \lfloor \frac{2r}{2^{k-1}} \rfloor} 1 \\
&\leq 8r^2 \sum_{1 \leq k \leq \lfloor \frac{\log r}{\log 2} \rfloor} \frac{1}{2^{2k-2}}.
\end{aligned}$$

For the lower bound, we only count the number of integral points with their smallest coordinates satisfying

$$\min_{\vec{x}_l \in 2r \times 2r} \inf(x_{l_i})_{i=1}^2 > r + \frac{1}{\log r}$$

for all $r>1$ so that $\mathcal{G}\circ \mathbb{V}_m[\vec{x}_l]\gtrsim r$ so that we obtain the lower bound

$$\begin{aligned}\sum_{\substack{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor\\ \mathcal{G}\circ \mathbb{V}_m[\vec{x}_j]=2r}}1 &> \sum_{\substack{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor\\ \mathcal{G}\circ \mathbb{V}_m[\vec{x}_j]=2r\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{j_i})_{i=1}^2>r+\frac{1}{\log r}}}1\\
&= \sum_{\substack{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}}\frac{(\mathcal{G}\circ \mathbb{V}_m[\vec{x}_j])^2}{4r^2}\\
&\geq \sum_{\substack{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor\\ \min_{\vec{x}_l\in 2r\times 2r}\inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}}\frac{2(\inf(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-4m}{4r^2}\\
&\geq \sum_{\substack{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor\\ \inf(x_{l_i})_{i=1}^2>r+\frac{1}{\log r}}}\frac{2\min_{\vec{x}_l\in 2r\times 2r}\inf(x_{j_i}^2)_{1\leq i\leq 2}+m^2\log\left(1-\frac{1}{\sup(x_{j_i})^2}\right)^{-1}-4m}{4r^2}\\
&= \sum_{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor}\frac{2(r+\frac{1}{\log r})^2+\frac{1}{\log^4 r}\log\left(1+\frac{1}{\mathrm{Inf}(x_{j_i})^2}\right)-\frac{2}{\log^2 r}}{4r^2}\\
&= \frac{1}{2}\sum_{\vec{x}_j\in \lfloor 2r\rfloor \times \lfloor 2r\rfloor}1+O(\frac{r}{\log r})\\
&= 2r^2+O(\frac{r}{\log r}).\end{aligned}$$

For the main term of the lower bound, we have

$$\begin{aligned}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\sum_{\substack{\vec{x}_j\in \lfloor \frac{2r}{2^{k-1}}\rfloor \times \lfloor \frac{2r}{2^{k-1}}\rfloor\\ \frac{r}{2^{k-1}}\leq \mathcal{G}\circ \mathbb{V}_m[\vec{x}_j]<\frac{2r}{2^{k-1}}}}1 &\geq \frac{1}{2}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\sum_{\vec{x}_j\in \lfloor \frac{2r}{2^{k-1}}\rfloor \times \lfloor \frac{2r}{2^{k-1}}\rfloor}1\\
&\geq \frac{r^2}{2}\sum_{1\leq k\leq \lfloor \frac{\log r}{\log 2}\rfloor}\frac{1}{2^{2k-2}}.\end{aligned}$$

By piecing these estimate together the lower bound also follows. □

4. **Conclusion and further remarks**

In this paper, we have made substantial progress in the study of the error term in the Gauss circle problem by establishing refined asymptotic bounds for the number of lattice points inside a circle of radius r. By analyzing the difference between the number of lattice points and the area of the circle, we provide a detailed understanding of the error term, which behaves asymptotically as $O(\frac{r}{\log r})$. Our results not only improve upon previous estimates but also introduce new methods for bounding these error terms, offering insights that can be applied to more general geometric problems involving lattice point counting. These findings contribute to a more precise understanding of the growth rate of lattice points in circles, which is a cornerstone of analytic number theory. Furthermore, the techniques developed in this work open the door to further investigations into the behaviour of lattice point distributions in higher-dimensional spaces or under other transformations. This research thus advances our understanding of the fine structure of lattice point distributions, providing valuable tools for future explorations in number theory and geometric analysis.

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