ARITHMETIC COMBINATORICS OF DISCONTINUITIES AND GAPS IN PRIME NUMBERS

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Abstract. This work investigates Landau’s problems in the context of arithmetic combinatorics, combinatorial number theory and general topology, we define gaps and discontinuities of integer topologies at the neighbourhood of infinity.

1. Introduction

Additive, multiplicative and combinatorial number theory are useful in prime numbers, it is possible to extend the scope of definitions and methods used in topologies on the integers together with additive and multiplicative number theory, as well as few other derivatives to those theories.

1.1. Polynomial residues and arithmetic progressions.

Definition 1.1 (Maximal variable residue over \( \mathbb{N} \)). \( N \) is said to be the maximal polynomial residue (or maximal residue) of \( f : X \to Y, \ N \subseteq X, Y \) when

\[
N = \max f|_N \iff \exists N \subseteq \mathbb{N} \forall M \subseteq \mathbb{N} \ f|_N[N], f|_N[M] \subseteq \mathbb{N}, \ M \subseteq N
\]

Lemma 1.1.0.1 (Uniqueness of the maximal residue). There is only one maximal variable residue and it is unique

Proof. Take \( f : X \to Y, \ N \subseteq X, Y \), assume \( N, M \subseteq \mathbb{N}, N \neq M \) are both maximal variable residues, then by definition

\[
\forall A \subseteq \mathbb{N} \ f|_N[N], f|_N[M], f|_N[A] \subseteq \mathbb{N}, \ A \subseteq N, M
\]

But then for \( A = N \) and \( A = M \) we have

\[
N \subseteq M, M \subseteq N \iff N = M
\]

Which is a contradiction to our assumption. \( \square \)

Conjecture 1 (Maximal residue of a polynomial expression). For the a polynomial function we have a linear maximal residue

\[
\forall n, m \in \mathbb{N} \ f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{m} \text{ where } i = 1, \ldots, n a_i \in \mathbb{N}
\]

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Then the maximal residue of $f$ is $\max f|_N(x) = (ax + b)$
Where we have $m|a$ and $b \in \mathbb{N}_a$ such that
\[
\sum_{k=1}^n a_k b^k \in \sum_{k=1}^n a_k N^k \cap mN - a_0
\]
Computationally speaking, that would require a time complexity of $O(a)$ in order to find such $b$

**Theorem 1.1.1** (Maximal residue of a quadratic residue expression). For the quadratic function
\[
\forall n, m \in \mathbb{N} \quad f(x) = \frac{x^2 - n}{m}, \quad f : \mathbb{R} \to \mathbb{R}
\]
The maximal quadratic residue is a linear function $\max f|_N(x) = (ax + b)$
Where $b^2$ is a perfect square for the quadratic residue $n$ with the restriction that $b < a \leq m$, and $a$ is such that $m|a^2$, $m|2ab$
\[
b^2 = \min N^2 \cap mN + n
\]
\[
a^2 = \min N^2_{m+1} \cap m \left( \frac{m}{b^2} \cap N \right)
\]
Proof. It is straightforward due to the previous conjecture.\[\]
Proof by contradiction would be to pick up a different expression such as $cN + d$ so that $a \neq c$ and that not necessarily $b = d$, then
\[
\exists p, q \in \mathbb{Q} \quad d = mp + b, \quad c = mq, \quad nN + a \equiv m(qN + p) + b \quad \text{such that}
\]
\[
\frac{(cN + d)^2 - n}{m} = \frac{(cN)^2 + 2cdN + d^2 - n}{m} \equiv \frac{m^2(qN)^2 + 2mqdN + (mp)^2 + 2mpb + b^2 - n}{m}
\]
\[
\equiv m(qN)^2 + 2qdN + mp^2 + 2pb + \frac{b^2 - n}{m}
\]
\[
\equiv m(qN + p)^2 + 2b(qN + b) + \frac{b^2 - n}{m}
\]
\[
\equiv (mqN + mp + 2b)(qN + p) + \frac{b^2 - n}{m}
\]
\[
\equiv (cN + d + b)(qN + p) + \frac{b^2 - n}{m}
\]
\[
\equiv q(cN + d + b)(N + \frac{d - b}{c}) \nsuch that q \not\in \mathbb{N} \Rightarrow q \not\in \mathbb{N} \lor \frac{p}{q} \not\in \mathbb{N}
\]
In the opposite case it would appear that $cN + d \subseteq aN + b$
We begin by stating a few simple observations in factorization and multiplication of the integers.

2.1. Set inclusion and set limits.

**Theorem 2.0.1** (Invariance of inclusion under union). For any two sets \( A \subseteq B \), we have union of set function to be invariant with respect to inclusion as follows

\[
\bigcup_{a \in A} \mu(a) \subseteq \bigcup_{b \in B} \mu(b)
\]

**Proof.** It is trivial to see that

\[
\exists C \subseteq B \quad A \cup C = B \Rightarrow \bigcup_{b \in B} \mu(b) = \bigcup_{a \in A} \mu(b) \bigcup_{c \in C} \mu(c) \supseteq \bigcup_{a \in A} \mu(a)
\]

\[\square\]

**Theorem 2.0.2** (Limit of a diverging sequence in \( \mathbb{N} \)). For the natural numbers \((\mathbb{N}_{a_n})\) bounded by the sequence \(a_n\) we get the conditional limit

\[
\lim_{n \to \infty} a_n = \infty \Rightarrow \lim_{n \to \infty} \mathbb{N}_{a_n} = \mathbb{N}
\]

**Proof.** Assume by contradiction that

\[
\exists n \in \mathbb{N} \quad n \notin \lim_{m \to \infty} \mathbb{N}_{a_m} \Rightarrow n \notin \bigcup_{m \geq 1} \mathbb{N}_{a_m}
\]

But there obviously exists \(m \geq 1\) such that \(a_m > n\) therefore necessarily

\[
\forall n \in \mathbb{N} \quad n \in \lim_{m \to \infty} \mathbb{N}_{a_m}
\]

\[\square\]

A trivial consequence of that is the divergence of \(\mathbb{N}_n\) as \(n \to \infty\), that is

\[
\lim_{n \to \infty} \mathbb{N}_n = \mathbb{N}
\]
2.2. Continuity and gaps in infinitely disconnected topological spaces.

Due to intuitive and trivial reasons in the discreteness of the natural numbers, and rigorous in their infinite countability, it must be then easy to see that in order for a set \( \subseteq \mathbb{N} \) to be countably infinite, it must be unbounded.

However for all kind of numbers it is possible to say that existence of "gaps" between two numbers continously up to infinity would then contradict that set being bounded, ad hoc it is possible to discover the following simple definition.

**Definition 2.1** (Discontinuity in the neighbourhood of infinity). For a subset \( A \subseteq B \) we say that \( A \) is infinitely discontinuous (has infinitely many gaps) in \( B \) iff

\[
A \propto B \iff \exists x \in B \setminus A \Rightarrow \forall y \in B \cap (x, \infty) \ y \notin A
\]

\[
A \propto B \iff \exists x \in A \ B \cap (x, \infty) \subseteq A
\]

This simple notion can be thought of as "There are gaps up to infinity" or complementary "There are no gaps in the neighbourhood of infinity", trivially the notion of "infinitely continuous" is just the set inclusion for a sufficiently large number.

We will also extend the definition for disconnected topological spaces

**Definition 2.2** (Infinitely disconnected topological space). A disconnected topological space \( A \) with an order topology on it is infinitely disconnected in \( U \) as it’s subspace if it’s right order topology is also disconnected and infinitely disconnected in \( U \) as it’s subspace

\[
A \propto U \iff \forall x \in A \exists y \in U \cap (x, \infty) \ y \notin A
\]

One could also define the same using topological cofiniteness and cointability.

It is trivially possible to express the same definition of infinite discontinuity in limit notation

**Notation 2.1** (Right order notation of discontinuity). We could use the limit to notate infinitely many gaps in a topological space \( A \) infinitely disconnected in \( B \)

\[
A \propto B \iff \forall x \in \mathbb{R} A \cap (x, \infty) \propto B \iff \forall x \in \mathbb{R} B \cap (x, \infty) \notin A
\]

Additionaly if the limit doesn’t approaches the empty set, then it’s trivial that \( A \) is countably infinite

**Proof.** It’s trivial.

\[
\forall a \in A \exists b \in B \cap (a, \infty) \ b \notin A \Rightarrow B \cap (a, \infty) \notin A
\]

\( \Box \)
We derive the following properties for infinite discontinuity

**Theorem 2.2.1** (Properties of discontinuity in the neighbourhood of infinity). Discontinuity holds the following properties for any $A, B \subseteq \mathbb{N}$, $|A| = |B| = \aleph_0$

1. Continuity of inclusion $A \subseteq B \Rightarrow B \not\propto A$
2. Commutative continuity of inclusion $A \subseteq B \not\propto C \Rightarrow B \not\propto A \not\propto C$
3. Transitivity of continuity $A \not\propto B \not\propto C \Rightarrow A \not\propto C$
4. Anticommutativity of discontinuity $A \not\propto B \not\propto \neg B \not\propto \neg A$
5. Anticommutativity of continuity $A \propto B \not\propto \neg B \not\propto \neg A$
6. Transitivity of discontinuity over set relations $(A \cup B) \not\propto C \Rightarrow A, B \not\propto C \Rightarrow (A \cap B) \not\propto C$
7. Distributivity of negation over discontinuity $(A \cup B) \not\propto C \Rightarrow A \propto C \setminus B$
8. Distributivity of negation over continuity $(A \cup B) \propto C \Rightarrow A \not\propto C \setminus B$
9. Exclusive continuity $(A \cap B) \not\propto A \Rightarrow B \not\propto A$
10. Common continuity $A \not\propto A \cup B \iff A \not\propto B$

**Proof.** It is fairly clear that for (1)

$$\forall a \in A \ a \in B \Rightarrow \exists b \in B \ \forall a \in A \ a > b \Rightarrow a \in B$$

Then due to that property, it follows for (2) that

$$B \not\propto A \ \forall b \in \mathbb{N} \ C \cap (b, \infty) \not\subseteq B \supseteq A \Rightarrow C \cap (b, \infty) \not\subseteq A$$

For (3) it is possible to combine the definition to get

$$\exists (a, b) \in A \times B \ \forall (b', c) \in (B \cap (a, \infty) \times C \cap (b, \infty)) \ b' \in A, c \in B$$

$$c > \max a, b \Rightarrow c \in C, c \in B \Rightarrow c \in A$$

It might seem quite intuitive that $A \propto B \propto C \Rightarrow A \propto C$, but a counter example to that would be $2N \propto 3N \propto 4N$ but $2N \not\propto 4N$

For (4) it is fairly obvious that

$$\forall a \in A \ \exists b \in B \cap (a, \infty) \ b \not\in A \Rightarrow \exists b \in (\neg A) \cap (a, \infty) \ b \not\in \neg B$$

It’s also obvious for (5) we have the identity

$$\exists a \in A \ B \cap (a, \infty) \subset A \Rightarrow \neg A \subset B \cup (\neg \infty, a] \Rightarrow \neg A \cap (a, \infty) \subset B \cap (a, \infty)$$

$$\exists x \in \mathbb{R} \ A \cap B \cap (x, \infty) = B \cap (x, \infty) \iff (A \cup \neg(B \cap (x, \infty))) \cap \mathbb{R} = R$$

$$\iff (\neg A \cap \neg(B \cap (x, \infty))) = \neg A$$

$$\iff ()$$

Assume the opposite that

$$(A \cap B) \not\propto A \Rightarrow A \subseteq B$$

Transitivity of over set relations (6) is straightforward due to

$$\forall x \in A \cup B \ C \cap (x, \infty) \not\subseteq A \cup B \Rightarrow C \cap (x, \infty) \not\subseteq A, B$$

$$\Rightarrow C \cap (x, \infty) \not\subseteq A \cap B$$

For distributivity of negation (7) it’s trivial that

$$\forall x \in A \cup B \ \exists c \in C \cap (x, \infty) \ c \not\in A \cup B \Rightarrow \exists c \in C$$
And for continuity
\[\exists x \in A \cup B \forall c \in C \cap (x, \infty) \ c \in A \cup B \Rightarrow (c \notin B \Rightarrow c \in A)\]
Consequently for exclusive and common continuity it happens that
\[\exists x \in A \cap B \forall a \in A \cap (x, \infty) \ a \in A \cap B \Rightarrow a \in A \land a \in B \Rightarrow a \in B\]
and
\[\exists a \in A \forall x \in (A \cup B) \cap (a, \infty) \ x \in A \text{ but also } x \in B \cap (a, \infty)\]

Another few properties just as trivial as the previous are for set operators and distributive discontinuity on them.

**Theorem 2.2.2** (Distributivity of discontinuity over set operators). For a given discontinuous (continuous) sets \((A \propto B, C \propto D)\) \((A \propto B, C \propto D)\) we have the distributive properties over set operators
\[((A \cap C) \propto (B \cap D))\]
\((A \cup C) \propto (B \cup D) ((A \cup C) \propto (B \cup D))\)

**Proof.**
\[\forall a \in A, b \in B \exists c \in C \cap (a, \infty), d \in D \cap (b, \infty) \ c \notin A, d \notin B\]
\[\forall x \in A \cap C \exists y \in B \cap D \ y \notin A \cap C\]
\[y \in B \land y \in D\]
Then by transitivity of discontinuity
\[\exists a \in A, b \in B \forall c \in C \cap (a, \infty), d \in D \cap (b, \infty) \ c \in A, d \in B \Rightarrow C \cap D \cap (\max (a, b), \infty) \subset A \cap B \subset A, B\]
For the second few identities we have
\[\forall a \in A, c \in C \exists b \in B \cap (a, \infty), d \in D \cap (c, \infty) \ b \notin A, d \notin C\]
Therefore trivially
\[\exists x \in (B \cup D) \cap (\max (a, c), \infty) \ x \notin A \cup C\]
And for the last identity
\[\exists a \in A, c \in C \forall b \in B \cap (a, \infty), d \in D \cap (c, \infty) \ b \in A, d \in C \Rightarrow \exists a \in A, c \in C \ B \cap (a, \infty) \subseteq A, D \cap (c, \infty) \subseteq C\]

**Theorem 2.2.3** (Discontinuity of infinite complementary sets). For a countably infinite set \(C \supseteq A \cup B\), where \(A, B\) are countably infinite unbounded sets with \(A\) infinitely discontinuous in \(B\), \(A\) must be infinitely discontinuous in \(C\)

**Proof.** If \(A\) was to be infinitely continuous in \(C\), then due to the previous theorem it would appear that due to continuity of inclusion
\[A \propto C \supset B \Rightarrow A \propto C \propto B \Rightarrow A \propto B\]
Which is a contradiction to \(A\) being infinitely discontinuous in \(B\)
It is useful to observe that for disjoint sets $A, B$ this holds true. Another important consequence of discontinuously complementary sets would be congruent to the previous one but with one subset being proprietary.

**Theorem 2.2.4** (Complementary infinite discontinuity). For a countably infinite set $C = A \cup B$ where $A$ is infinitely discontinuous in $C$, if $B$ is infinitely discontinuous in $A$, then $B$ is infinitely discontinuous in $C$.

**Proof.** Assume the opposite, then by common continuity

$$B \not\propto C \Rightarrow B \not\propto A \cup B \Rightarrow B \not\propto A$$

Which is a contradiction. □

As well as in the discontinuity of complementary sets, it is trivial that for $A \cap B = \emptyset$ complementary discontinuity holds true.

**Theorem 2.2.5** (Preservance of discontinuity of set functions). For a given set function $\mu : F \rightarrow \mathbb{R}$ with $F$ being a family of sets over $\Omega$, for every $F \in F$ that is discontinuous in $C \in F, \mu(F)$ is also discontinuous in $\mu(C)$.

**Proof.** It's trivial that

$$F \propto C \Rightarrow \forall f \in F \exists c \in C \cap (f, \infty) \ c \notin F \Rightarrow \mu(c) \notin \mu(F)$$

□

2.3. **Discontinuous Furstenberg’s topology and linear fractional transformations.** It’s possible to determine the gaps in Furstenberg’s topology together with another multiples of them, in particular the following observation is fairly intuitive as well as important when dealing with gaps in composite numbers.

**Theorem 2.2.6** (Infinitely discontinuous composite multiples). Infinitely many composite multiples are discontinuous in the natural numbers

$$\forall A \subseteq \mathbb{N} + 1 \ A\mathbb{N} = \bigcup_{a \in A} a\mathbb{N} \propto \mathbb{N}$$

**Proof.** It is fairly trivial that $\forall a, b \in \mathbb{N} \ a\mathbb{N} \cap b\mathbb{N} = \text{lcm}(a, b)\mathbb{N} \subseteq ab\mathbb{N}$. Then inductively

$$\bigcap_{a \in A} a\mathbb{N} = \text{lcm}(A)\mathbb{N} \subseteq \left( \prod_{a \in A} a \right) \mathbb{N} \propto \mathbb{N}$$

Since trivially for any $A \subseteq \mathbb{N}$ such that $\text{lcm}(A) \neq 1$ we have

$$\text{lcm}(A)\mathbb{N} + 1 \cap \text{lcm}(A)\mathbb{N} = \emptyset \Rightarrow \text{lcm}(A)\mathbb{N} \propto \mathbb{N}$$

But then by complementary discontinuity 2.2.4 the complement must be infinitely discontinuous

$$\bigcup_{a \in A} a\mathbb{N} + \mathbb{N}_a \propto \mathbb{N}$$

But then obviously due to commutative continuity of inclusion 2 it must be that

$$\bigcup_{a \in A} a\mathbb{N} + 1 \subseteq \bigcup_{a \in A} a\mathbb{N} + \mathbb{N}_a \propto \mathbb{N} \Rightarrow \bigcup_{a \in A} a\mathbb{N} + 1 \propto \mathbb{N}$$
Then arithmetically follows
\[ \bigcup_{a \in A} a \mathbb{N} \propto \mathbb{N} - 1 \not\propto N \Rightarrow \bigcup_{a \in A} a \mathbb{N} \propto \mathbb{N} \]
So that additionally or exclusively when \( A \subseteq \mathbb{N} + 1 \) then \( \bigcup_{a \in A} a \mathbb{N} \propto \mathbb{N} \)

Implicitly due to complementary discontinuity it must be that composite numbers and prime numbers \( \mathbb{NN}, \mathbb{P} \) are infinitely discontinuous in \( \mathbb{N} \) (Infinitude of prime numbers)
Then the complementary theorem can be derived.

**Theorem 2.2.7** (Discontinuity of Furstenberg’s topology). *Furstenberg’s topologies are discontinuous in the integers*

\[ \forall A \subseteq \mathbb{Z} \setminus \{ \pm 1 \}, B \propto \mathbb{N} \ A \mathbb{Z} \oplus B \propto \mathbb{N} \]
Where \( B \) is dependent on \( A \subseteq \mathbb{Z} \setminus \{ \pm 1 \} \) such that \( B \subseteq \mathbb{N} \)

*Proof.* Then taking the complement of the composite multiples
\[ \forall A \subseteq \mathbb{Z} \setminus \{ \pm 1 \}, B \subseteq \mathbb{N} \ A \mathbb{Z} = \bigcup_{a \in A} a \mathbb{Z} = \bigcup_{a \in A} a \mathbb{Z} = B \subseteq \mathbb{N} \]
Then by the previous theorem and complementary discontinuity 2.2.4 \( A \mathbb{Z} \oplus B \propto \mathbb{Z} \)

Next it is possible to define the tuples and pairings of integer sets repeating in the natural numbers.

**Definition 2.3** (n-Tuples topology). n-Tuples topology of a topology \( \mathcal{A} \) on the positive integers \( \mathbb{N} \) with morphisms \( \tau_0, \ldots, \tau_n : \mathcal{A} \to \mathcal{P}(\mathbb{N}) \) defines
\[ \mathcal{A}(\tau_0, \ldots, \tau_n) = \mathcal{A} \cap \tau_0(\mathcal{A}) \cap \cdots \cap \tau_n(\mathcal{A}) \]

It’s possible to theorize that there are no infinitely discontinuous arithmetic n-tuples, that is

**Conjecture 2** (Infinitely discontinuous pair topologies of n-tuples). *Infinitely discontinuous n-tuple topology of \( \mathcal{A} \) in some \( \tau_m(\mathcal{A}) \) with set function \( \tau_n, n \in \mathbb{N} \) have no \( \mathbb{N} \) tuple topology, for any \( N \subseteq \mathbb{N} \)
\[ \mathcal{A} \bigcap_{n \in \mathbb{N}} \tau_n(\mathcal{A}) \propto \tau_m(\mathcal{A}) \Rightarrow \mathcal{A} \bigcap_{n \in \mathbb{N}} \tau_n(\mathcal{A}) = \emptyset \]

**Theorem 2.3.1** (Discontinuous exception in a discontinuous intersection). *For any infinitely discontinuous \( A_n, B \propto \mathbb{N} \) for all \( n \in \mathbb{N} \) with \( n \neq m \), \( A_n \cap A_m \) infinitely discontinuous in \( \mathbb{N} \)
\[ \bigcap_{n \in \mathbb{N}} A_n \propto B \Rightarrow \exists n \in \mathbb{N} A_n \propto B \]
Proof. For $S = \bigcap_{n \in \mathbb{N}} A_n$ there must be some $C$ such that $S \not\propto C$, then by definition
\[
\forall s \in C \exists b \in B \cap (s, \infty) \ b \not\in S
\]
But if all $A_n$ were infinitely continuous in $B$, it would necessarily mean that their intersection

**Lemma 2.3.1.1** (Arithmetic Invariance of the integers). For evenly spaced integer topology of the form $\mathbb{Z} \subset \mathbb{Z} - a$ where $a \in A \subseteq \mathbb{Z}$ we have $\mathbb{Z} - a \not\propto \mathbb{Z}$

**Conjecture 3** (Infinitely many certain arithmetic pairs). There exists a countably infinite arithmetic pairs topology of $A$ with an infinitely countable set $A$ that is cyclical in the neighbourhood of infinity
\[
A \not\propto O(n) \Rightarrow \exists n \in \mathbb{N} \ |A \cap A + n| = \aleph_0
\]
Proof. By the previous lemma it follows that $A \cap A \cap \mathbb{N} = A \cap \mathbb{Z} = A$, but then assume by contradiction that for all $n \in \mathbb{N} A \cap A - n$ is finite, then the union of all such pairs would have to be finite which is a contradiction since $|A| = \aleph_0$

We will also study fractions of linear and polynomial expressions over the integers.

**Lemma 2.3.1.2** (Infinite discontinuity of linear fractional transformations with same free terms). Linear fractional transformation with the same free term have infinitely many solutions over the integers with gaps at $\mathbb{Z}_{\gcd(a,b)}$
Proof. Take $a, b, c \in \mathbb{Z}$, then for all $n \in \mathbb{Z}, z \in \gcd(a,b)\mathbb{Z}$
\[
\frac{a + c}{nb + c} \in \mathbb{Z}
\]

**Conjecture 4** (Infinitely discontinuity linear fractional transformations). Linear fractional transformations over the integers have infinitely discontinuously many solutions in the integers with coprime different free terms in the denominator and numerator
2.4. Gaps and inductive discontinuity of prime tuples.

Theorem 2.3.2 (Infinitely discontinuous pairs of composite numbers). There are infinitely discontinuously many pairs of composite numbers
\[ \forall n \in \mathbb{N} \mathbb{N} \cap \mathbb{N} + n \propto \mathbb{N} \]

Proof. We will notate by \( \mathbb{N} \) the composite numbers, then assuming that there are infinitely continuously many composite numbers of the form \( \mathbb{N} + n \in \mathbb{N} \), we will get
\[ \mathbb{N} + n \propto \mathbb{N} \]

Then necessarily for all factors \( m \in \mathbb{N}_1 \), the composites with a difference of \( n \) must be infinitely continuous in those multiples, formally due to transitivity of continuity
\[ \forall m \in \mathbb{N}_1 \mathbb{N} + n \propto \mathbb{N} \propto m \mathbb{N}_1 = \mathbb{N} \]

Where \( \mathbb{N}_1 = \mathbb{N} \setminus \{1\} \) but then according to Dirichlet’s theorem, we could find for all \( n \in \mathbb{N} \) such coprime \( m \in \mathbb{N} \) forming Golomb’s topology\(^2\) in which the composite numbers are infinitely discontinuous.
\[ \mathbb{N} \propto m \mathbb{N}_1 = \mathbb{N} \]

Contradiction, next for all \( m \in \mathbb{N}_1 \) such that \( (n, m) > 1 \) it happens that there are infinitely many composite numbers of the form \( m \mathbb{N}_1 - n \) since we can always factor out \( \text{gcd}(n, m) > 1 \) in which case \( \mathbb{N} \propto m \mathbb{N}_1 - n \)

Then it’s possible to figure out for what differences between the prime numbers are infinitely continuous in the primes

Theorem 2.3.3 (infinitely continuous prime tuples with an even difference). For any finite subset \( S \) of the even natural numbers, the arithmetic differences of \( \mathbb{P} \) indexed on it’s complement are infinitely continuous in the primes
\[ \bigcup_{n \in 2\mathbb{N} \setminus S} \mathbb{P} - n \propto \mathbb{P} \]

Proof. For any finite \( S \subseteq 2\mathbb{N} \) and prime number \( p \) we could take sufficiently large \( n \) such that \( n > |S| \) so that for all \( s \neq s' \in S \)
\[ p - s = p_n - (s + t_n) \text{ where } p = p_n - t_n, \ t_n \in 2\mathbb{N} \text{ and } s + t_n \neq s' \in S \]

For infinitely many primes \( p_n > p \) meaning that \( \exists t_n \in 2\mathbb{N} \ t_n + s \in 2\mathbb{N} \setminus S \) then it’s sufficient to say that there are more than enough primes to choose from such that the gap between the nth prime is not in \( S \setminus S \)

Lemma 2.3.3.1 (Discontinuous transitivity of prime composite differences).
\[ |\mathbb{N} + k \cap \mathbb{N}| = |\mathbb{N} + k \cap \mathbb{P}| = \aleph_0 \]

Proof. Trivially according to the infinitude discontinuity of pairs of composite numbers as of 2.3.2 and due to complementary discontinuity 2.2.4
Theorem 2.3.4 (Infinitely many certain arithmetic prime pairs). There are infinitely many \( n \in 2\mathbb{N} \) for which there are infinitely many pairs of prime numbers of the form \((p, p + n)\).

Proof. The arithmetic n-tuple topology indexed on any complemented finite set is infinitely continuous in the primes as stated by 2.3.3, therefore for any finite \( S \subset 2\mathbb{N} \) we have

\[
\bigcap_{n \in 2\mathbb{N} \setminus S} \ NN - n \propto P \Rightarrow \exists s \in 2\mathbb{N} \setminus S \ NN - s \propto P
\]

Inductively, the set \( S \) could always be extended so that the identity in 2.3.3 is satisfied, consequently there could always be found an even \( n \in 2\mathbb{N} \) such that there are infinitely many (prime, composite) pairs with that difference. But due to the infinitude of such pairs as of the previous lemma, and by complementary discontinuity of theorem 2.2.4 it follows that there are infinitely discontinuously many primes twins of infinitely many even differences

\[
\ NN - n \propto P \Rightarrow \ NN - n \cap P \propto P \Rightarrow P \cap P - n \propto P
\]

\(\square\)

It is possible to theorize the same for infinitely discontinuous complementary even integers

**Conjecture 5** (Infinitely discontinuously many certain arithmetic prime pairs). For any infinitely discontinuous \( S \subset 2\mathbb{N} \), the arithmetic n-tuple topology of \( P \) indexed on it’s complement is infinitely continuous in the primes

\[
\bigcup_{n \in 2\mathbb{N} \setminus S} P - n \not\propto P
\]
2.5. **Goldbach’s conjecture and equidistant pairs.**

Equidistant pairs in n-tuple topologies of the prime numbers finds application in Goldbach’s conjecture, we find a different representation for the summation of composite numbers, in prime numbers.

**Lemma 2.3.4.1** (Prime complementary partition for the equidistant integers). *The common equidistances from the composite numbers are all the possible combinations ( + , −) with respect to \( n \in \mathbb{N} \) as arithmetic differences in the n-tuple topology of the composite numbers, that is*

\[
\bigcap_{n \in \mathbb{N}} \mathbb{N} - n \cup \mathbb{N} + n = \bigcup_{N \in P(\mathbb{N})} \bigcap_{n \in \mathbb{N} \setminus (\mathbb{N} \setminus N)} \mathbb{N} + n
\]

*Proof.* With the first few terms it’s simple to see that

\[
\bigcap_{t \in n,m} (\mathbb{N} - t \cup \mathbb{N} + t) = \mathbb{N} \pm n \cap \mathbb{N} \pm m
\]

\[
= \bigcup_{N \in P(n,m)} \bigcap_{n \in \mathbb{N} \setminus (\mathbb{N} \setminus N \setminus \{n,m\})} \mathbb{N} + n
\]

Inductively the identity follows for all \( n \in \mathbb{N} \). \( \square \)

Temporarily introduce a singleton notation as follows

\[
\bigcap_{n \in \mathbb{N}} \mathbb{N} - n \cap \mathbb{N} + n = \bigcap_{n \in \mathbb{N}} \mathbb{N} + \text{rnd}(\pm)n
\]

Intuitively the \( \text{rnd} \) function is a non deterministic random function that for each \( n \in \mathbb{N} \) may give a different sequence of \( \pm \).

Then it’s possible to find that there are no such composite n-tuple topology of equidistances

**Theorem 2.3.5** (Equidistances from the composite numbers are uncommon). *There are no natural numbers in infinitely many equidistances from the composite numbers*

\[
\bigcap_{n \in \mathbb{N}} \mathbb{N} - n \cup \mathbb{N} + n = \emptyset
\]

*Proof.* It’s intuitively simple to see that there are no common composite numbers with infinitely many differences of \( n \in \mathbb{N} \) partitioned in the previous lemma, let \( N \in P(\mathbb{N}) \), then for infinitely many positive integers in \( N \) it follows

\[
\bigcap_{n \in N} \mathbb{N} + n = \emptyset
\]

That is trivial because if we assume by contradiction that there is such tuple, then there is \( n \in \mathbb{N} \) that is in that intersection, but it would follow that \( \forall m > n \ n \notin \mathbb{N} + m \), therefore in the intersection it cannot be, there are no tuples of that form.

In case there are finitely many positive integers, then the negative integers are infinitely continuous in \( \mathbb{N} \), but that means there is a neighbourhood of infinity in which the natural numbers construct such tuples.

\[
\bigcap_{n \in \mathbb{N} \cap (a,\infty)} \mathbb{N} - n = \emptyset
\]
But that must also be trivial because $NN$ is not infinitely continuous in $N$ (Due to the infinitude of prime numbers and complementary discontinuity 2.2.4), then for any $m \in NN - (n + 1)$ there must be such $n \in N \cap (a + 1, \infty)$ so that $m \notin NN - n$

In other words, there are no composite numbers that are simultaneously $nm - 1, nm - 2, \cdots$ for any different composite $nm \in NN$

Intuitively the last statement is not unique for the prime/composite numbers only, in fact for any kind of numbers $S$ with random like sequence of gaps $g_n = s_{n+1} - s_n$, there is no randomless $N \subseteq N$ such that there is an $s \in S$ for which \(\forall n \in N s - n \in S\) are simultaneously

**Theorem 2.3.6** (Equidistance of the natural numbers from prime numbers). The natural numbers are equidistant from some prime numbers, formally

$$\forall n \in N \exists m \in N \ n \pm m \in P$$

Proof. An equivalent statement to equidistance from the natural numbers would be

$$\bigcup_{n \in N} P - n \cap N \cap P + n = N$$

Then it follows that

$$N \cap \bigcup_{n \in N} P - n \cap P + n = N$$

Which would mean that

$$N \subseteq \bigcup_{n \in N} P - n \cap P + n \Leftarrow \bigcap_{n \in N} NN - n \cup NN + n = \emptyset$$

But as proven in the previous theorem, the complement must be empty

$$\bigcap_{n \in N} NN - n \cup NN + n = \emptyset$$

Therefore for any natural numbers there must be a pair of equidistant prime numbers. \(\square\)

It is then possible to evaluate the even natural numbers in their prime sum

**Theorem 2.3.7** (Even natural numbers have a prime sum form).

$$2N = P \oplus P$$

Proof. Following the previos theorem for all natural numbers there are equidistant odd prime numbers

$$\forall n \in N \exists m \in N \ n \pm m \in P \Rightarrow p_1 + m = z = p_2 - m, \ p_1, p_2 \in P$$

Then to summarize

$$(p_1 + m) + (p_2 - m) = p_1 + p_2 = 2z$$\(\square\)

A rather trivial consequence of that would be that for odd numbers there is a sum, in particular according completing Goldbach’s weak conjecture

$$2N + 1 = P \oplus 2N = P \oplus P \oplus P$$

In a more rigorous writing, when considering $P$ to include the prime even number 2, it simply means that $N + 1 \setminus (NN \cap P + 2N \setminus \{2\}) = P \oplus P$
2.6. **Prime remainder integers topology.**

**Definition 2.4** (Prime remainder integer topology). The prime remainder integer topology on the non negative integers is the collection of all $a \mathbb{N} + p$ with $a > p > 0$ and $p$ prime.

It is possible to observe a few useful properties of this topology in certain forms.

**Theorem 2.4.1** (Maximal quadratic prime residue). *The prime remainder integer topology is the maximal quadratic residue of a quadratic residue expression with a quadratic residue 1 modulo $p$ where $p$ is prime, in particular*

\[
x^2 \equiv 1 \pmod{p + 1} \iff x \in (p + 1) \mathbb{N} \oplus \mathbb{P}_{p+1} \]

\[
n^2 \equiv 1 \pmod{m}
\]

*Proof.* By theorem 1.1.1 it is obvious that $(px + m_p)^2 \equiv 1 \pmod{p}$ for $p > m_p \in \mathbb{P}$.

---

1. By the set $A_n$ we mean the set of elements of $A$ lesser than $n$
2.7. Discontinuous Furstenberg’s topology in the form \( \mathbb{N}^2 + a \). The last Landau’s problem is determinable under arithmetic of integer topologies and their discontinuities.

We find the composite representation of the form \( \mathbb{N}^2 + a \) in order to prove it is discontinuous.

**Lemma 2.4.1.1** (Prime ideals of the form \( \mathbb{N}^2 + a \)). There are infinitely discontinuously many \( n \in \mathbb{N} \) such that \( n\mathbb{N} \) is a prime ideal in \( \mathbb{N}^2 + a \) denoted by \( (\mathbb{N}^2 + a)\mathbb{N} \) for \( a \in \mathbb{N} \)

**Proof.** Then \( x^2 + a, y^2 + a \in \mathbb{N}^2 + a \) such that \( (x^2 + a)(y^2 + a) \in n\mathbb{N} \), then there are infinitely many \( n \) of the form \( x^2 + a \) or \( y^2 + a \), that is \( n \) must be of the form \( \mathbb{N}^2 + a \) which is discontinuous. \( \square \)

**Lemma 2.4.1.2** (Quadratic residue modulo \( \mathbb{N}^2 + a \)). Solutions in \( \mathbb{N}_m \) satisfying the quadratic residue expression \( x^2 \equiv a \pmod{m} \) with \( m \in \mathbb{N}^2 + a \) are exactly \( \sqrt{m-a}, m - \sqrt{m-a} \)

**Proof.** It’s easy to check that \( \sqrt{m-a}^2 + a \equiv m^2 - 2am \equiv 0 \equiv m^2 + m - 2m\sqrt{m-a} - 1 \equiv (m - \sqrt{m-a})^2 + a \pmod{m} \) \( \square \)

**Theorem 2.4.2** (Infinitely discontinuous composite numbers in the form \( \mathbb{N}^2 + a \)). There are infinitely discontinuously many composite numbers in the form \( \mathbb{N}^2 + a \), they can be represented by squares of Furstenberg’s topology with \( t_m = \{ \sqrt{m-a}, m - \sqrt{m-a} \} \)

\[ \bigcup_{m \in \mathbb{N}^2 + a} \left( (m\mathbb{N}_0 + t_m) \setminus \{ \sqrt{m-a} \} \right)^2 + a \]

**Proof.** For each composite \( m\mathbb{N} \) it is possible to find it’s representation in the form of \( n^2 + a \), then according to 1.1.1 and the previous lemmas

\[ \mathbb{N}\mathbb{N} \cap \mathbb{N}^2 + a = \bigcup_{m \in \mathbb{N}^2 + a} m \left( \mathbb{N}_1 \cap \frac{\mathbb{N}^2 + a}{m} \right) = \bigcup_{m \in \mathbb{N}^2 + a} \left( (m\mathbb{N}_0 + t_m) \setminus \{ \sqrt{m-a} \} \right)^2 + a \]

With \( t_m \in \mathbb{N}_m \) such that \( t_m^2 + a \mid m \), in particular \( t_m \in \{ \sqrt{m-a}, m - \sqrt{m-a} \} \) due to the previous lemma.

It must be obvious that due to discontinuity of Furstenberg’s topology 2.2.7 with preservance of discontinuity in 2.2.5 the following holds

\[ \bigcup_{m \in \mathbb{N}^2 + a} m\mathbb{N}_0 + t_m \propto \mathbb{N} \Rightarrow \bigcup_{m \in \mathbb{N}^2 + a} \left( (m\mathbb{N}_0 + t_m) \setminus \{ \sqrt{m-a} \} \right)^2 + a \propto \mathbb{N}^2 + a \]

Therefore \( \mathbb{N}\mathbb{N} \cap \mathbb{N}^2 + a \propto \mathbb{N}^2 + a \) \( \square \)

Then the infinitude of prime numbers in that form follows.
Theorem 2.4.3 (Infinitely discontinuous prime numbers in $N^2 + a$). There are infinitely discontinuously many prime numbers of the form $N^2 + a$.

Proof. It’s consequent from the previous theorem that $NN \cap N^2 + a$ is infinitely discontinuous in $N^2 + a$, therefore by complementary discontinuity theorem 2.2.4 $P \cap N^2 + a$ is infinitely discontinuous in $N^2 + a$. □

It’s also possible to conjure the same for Bunyakovsky’s conjecture.

2.8. Complementary discontinuity of primes in irreducible polynomials. It’s possible to generalize the previous for the polynomial ring $Z[X]$ with coprime coefficients, then conjure that

Conjecture 6 (Infinitely discontinuous composite numbers in the polynomial ring $Z\langle X \rangle$). There are infinitely discontinuously many composite numbers in the form $f(n)$ for $f \in Z[X], f|_N(x) = \sum_{i=1}^{n} a_i x^i + a_0$ that can be represented by divisors of that form with $f$ on them with $R(f)$ as it’s range

$$\bigcup_{m \in R(f)} \sum_{i=1}^{n} a_i (mN \oplus N_m) + a_0$$
References
