Series representations for π^3 involving the golden ratio

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Abstract

In this note we give some representations of π^3 involving infinite sums and the golden ratio.

Introduction

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

and

$$\varphi = \frac{1+\sqrt{5}}{2}$$

Remark 1: φ is the Golden ratio.

In this note we give some series for π^3 .

List of formulae involving π^3

Entry 1.

$$\pi^3 = 32 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$$

Entry 2.

$$\pi^3 = 32 \sum_{n=1}^{\infty} (-1)^{n-1}\, n \left(\frac{1}{(2n-1)^3} + \frac{1}{(2n+1)^3} \right)$$

$$\pi^3 = 32 \sum_{n=1}^{\infty} (-1)^{n-1}\, n^2 \left(\frac{1}{(2n-1)^4} + \frac{1}{(2n+1)^4} \right)$$

Entry 3.

$$\pi^3 = 16 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2k+1)^{-3}$$

$$\pi^3 = \frac{64}{3} \sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} (-2)^k \binom{n}{k} (2k+1)^{-3}$$

Entry 4.

$$\pi^3 = 3\sqrt{105}\left(1+\sum_{n=1}^{\infty}\frac{1}{(n+1)^6\left(\sqrt{H_n^{(6)}}+\sqrt{H_{n+1}^{(6)}}\right)}\right)$$

Remark: The generalized harmonic numbers $H_n^{(k)}$ are defined by $H_n^{(k)} = \sum_{m=1}^{n} m^{-k}, k \in \mathbb{N}$.

Entry 5.

$$\pi^3 = 32 - 2\sum_{n=0}^{\infty}(-1)^n\,(n+1)(n+2)2^{-n}(1-2^{-n-2})\,\zeta(n+3)$$

$$\pi^3 = 32 - 4\sum_{n=1}^{\infty}n(n+1)\,2^{-n}\big(1-(1-2^{-n-1})\zeta(n+2)\big)$$

$$\pi^3 = 32 - \frac{1}{8}\sum_{n=1}^{\infty}n(n+1)\,2^{-4n}\big(48\zeta(2n+2)+\zeta(2n+4)\big)$$

$$\pi^3 = 32 - \frac{1}{4}\sum_{n=0}^{\infty}(n+1)\,(2n+3)2^{-4n}\zeta(2n+4)$$

$$\frac{8\pi^3}{3\sqrt{3}} = \sum_{n=0}^{\infty}(n+1)\,(n+2)\left(\left(\frac{2}{3}\right)^n-\left(\frac{1}{3}\right)^n\right)\,\zeta(n+3)$$

$$\frac{\pi^3}{4\sqrt{3}} = \sum_{n=0}^{\infty}(n+1)\,(2n+3)3^{-2n}(1-2^{-2n-4})\,\zeta(2n+4)$$

Remark 2: $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$, $n > 1$, is the Riemann zeta function.

Entry 6.

$$\pi^3 = 64\sum_{n=0}^{\infty}\frac{48n^2+48n+13}{(4n+1)^3(4n+3)^3}$$

Entry 7.

$$\pi^3 = 256\sum_{n=0}^{\infty}\frac{(3n+1)^2}{(4n+1)^3(4n+3)^3}+192\sum_{n=0}^{\infty}\frac{(2n+1)(2n+3)}{(4n+1)^3(4n+3)^3}$$

$$\pi^3 = 256\sum_{n=0}^{\infty}\frac{(3n+2)^2}{(4n+1)^3(4n+3)^3}+192\sum_{n=0}^{\infty}\frac{(2n-1)(2n+1)}{(4n+1)^3(4n+3)^3}$$

Entry 8. If $m = 1,2,3,\ldots$ we have

$$\pi^3 = 32\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{m+3}} - 64\sum_{n=1}^{\infty}(-1)^{n-1}\sum_{k=1}^{m}\frac{n}{(2n+1)^{k+3}}$$

Entry 9.

$$\pi^3 = 8 + 8\sum_{n=0}^{\infty} 8^n \left(8\left(\sin\left(\frac{\pi}{2^{n+2}}\right)\right)^3 - \left(\sin\left(\frac{\pi}{2^{n+1}}\right)\right)^3\right)$$

$$\pi^3 = 16\sqrt{2} + 8\sum_{n=1}^{\infty} 8^n \left(8\left(\sin\left(\frac{\pi}{2^{n+2}}\right)\right)^3 - \left(\sin\left(\frac{\pi}{2^{n+1}}\right)\right)^3\right)$$

Entry 10.

$$\frac{1}{\pi^3} = \frac{1}{64} + \frac{1}{64}\sum_{n=1}^{\infty}\left(\binom{2n}{n}2^{-2n}\right)^6\left(\frac{48n^4+96n^3+60n^2+12n+1}{(n+1)^3}\right)$$

Entry 11.

$$\pi^3 = \frac{32}{3}\sum_{n=1}^{\infty}(-1)^{n-1}\,n(4n^2-1)\left(\frac{1}{(2n-1)^5}+\frac{1}{(2n+1)^5}\right)$$

Entry 12.

$$\pi^3 = 24\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{m=1}^{k+1}\frac{1}{m^2(2k-2m+3)}$$

Entry 13.

$$\pi^3 = \frac{81}{8}\sqrt{3}\sum_{n=0}^{\infty}2^{-n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{m=0}^{k}\binom{k}{m}(k+m+1)^{-3}$$

Entry 14. Complex series , $i = \sqrt{-1}$.

$$\pi^3 = -384i\sum_{n=1}^{\infty}\frac{1}{n+2}\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^{n+2}\sum_{k=1}^{n}\frac{H_k}{k+1}$$

$$\pi^3 = -1296i\sum_{n=1}^{\infty}\frac{1}{n+2}\left(1-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{n+2}\sum_{k=1}^{n}\frac{H_k}{k+1}$$

Remark 3: $H_k = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}$, is the harmonic numbers.

Entry 15. For $0 < a < \frac{\sqrt{5}-1}{2}$ we have

$$\pi^3 = \frac{81\sqrt{3}}{8}\sum_{n=0}^{\infty}(-1)^n a^{n+1}\sum_{k=0}^{n}\binom{n}{k}a^k\left(\frac{(\ln a)^2}{n+k+1}-\frac{2\ln a}{(n+k+1)^2}+\frac{2}{(n+k+1)^3}\right)$$
$$+\frac{27\sqrt{3}}{4}\sum_{n=0}^{\infty}(1-a)^{n+3}\sum_{k=0}^{n}\frac{H_{k+1}}{k+2}\sum_{m=0}^{n-k}\binom{n-k}{m}\frac{(-3)^{-m}(1-a)^m}{n+m+3}$$

Entry 16.

$$\pi^3 = 12 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \frac{(-1/2)^k}{2k+1} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} (m+1)^{-2}$$
$$+ 8 \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \frac{(-2/9)^k}{2k+1} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} (m+1)^{-2}$$

Entry 17. For $0 < a < \pi/2$, we have

$$\pi^3 = 8 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+3} E_n}{(2n)!\,(2n+3)} + 16 \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(2n+1)a}}{2n+1} \left(a^2 + \frac{2a}{2n+1} + \frac{2}{(2n+1)^2} \right)$$

where E_n are the Euler numbers:

$$E_n = \{1,1,5,61,1385,50521, \ldots\}$$

Example $a = 1$,

$$\pi^3 = 8 \sum_{n=0}^{\infty} \frac{(-1)^n E_n}{(2n)!\,(2n+3)} + 16 \sum_{n=0}^{\infty} \frac{(-1)^n e^{-(2n+1)} \left(4(n+1)^2 + 1\right)}{(2n+1)^3}$$

Entry 18.

$$\pi^3 = 32 - 64 \sum_{n=1}^{\infty} \frac{48n^2 + 1}{(16n^2 - 1)^3}$$

Entry 19.

$$\pi^3 = 32 - 128 \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{k=1}^{n} \frac{k(k+1)}{(4n - 4k + 5)^{k+2}}$$

Entry 20.

$$\frac{4\pi^3}{81\sqrt{3}} = \sum_{n=0}^{\infty} \left(\frac{1}{(3n+1)^3} - \frac{1}{(3n+2)^3} \right)$$

$$\frac{4\pi^3}{81\sqrt{3}} = \sum_{n=0}^{\infty} \frac{27n^2 + 27n + 7}{(3n+1)^3 (3n+2)^3}$$

$$\frac{4\pi^3}{81\sqrt{3}} = \sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)^3} + \sum_{n=0}^{\infty} \frac{3(2n+1)}{(3n+1)^3 (3n+2)^2}$$

$$\frac{4\pi^3}{81\sqrt{3}} = \sum_{n=0}^{\infty} \frac{3(2n+1)}{(3n+1)^2 (3n+2)^3} + \sum_{n=0}^{\infty} \frac{1}{(3n+1)^3 (3n+2)}$$

Entry 21. For $a > 1/2$ we have

$$\frac{4\pi^3}{81\sqrt{3}} = \frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{1+a}\right)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}a^{-k}\sum_{m=0}^{k}\binom{k}{m}(k+m+1)^{-3}$$

Entry 22. For $a > 0$ we have

$$\frac{4\pi^3}{81\sqrt{3}} = \frac{1}{(1+a)^2}\sum_{n=0}^{\infty}\left(\frac{a}{1+a}\right)^n\sum_{k=0}^{n}\binom{n+k+1}{n-k}\frac{a^{-k}}{(1+a)^k}\sum_{m=0}^{n-k}\binom{n-k}{m}\frac{(-1)^m a^{-m}}{(k+m+1)^3}$$

For $a = 1$ we have

$$\frac{\pi^3}{9\sqrt{3}} = \sum_{n=0}^{\infty}3^{-n}\sum_{k=0}^{n}\binom{n+k+1}{n-k}\left(\frac{4}{3}\right)^k\sum_{m=0}^{n-k}\binom{n-k}{m}\frac{(-1)^m 2^m}{(k+m+1)^3}$$

Entry 23. For $h_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ we have

$$\frac{\pi^3}{96\sqrt{3}} = \sum_{n=1}^{\infty}(-1)^{n-1}3^{-n}\sum_{k=1}^{n}\frac{h_k\, h_{n-k+1}}{k}$$

$$\frac{\pi^3}{12^3} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(2-\sqrt{3}\right)^{2n+1}\sum_{k=1}^{n}\frac{h_k}{k(2n-2k+1)}$$

Entry 24.

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\tan^{-1}\left(\frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\tan^{-1}\left(\frac{n}{n+2}\right)$$

$$\frac{\pi^3}{24} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\tan^{-1}\left(\frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\tan^{-1}\left(\frac{n}{n+2}\right)$$

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty}\frac{1}{n^2}\tan^{-1}\left(\frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\frac{1}{n^2}\tan^{-1}\left(\frac{n}{n+2}\right)$$

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty}\frac{1}{n^2}\tan^{-1}\left(\frac{1}{n}\right) + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

$$\frac{\pi^3}{48} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\tan^{-1}\left(\frac{1}{n}\right) + \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2}\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2+k+1}\right)$$

Entry 25.

$$\pi^3 = \sqrt{2}\sum_{n=0}^{\infty}(-1)^n 2^{-n}(n+1)\left(\frac{(\ln 2)^2}{2n+3}+\frac{4\ \ln 2}{(2n+3)^2}+\frac{8}{(2n+3)^3}\right)$$
$$+\sqrt{2}\sum_{n=0}^{\infty}2^{-n}(n+1)\sum_{k=0}^{n}\binom{n}{k}(-2)^k\left(\frac{(\ln 2)^2}{2k+1}-\frac{4\ \ln 2}{(2k+1)^2}+\frac{8}{(2k+1)^3}\right)$$

Entry 26. For $0 < a < 1 < b$, we have

$$\pi^3 = 16\sum_{n=0}^{\infty}(-1)^n (n+1)a^{2n+1}\left(\frac{(\ln a)^2}{2n+1}-\frac{2\ \ln a}{(2n+1)^2}+\frac{2}{(2n+1)^3}\right)$$
$$+16\sum_{n=0}^{\infty}(-1)^n (n+1)b^{-2n-3}\left(\frac{(\ln b)^2}{2n+3}+\frac{2\ \ln b}{(2n+3)^2}+\frac{2}{(2n+3)^3}\right)$$
$$+16\sum_{n=0}^{\infty}\frac{(n+1)}{(1+c)^{n+2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^k c^{n-k}\left(b^{2k+1}\left(\frac{(\ln b)^2}{2k+1}-\frac{2\ \ln b}{(2k+1)^2}+\frac{2}{(2k+1)^3}\right)\right.$$
$$\left.-\,a^{2k+1}\left(\frac{(\ln a)^2}{2k+1}-\frac{2\ \ln a}{(2k+1)^2}+\frac{2}{(2k+1)^3}\right)\right)$$

where

$$c = \frac{a^2+b^2}{2}$$

Entry 27.

$$\pi^3 = \frac{128}{9}\sum_{n=0}^{\infty}\left(\frac{7}{9}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2}{7}\right)^k\sum_{m=0}^{k}\binom{k}{m}(-1)^m\left(\sum_{r=0}^{m}\binom{m}{r}\frac{1}{2r+1}\right)^3$$

$$\pi^3 = 24\sqrt{3}\sum_{n=0}^{\infty}\left(\frac{54}{91}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{37}{54}\right)^{n-k}\sum_{m=0}^{k}\binom{k}{m}(-1)^m\left(\sum_{r=0}^{m}\binom{m}{r}\frac{3^{-r}}{2r+1}\right)^3$$

Entry 28.

$$\pi^3 = 512\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\sum_{m=0}^{k}\binom{k}{m}\frac{\left(\sqrt{2}-1\right)^{2m+1}}{2m+1}\right)^3$$

$$\pi^3 = 1728\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\sum_{m=0}^{k}\binom{k}{m}\frac{\left(2-\sqrt{3}\right)^{2m+1}}{2m+1}\right)^3$$

Entry 29.

$$\pi^3 = 216 \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m 2^{-2m-1}}{2m+1} \right)^3$$

$$\pi^3 = 16\sqrt{2} \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m 2^{-m}}{2m+1} \right)^3$$

$$\pi^3 = \frac{81\sqrt{3}}{8} \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m (3/4)^m}{2m+1} \right)^3$$

$$\pi^3 = 512 \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m}{2m+1} \left(\frac{\sqrt{2-\sqrt{2}}}{2} \right)^{2m+1} \right)^3$$

$$\pi^3 = 2\sqrt{2} \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m 2^{1-m}}{2m+1} \right)^3$$

Entry 30.

$$\pi^3 = 24\sqrt{3} - 24\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-n}}{2n+3} \left(\sum_{k=0}^{n} \frac{(-3)^{-k}}{2k+1} \right) \left(\sum_{k=0}^{n+1} \frac{(-3)^{-k}}{2k+1} \right) - 24\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-3n-3}}{(2n+3)^3}$$

Series: π^3 and φ

Entry 31.

$$\frac{2\pi^3}{125} \sqrt{2 + \frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} (-1)^n \, 2^{-n} \sum_{k=0}^{n} (-3)^k \binom{n}{k} \sum_{m=0}^{k} \binom{k}{m} \frac{(-1)^m \varphi^{k-m}}{(k+m+1)^3}$$

$$\frac{6\pi^3}{125} \sqrt{2 + \frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \varphi^k \binom{n}{n-k} \sum_{m=0}^{n} \binom{n-k}{m} \frac{(-1)^m}{(k+2m+1)^3}$$

$$\frac{3\pi^3}{125} \sqrt{2 + \frac{2}{\sqrt{5}}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \varphi^{n-k}}{(n+k+1)^3}$$

Entry 32.

$$\pi^3 = 32 \sum_{n=0}^{\infty} \varphi^{-n-2} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \varphi^{-k}}{(2k+1)^3}$$

$$\pi^3 = 32 \sum_{n=0}^{\infty} \varphi^{-2n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \varphi^{k}}{(2k+1)^3}$$

Entry 33.

$$\pi^3 = 8 \sum_{n=0}^{\infty} \frac{(-1)^n \, \varphi^{-2n-3} \, E_n}{(2n)!\,(2n+3)} + 16 \sum_{n=0}^{\infty} (-1)^n \, e^{-(2n+1)/\varphi} \left(\frac{\varphi^{-2}}{2n+1} + \frac{2\varphi^{-1}}{(2n+1)^2} + \frac{2}{(2n+1)^3} \right)$$

where E_n are the Euler numbers.

Entry 34.

$$\pi^3 = 125\sqrt{7-4\varphi} \sum_{n=0}^{\infty} (-1)^n \, (7-4\varphi)^{n+1} \sum_{k=0}^{n} \frac{1}{2k+1} \sum_{m=0}^{n-k} \frac{1}{(2m+1)(2n-2k-2m+1)}$$

Entry 35.

$$\pi^3 = 64 \sum_{n=0}^{\infty} (-1)^n \, \varphi^{-6n-3} \sum_{k=0}^{n} \left(\frac{1}{6k+1} + \frac{2\varphi^{-2}}{6k+3} + \frac{\varphi^{-4}}{6k+5} \right) \sum_{m=0}^{n-k} \left(\frac{1}{6m+1} + \frac{2\varphi^{-2}}{6m+3} + \frac{\varphi^{-4}}{6m+5} \right) \left(\frac{1}{6n-6k-6m+1} + \frac{2\varphi^{-2}}{6n-6k-6m+3} + \frac{\varphi^{-4}}{6n-6k-6m+5} \right)$$

Entry 36.

$$\pi^3 = 64 \sum_{n=0}^{\infty} (-1)^n \, \varphi^{-12n-6} \sum_{k=0}^{n} \left(\frac{2}{6k+1} + \frac{\varphi^{-4}}{6k+3} + \frac{2\varphi^{-8}}{6k+5} \right) \sum_{m=0}^{n-k} \left(\frac{2}{6m+1} + \frac{\varphi^{-4}}{6m+3} + \frac{2\varphi^{-8}}{6m+5} \right) \left(\frac{2}{6n-6k-6m+1} + \frac{\varphi^{-4}}{6n-6k-6m+3} + \frac{2\varphi^{-8}}{6n-6k-6m+5} \right)$$

Entry 37.

$$\pi^3 = 64 \sum_{n=0}^{\infty} (-1)^n \, \varphi^{-30n-9} \sum_{k=0}^{n} f(k) \sum_{m=0}^{n-k} f(m) \, f(n-k-m)$$

where

$$f(n) = \frac{3}{10n+1} + \frac{\varphi^{-2}}{6n+1} - \frac{3\varphi^{-6}}{10n+3} + \frac{4\varphi^{-12}}{5(6n+3)} - \frac{3\varphi^{-18}}{10n+7} + \frac{\varphi^{-22}}{6n+5} + \frac{3\varphi^{-24}}{10n+9}$$

Future Research

Entry 38.

$$\pi^3 = 32\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{\cosh(x+y+z)+\cosh(x+y-z)+\cosh(x-y+z)+\cosh(x-y-z)}$$

Entry 39.

$$\pi^3 = 8\int_0^\infty \frac{(\ln x)^2}{1+x^2}dx = 16\int_0^1 \frac{(\ln x)^2}{1+x^2}dx = 16\int_1^\infty \frac{(\ln x)^2}{1+x^2}dx = 16\int_0^\infty \tan^{-1}\left(e^{-\sqrt{x}}\right)dx$$
$$= 32\int_0^\infty x\ \tan^{-1}(e^{-x})\,dx = -32\int_0^1 \frac{\ln x\ \tan^{-1}x}{x}dx = 32\int_1^\infty \frac{\ln x}{x}\ \tan^{-1}\left(\frac{1}{x}\right)dx$$
$$= -64\int_0^{\frac{\pi}{4}} \frac{x\ \ln(\tan x)}{\sin(2x)}dx = -32\int_0^{\ln(1+\sqrt{2})} \frac{\sin^{-1}(\tanh x)}{\tanh x}\ln(\sinh x)\,dx$$
$$= 32\int_0^\infty x\ \sin^{-1}\left(\sqrt{\frac{1-\tanh x}{2}}\right)dx = 32\int_0^1 \frac{\tanh^{-1}x}{1-x^2}\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)dx$$
$$= 16\int_0^{\frac{\pi}{2}} \frac{x\ \tanh^{-1}(\cos x)}{\sin x}dx = 16\int_0^\infty \sin^{-1}\left(\sqrt{\frac{1-\tanh\sqrt{x}}{2}}\right)dx = 2\int_0^1 \frac{(\ln x)^2}{\sqrt{x}\,(1+x)}dx$$
$$= 32\int_{-\infty}^\infty e^{2x}\ \tan^{-1}\left(e^{-e^{x}}\right)dx = 32\int_{-\infty}^\infty e^{-2x}\ \tan^{-1}\left(e^{-e^{-x}}\right)dx = 8\int_0^\infty \frac{(\ln\sinh x)^2}{\cosh x}dx$$
$$= 16\int_0^{\frac{\pi}{4}} (\ln\tan x)^2\,dx = 8\int_{-\infty}^\infty \frac{x^2\,e^x}{1+e^{2x}}dx$$

References

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