(Anti) de Sitter Geometry, Complex Conformal Gravity-Maxwell Theory from a $Cl(4, C)$ Gauge Theory of Gravity and Grand Unification

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Abstract

We present the deep connections among (Anti) de Sitter geometry, complex conformal gravity-Maxwell theory, and grand unification, from a gauge theory of gravity based on the complex Clifford algebra $Cl(4, C)$. Some desirable results are found, like a plausible *cancellation* mechanism of the cosmological constant involving an algebraic constraint between e^a_μ, b^a_μ (the real and imaginary parts of the complex vierbein).

Keywords: Clifford Algebras, Conformal Gravity, Grand Unification, (Anti) de Sitter groups.

1 Introduction

In recent years conformal gravity has been advanced as a candidate towards a satisfactory quantum theory that is both renormalizable and unitary (despite higher derivatives) due to a modification of the Hilbert space inner products; i.e. unitarity is being obtained because the theory is a PT symmetric rather than a Hermitian theory. For a recent discussion, details and references see [2]. Weinberg's Asymptotic Safety program for quantum Einstein gravity based on the existence of an interacting (non-Gaussian) ultraviolet fixed point of the nonperturbative renormalization group flow of the average quantum effective action has also received a lot of attention in recent years [3].

Clifford algebras are essential tools in many aspects in Physics [1]. In this letter we shall present the deep connections among (Anti) de Sitter geometry, complex conformal gravity-Maxwell theory, and grand unification, starting from a gauge theory of gravity based on the complex Clifford algebra $Cl(4, C)$. The key role of Clifford algebras will be self-evident. We will show that the complexified $4D$ Conformal Gravity-Maxwell theory turns out to be isomorphic to a gauge theory of gravity based on the complex Clifford $CL(4, C)$ algebra. This is attained by simply extending the de Sitter algebra to the Clifford algebra case. The Clifford algebraic version of the de Sitter algebra $so(4, 1)$ is realized via the of $Cl(4, 1, R)$ algebra, and which in turn, leads to a complexification of the Conformal Gravity-Maxwell theory in $4D$ due to the isomorphism $Cl(4, C) \sim Cl(4, 1, R)$. Similar results follow in the Anti de Sitter case after constructing a $Cl(3, 2, R)$ gauge theory of gravity. Because $Cl(3, 1, R)$ is a subalgebra of both $Cl(4, 1, R)$ *and* $Cl(3, 2, R)$, after reduction of one spatial and temporal dimension, respectively, one recovers $4D$ conformal gravity in both cases. Despite that this reduction of one dimension mimics the holographic principle we have not invoked holography in this work

Some desirable results are found, like a very plausible *cancellation* mechanism of the cosmological constant involving an algebraic constraint between e^a_μ, b^a_μ (the real and imaginary parts of the complex vierbein). We expect to find other novel consequences emerging from the physics behind complex conformal gravity-Maxwell theory and Clifford gauge theories of gravity.

2 Clifford Gauge Theories of Gravity

In this letter we shall generalize the Macdowell-Mansouri-Chamseddine-West (MMCW) [5] formulation of ordinary $4D$ gravity, based on the de Sitter group $SO(4, 1)$, to the Clifford group associated to the real $Cl(4, 1, R)$ algebra. This construction also applies to the Anti de Sitter group $SO(3, 2)$ and the real $Cl(3, 2, R)$ algebra. Given a Clifford algebra defined by the anticommutators $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}$, with $\eta^{ab} = diag(-1, +1, +1, \cdots, +1)$, the 2^D Clifford-algebra generators in D-dimensions are given respectively by the wedge products (signed antisymmetrized sums of products of gamma matrices)

$$\mathbf{1},\ \gamma^a,\ \gamma^{a_1} \wedge \gamma^{a_2},\ \ \cdots,\ \gamma^{a_1} \wedge \gamma^{a_2} \wedge \cdots \wedge \gamma^{a_D} \tag{2.1}$$

where $\mathbf{1}$ is the unit element of the algebra and the numerical combinatorial factors due to the antisymmetrization of indices can be omitted by imposing the ordering prescription $a_1 < a_2 < a_3 \cdots < a_D$.

The evaluation of the commutators of the above Clifford-algebra generators can be found in [6]. In general for $pq = odd$ one has

$$[\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}] \ = \ 2\gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!}\ \delta^{[a_1 a_2}_{[b_1 b_2}\ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!}\ \delta^{[a_1 ....a_4}_{[b_1 ....b_4}\ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \ - \ ...... \tag{2.2}$$

for $pq = even$ one has

$$[\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}] \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \ + \ ...... \quad (2.3)$$

The anti-commutators of the Clifford algebra generators can also be found in [6], and one has the reciprocal situation as in eqs-(2.2,2.3), one has instead that for $pq = even$

$$\{\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\} \ = \ 2 \gamma^{a_1 a_2 ......a_q}_{b_1 b_2 .....b_p} \ -$$

$$\frac{2p!q!}{2!(p-2)!(q-2)!} \ \delta^{[a_1 a_2}_{[b_1 b_2} \ \gamma^{a_3 ....a_q]}_{b_3 .....b_p]} \ + \ \frac{2p!q!}{4!(p-4)!(q-4)!} \ \delta^{[a_1 ....a_4}_{[b_1 ....b_4} \ \gamma^{a_5 ....a_q]}_{b_5 .....b_p]} \ - \ ...... \quad (2.4)$$

And for $pq = odd$ one has

$$\{\gamma_{b_1 b_2 .....b_p}, \gamma^{a_1 a_2 ......a_q}\} \ = \ - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \ \delta^{[a_1}_{[b_1} \ \gamma^{a_2 a_3 ....a_q]}_{b_2 b_3 .....b_p]} \ -$$

$$\frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \ \delta^{[a_1 ....a_3}_{[b_1 ....b_3} \ \gamma^{a_4 ....a_q]}_{b_4 .....b_p]} \ + \ ...... \quad (2.5)$$

The aim to is to construct the Clifford-algebra generalization of MMCW procedure to construct a gauge theory of gravity with a cosmological constant by gauging the de Sitter group. The natural generalization of the de Sitter group $SO(4,1)$ is the Clifford group based on the $Cl(4,1)$ algebra. Consequently, let us begin by writing the $Cl(4,1)$-valued gauge field in $4D$

$$\mathbf{A}_\mu \ = \ \mathcal{A}^M_\mu \ \Gamma_M \ = \ \mathcal{A}_\mu \mathbf{1} \ + \ \mathcal{A}^m_\mu \gamma_m \ + \ \mathcal{A}^{m_1 m_2}_\mu \gamma_{m_1 m_2} \ + \ \mathcal{A}^{m_1 m_2 m_3}_\mu \gamma_{m_1 m_2 m_3} +$$

$$\mathcal{A}^{m_1 m_2 m_3 m_4}_\mu \gamma_{m_1 m_2 m_3 m_4} + \mathcal{A}^{m_1 m_2 m_3 m_4 m_5}_\mu \gamma_{m_1 m_2 m_3 m_4 m_5}, \ \ m = 1,2,3,4,5 \quad (2.6)$$

In the next subsection we will *split* the field decomposition in the expansion in eq-(2.6) in terms of the fields $\mathcal{A}_\mu, \mathcal{A}^m_\mu, \mathcal{A}^{m_1 m_2}_\mu$ and their *duals* $\tilde{\mathcal{A}}_\mu, \tilde{\mathcal{A}}^m_\mu, \tilde{\mathcal{A}}^{m_1 m_2}_\mu$ as a result of the duality property of the gamma matrices in *odd* dimensions $D = r+s =$ odd given by the relations $\gamma_{m_1 m_2 \cdots m_r} \sim \epsilon_{m_1 m_2 \cdots m_r n_1 n_2 \cdots n_s} \gamma^{n_1 n_2 \cdots n_s}$. The proportionality factor is given by a phase factor that can be $\pm 1, \pm i$ depending on the odd dimensions and the metric signature. As a result of this splitting we shall show how the Clifford $Cl(4,1)$ gauge theory of gravity is equivalent to a theory of complexified conformal gravity-Maxwell theory involving conformal gravity-Maxwell theory and their purely imaginary dual fields; i.e. such theory is tantamount to a Clifford gauge theory of gravity based on the *complex* $Cl(4, C)$ algebra.

For now, let us perform a $4+1$ split of the indices in $\mathcal{A}^M_\mu \Gamma_M$ in eq-(2.6) of the form

$$\mathcal{A}^M_\mu \Gamma_M \ \equiv \ \mathcal{A}^I_\mu \Gamma_I \ + \ \mathcal{A}^{I5}_\mu \Gamma_{I5} \tag{2.7}$$

with

$$\mathcal{A}^I_\mu \ \Gamma_I \ = \ \mathcal{A}_\mu \mathbf{1} \ + \ \mathcal{A}^i_\mu \gamma_i \ + \ \mathcal{A}^{i_1 i_2}_\mu \gamma_{i_1 i_2} \ + \ \mathcal{A}^{i_1 i_2 i_3}_\mu \gamma_{i_1 i_2 i_3} + \mathcal{A}^{i_1 i_2 i_3 i_4}_\mu \gamma_{i_1 i_2 i_3 i_4} \tag{2.8}$$

whose indices span only four directions $i = 1, 2, 3, 4$, and whereas the indices in $\mathcal{A}^{I5}_\mu \Gamma_{J5}$ span all of the five directions. In this way one can split the Γ^M's into generalized boosts/rotations and generalized translations. The latter translations have the following one-to-one correspondence with the following gamma generators

$$P \leftrightarrow \gamma_5, \quad P_i \leftrightarrow \gamma_{i5}, \ P_{ij} \leftrightarrow \gamma_{ij5}, \ P_{ijk} \leftrightarrow \gamma_{ijk5}, \ P_{ijkl} \leftrightarrow \gamma_{ijkl5} \tag{2.9}$$

where P can be interpreted (from the four-dim point of view) as the scalar part of the momentum polyvector $\mathbf{P}$ in $4D$. P_i is the vector part; P_{ij} is the bi-vector part; P_{ijk} is the tri-vector part, and P_{ijkl} is the quad-vector part.

Hence, after naming $\{\Gamma^I\} = \mathbf{J}$, and $\{\Gamma^{I5}\} = \mathbf{P}$ as the sets of generalized boosts/rotations and translation generators, respectively, one obtains the sought-after commutation relations of the form

$$[\mathbf{J}, \ \mathbf{J}] \ \sim \ \mathbf{J}, \ \ [\mathbf{J}, \ \mathbf{P}] \ \sim \ \mathbf{P}, \ [\mathbf{P}, \ \mathbf{P}] \ \sim \ \mathbf{J} \tag{2.10}$$

after direct application of the commutators in eqs-(2.2,2.3). Therefore, the split (2.7) has precisely the same algebraic form as the gauge fields in the de Sitter gauge theory of gravity

$$A_\mu \ = \ \omega^{i5}_\mu \ J_{i5} \ + \ \omega^{ij}_\mu \ J_{ij} \ = \ \rho \, \omega^{i5}_\mu \ P_i \ + \ \omega^{ij}_\mu \ J_{ij} \ = \ e^i_\mu \ P_i \ + \ \omega^{ij}_\mu \ J_{ij} \tag{2.11}$$

which is given in terms of the vierbein $e^i_\mu = \rho \, \omega^{i5}_\mu$ and the spin connection ω^{ij}_μ. ρ is a length scale which can be identified with the de Sitter throat size. The *noncommutative* translation generators are P_i, and the Lorentz generators are J_{ij}, with $i, j = 1, 2, 3, 4$. In the limit $\rho \to \infty$ the P_i will commute and one recovers the Poincare algebra. Therefore, we have attained the desired goal of extending the de Sitter algebra to the Clifford algebra case.

The next step is to construct a Clifford extension of a de Sitter gauge theory of gravity (with a cosmological constant). In the conclusion we discuss how a similar procedure follows in the Anti de Sitter case. From eqs-(2.2,2.3), one can evaluate the commutators appearing in the definition of the $Cl(4,1)$-algebra-valued two-form field strength $\mathbf{F} = \mathbf{dA} + [\mathbf{A}, \mathbf{A}]$. Thus, the $Cl(4,1)$-algebra-valued field strength components are given respectively by

$$\mathcal{F}_{\mu\nu} \ = \ \partial_{[\mu} A_{\nu]}, \quad \mu, \nu = 1, 2, 3, 4 \tag{2.12a}$$

$$\mathcal{F}^k_{\mu\nu} \ = \ \partial_{[\mu} A^k_{\nu]} \ + 4 \ (\mathcal{A}_{\mu r} \ \mathcal{A}^{mn}_\nu \ - \ \mu \leftrightarrow \nu) \ \delta^r_{[m} \ \delta^k_{n]} \ -$$

$$48\ (\mathcal{A}_{\mu rst}\ \mathcal{A}_{\nu}^{mnpq}\ -\ \mu \leftrightarrow \nu)\ \delta^{rst}_{[mnp}\ \delta^{k}_{q]} \tag{2.12b}$$

$$\begin{aligned}\mathcal{F}^{k_1k_2}_{\mu\nu}\ =\ & \partial_{[\mu}A^{k_1k_2}_{\nu]}\ -\ (\mathcal{A}_{\mu r}\ \mathcal{A}_{\nu}^{m}\ -\ \mu \leftrightarrow \nu)\ \eta^{r[k_1}\ \delta^{k_2]}_{m}\ + \\ & 4\ (\mathcal{A}_{\mu rs}\ \mathcal{A}_{\nu}^{mn}\ -\ \mu \leftrightarrow \nu)\ \delta^{[r}_{[m}\ \eta^{s][k_1}\ \delta^{k_2]}_{n]}\ + \\ & 18\ (\mathcal{A}_{\mu rst}\ \mathcal{A}_{\nu}^{mnp}\ -\ \mu \leftrightarrow \nu)\ \delta^{[rs}_{[mn}\ \eta^{t][k_1}\ \delta^{k_2]}_{p]}\ - \\ & 96\ (\mathcal{A}_{\mu rstu}\ \mathcal{A}_{\nu}^{mnpq}\ -\ \mu \leftrightarrow \nu)\ \delta^{[rst}_{[mnp}\ \eta^{u][k_1}\ \delta^{k_2]}_{q]}\ - \\ & 600\ (\mathcal{A}_{\mu rstuv}\ \mathcal{A}_{\nu}^{mnpqk}\ -\ \mu \leftrightarrow \nu)\ \delta^{[rstu}_{[mnpq}\ \eta^{v][k_1}\ \delta^{k_2]}_{k]}\end{aligned} \tag{2.12c}$$

$$\begin{aligned}\mathcal{F}^{k_1k_2k_3}_{\mu\nu}\ =\ & \partial_{[\mu}A^{k_1k_2k_3}_{\nu]}\ -\ 2\ (\mathcal{A}_{\mu rs}\ \mathcal{A}_{\nu}^{mn}\ -\ \mu \leftrightarrow \nu)\ \delta^{[r}_{[m}\ \eta^{s][k_1}\ \delta^{k_2k_3]}_{np]}\ + \\ & 80\ (\mathcal{A}_{\mu rstu}\ \mathcal{A}_{\nu}^{mnpqk}\ -\ \mu \leftrightarrow \nu)\ \delta^{[rst}_{[mnp}\ \eta^{u][k_1}\ \delta^{k_2k_3]}_{qk]}\ + \\ & \frac{4}{3}\ (\mathcal{A}_{\mu r}\ \mathcal{A}_{\nu}^{mnpq}\ -\ \mu \leftrightarrow \nu)\ \delta^{r}_{[m}\ \delta^{k_1k_2k_3}_{npq]}\end{aligned} \tag{2.12d}$$

$$\begin{aligned}\mathcal{F}^{k_1k_2k_3k_4}_{\mu\nu}\ =\ & \partial_{[\mu}A^{k_1k_2k_3k_4}_{\nu]}\ -\ \frac{1}{12}\ (\mathcal{A}_{\mu r}\ \mathcal{A}_{\nu}^{mnp}\ -\ \mu \leftrightarrow \nu)\ \delta^{[k_2k_3k_4}_{[mnp}\ \eta^{k_1]r}\ + \\ & \frac{2}{3}\ (\mathcal{A}_{\mu rs}\ \mathcal{A}_{\nu}^{mnpq}\ -\ \mu \leftrightarrow \nu)\ \delta^{[r}_{[m}\ \eta^{s][k_1}\ \delta^{k_2k_3k_4]}_{npq]}\ + \\ & 5\ (\mathcal{A}_{\mu rst}\ \mathcal{A}_{\nu}^{mnpqk}\ -\ \mu \leftrightarrow \nu)\ \delta^{[rs}_{[mn}\ \eta^{t][k_1}\ \delta^{k_2k_3k_4]}_{pqk]}\end{aligned} \tag{2.12e}$$

$$\begin{aligned}\mathcal{F}^{k_1k_2k_3k_4k_5}_{\mu\nu}\ =\ & \partial_{[\mu}A^{k_1k_2k_3k_4k_5}_{\nu]} - \frac{1}{6}\ (\mathcal{A}_{\mu rs}\ \mathcal{A}_{\nu}^{mnpqk} - \mu \leftrightarrow \nu)\ \delta^{[r}_{[m}\ \eta^{s][k_1}\ \delta^{k_2k_3k_4k_5]}_{npqk]} + \\ & \frac{1}{5}\ (\mathcal{A}_{\mu rst}\ \mathcal{A}_{\nu}^{mnpq}\ -\ \mu \leftrightarrow \nu)\ \delta^{[r}_{[m}\ \eta^{st]|[k_1k_2}\ \delta^{k_3k_4k_5]}_{npq]}\end{aligned} \tag{2.12f}$$

The indices' range in the above equations is $1,2,3,4,5$, and one has the following definitions

$$\eta_{m_1m_2|n_1n_2}\ =\ \eta_{m_1n_1}\ \eta_{m_2n_2}\ -\ \eta_{m_1n_2}\ \eta_{m_2n_1} \tag{2.13a}$$

$$\eta_{m_1m_2m_3|n_1n_2n_3}\ =\ \eta_{m_1n_1}\ \eta_{m_2n_2}\ \eta_{m_3n_3}\ \mp\ \cdots \tag{2.13b}$$

$$\eta_{m_1m_2m_3m_4|n_1n_2n_3n_4}\ =\ \eta_{m_1n_1}\ \eta_{m_2n_2}\ \eta_{m_3n_3}\ \eta_{m_4n_4}\ \mp\ \cdots \tag{2.13c}$$

and so forth. The expressions in eqs-(2.13) can be rewritten as the determinants of a square $N \times N$ matrix whose entries are $\eta_{m_rn_s}; r,s = 1,2,3,\cdots,N$. The mixed generalized Kronecker symbols $\delta^{m_1m_2\cdots m_N}_{n_1n_2\cdots n_N}$ can also be rewritten as the determinants of a square $N \times N$ matrix whose entries are $\delta^{m_r}_{n_s}; r,s = 1,2,3,\cdots,N$.

The evaluation of the field strength components in eqs-(2.12) required taking the scalar parts of the geometric product of the gammas, which is denoted by the brackets. For example, $< \Gamma_M \Gamma_N >$ denotes the scalar part (those terms multiplying the Clifford algebra unit element $\mathbf{1}$) of the geometric product of two gamma matrices. It was required in order to perform the necessary contractions. In particular one has

$$< \gamma_m \ \gamma_n \ > \ = \eta_{mn}, \quad < \gamma_{m_1 m_2} \ \gamma_{n_1 n_2} \ > \ = \ \eta_{m_1 n_1} \ \eta_{m_2 n_2} - \eta_{m_1 n_2} \ \eta_{m_2 n_1}$$

$$< \gamma_{m_1} \ \gamma_{m_2} \ \gamma_{m_3} \ > \ = \ 0, \quad < \gamma_{m_1 m_2 m_3} \ \gamma_{n_1 n_2 n_3} \ > \ = \ \eta_{m_1 n_1} \ \eta_{m_2 n_2} \ \eta_{m_3 n_3} \mp \cdots$$

$$< \gamma_{m_1 m_2 m_3 m_4} \ \gamma_{n_1 n_2 n_3 n_4} \ > \ = \ \eta_{m_1 n_1} \ \eta_{m_2 n_2} \ \eta_{m_3 n_3} \ \eta_{m_4 n_4} \ \mp \ \cdots \tag{2.14}$$

and so forth.

The terms that one is familiar with are those appearing in eq-(2.12b)

$$\partial_{[\mu} A_{\nu]}^{k_1 k_2} \ + \ (\mathcal{A}_{\mu rs} \ \mathcal{A}_\nu^{mn} \ - \ \mu \leftrightarrow \nu) \ \delta_{[m}^{[r} \ \eta^{s][k_1} \ \delta_{n]}^{k_2]} \tag{2.15}$$

which have the *same* functional form as the de Sitter field strength. To see this one just needs to recall that given the spin connection one-form $\omega^{k_1 k_2} \equiv \omega_\mu^{k_1 k_2} dx^\mu$, and the exterior derivative d, the field strength two-form $F^{k_1 k_2} \equiv F_{\mu\nu}^{k_1 k_2} dx^\mu \wedge dx^\nu$ is defined as

$$F^{k_1 k_2} \ = \ d\omega^{k_1 k_2} \ + \ \omega_r^{k_1} \wedge \omega^{r k_2}, \quad r, k_1, k_2 = 1, 2, 3, 4, 5 \tag{2.16}$$

Upon performing the $4+1$ split of the indices in eq-(2.16) one can isolate the curvature from the torsion terms as follows

$$F^{ab} \ = \ d\omega^{ab} \ + \ \omega_c^a \wedge \omega^{cb} \ + \ \omega_5^a \wedge \omega^{5b} \ = \ d\omega^{ab} \ + \ \omega_c^a \wedge \omega^{cb} \ + \frac{1}{\rho^2} \ e^a \wedge e^b \ =$$

$$R^{ab} \ + \ \frac{1}{\rho^2} \ e^a \wedge e^b \ = \ R^{ab} \ + \ \Lambda \ e^a \wedge e^b, \quad a, b, c = 1, 2, 3, 4 \tag{2.17}$$

after identifying ρ^{-2} with the cosmological constant Λ, and R^{ab} with the curvature two-form associated with the spin connection ω_μ^{ab}. The torsion two-form is given in terms of the mixed components involving the spin connection ω_μ^{ab} and the vierbein $e_\mu^a = \rho \omega_\mu^{a5}$, where ρ is the de Sitter throat size, as follows

$$F^{a5} \ = \ d\omega^{a5} \ + \ \omega_c^a \wedge \omega^{c5} \ = \ \frac{1}{\rho} \ (de^a \ + \ \omega_c^a \wedge e^c) \ \Rightarrow$$

$$T^a \ \equiv \ T_{\mu\nu}^a \ dx^\mu \wedge dx^\nu \ = \ \rho \ F_{\mu\nu}^{a5} \ dx^\mu \wedge dx^\nu \ = \ de^a \ + \ \omega_c^i \wedge e^c \tag{2.18}$$

After constraining the torsion to zero which allows to express the spin connection in terms of the vielbein $\omega = \omega(e)$, the $4D$ MMCW Lagrangian (density) [5] is defined by $F^{ab} \wedge F^{cd} \ \epsilon_{abcd}$ which decomposes into a sum of the $4D$ Gauss-Bonnet topological invariant term $R^{ab} \wedge R^{cd} \epsilon_{abcd}$; the $4D$ Einstein-Hilbert term $\rho^{-2} R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}$, and the $4D$ cosmological constant term $\rho^{-4} e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}$.

One may notice that after one introduces a matrix representation of the Clifford algebra generators, some of the Γ_M matrices are Hermitian, and some are anti-Hermitian, due to the chosen signature $\eta_{ab} = diag(-1,1,1,1,1)$. For example, because $(\gamma_1)^2 = -1$ the γ_1 matrix is anti-Hermitian, while $\gamma_2, \gamma_3, \gamma_4, \gamma_5$ are Hermitian. γ_{mn} is anti-Hermitian when $m, n = 2,3,4,5$, and Hermitian when $m = 1$ or $n = 1$; and so forth. Hence, if all of the gauge field components $\mathcal{A}^M_\mu$ are chosen to be real-valued, the matrix $\mathcal{A}^M_\mu \Gamma_M$ has a mixture of Hermitian and anti-Hemitian pieces. The Clifford-valued field strength $\mathbf{F}_{\mu\nu} = \mathcal{F}^M_{\mu\nu}\Gamma_M$, with $\mathcal{F}^M_{\mu\nu}$ real, will also have a mixture of Hermitian and anti-Hermitian pieces. There are two (real-valued) gauge-invariant Lagrangians (densities) which can be constructed in $4D$

$$\mathcal{L}_1 \;=\; \mathcal{F}^M_{\mu\nu}\, \mathcal{F}^{\mu\nu}_M, \;\; \mathcal{L}_2 \;=\; \epsilon^{\mu_1\mu_2\mu_3\mu_4}\, \mathcal{F}^M_{\mu_1\mu_2}\, \mathcal{F}_{\mu_3\mu_4 M}, \;\; \mu,\nu = 1,2,3,4 \tag{2.20}$$

where M is a Clifford-valued polyvector index spanning the 2^5-dimensional $Cl(4,1,R)$ algebra. These Lagrangians have the same functional form $Tr(F \wedge^* F), Tr(F \wedge F)$ as in ordinary Yang-Mills theory.[1]

Complex Conformal Gravity-Maxwell Theory from a $Cl(4,C)$ Gauge Theory of Gravity

Let us focus now on the following terms

$$\mathcal{A}_\mu \mathbf{1} \;+\; \mathcal{A}^m_\mu \gamma_m \;+\; \mathcal{A}^{m_1m_2}_\mu \gamma_{m_1m_2}, \;\; m, m_1, m_2 = 1,2,3,4,5 \tag{2.21}$$

in eq-(2.6). A $4+1$ splitting of the indices above yields

$$\mathcal{A}_\mu \mathbf{1} \;+\; \mathcal{A}^a_\mu \gamma_a \;+\; \mathcal{A}^5_\mu \gamma_5 \;+\; \mathcal{A}^{ab}_\mu \gamma_{ab} \;+\; \mathcal{A}^{a5}_\mu \gamma_{a5}, \;\; a, b = 1,2,3,4 \tag{2.22}$$

As explained earlier, as a result of the dualization process of the gamma matrices in *odd* dimensions, the remaining terms in eq-(2.6)

$$\mathcal{A}^{m_1m_2m_3}_\mu \gamma_{m_1m_2m_3} +$$
$$\mathcal{A}^{m_1m_2m_3m_4}_\mu \gamma_{m_1m_2m_3m_4} \;+\; \mathcal{A}^{m_1m_2m_3m_4m_5}_\mu \gamma_{m_1m_2m_3m_4m_5}, \;\; m = 1,2,3,4,5 \tag{2.23}$$

lead to an expansion expressed in terms of the *dual* fields as follows

$$\tilde{\mathcal{A}}_\mu \mathbf{1} \;+\; \tilde{\mathcal{A}}^{m_1}_\mu \gamma_{m_1} \;+\; \tilde{\mathcal{A}}^{m_1m_2}_\mu \gamma_{m_1m_2}, \;\; m_1, m_2 = 1,2,3,4,5 \tag{2.24a}$$

with

$$\tilde{\mathcal{A}}_\mu \sim \epsilon_{m_1m_2m_3m_4m_5}\, \mathcal{A}^{m_1m_2m_3m_4m_5}_\mu, \;\; \tilde{\mathcal{A}}^{m_1} \sim \epsilon^{m_1}_{\;\;m_2m_3m_4m_5}\, \mathcal{A}^{m_2m_3m_4m_5}_\mu,$$
$$\tilde{\mathcal{A}}^{m_1m_2} \sim \epsilon^{m_1m_2}_{\;\;m_3m_4m_5}\, \mathcal{A}^{m_3m_4m_5}_\mu \tag{2.24b}$$

[1] Due to $\epsilon^{\mu_1\mu_2\mu_3\mu_4} = \epsilon^{\mu_3\mu_4\mu_1\mu_2}$, the Lagrangian $\mathcal{L}_2$ is not zero

The key phase factors appearing in the dualization process of the gamma matrices are $\pm i$ due to the signature $(-,+,+,+,+)$, and can be reabsorbed into the definition of the dual fields. Thus the dual fields are purely imaginary.[2] The $4+1$ decomposition of eq-(2.24a) yields

$$\tilde{\mathcal{A}}_\mu \mathbf{1} \; + \; \tilde{\mathcal{A}}^a_\mu \gamma_a \; + \; \tilde{\mathcal{A}}^5_\mu \gamma_5 \; + \; \tilde{\mathcal{A}}^{ab}_\mu \gamma_{ab} \; + \; \tilde{\mathcal{A}}^{a5}_\mu \gamma_{a5}, \quad a,b = 1,2,3,4 \tag{2.25}$$

This dualization process is also consistent with the isomorphisms of the following Clifford algebras

$$Cl(4,C) \sim M(4,C) \sim Cl(4,1,R) \sim Cl(2,3,R) \sim Cl(0,5,R) \tag{2.26a}$$

$$Cl(4,C) \sim M(4,C) \sim Cl(3,1,R) \oplus \mathbf{i}\; Cl(3,1,R), \quad Cl(3,1,R) \sim M(4,R) \tag{2.26b}$$

where $M(4,R), M(4,C)$ is the 4×4 matrix algebra over the reals and complex numbers, respectively.

Therefore, the Clifford gauge theory of gravity based on the real $Cl(4,1,R)$ algebra involving the real valued 16 gauge fields in eq-(2.22), *and* their 16 imaginary valued duals in eq-(2.25), can be accommodated into the 16 complex-valued fields based on a *complex* Clifford $Cl(4,C)$ algebra, and given by $\mathbf{A}_\mu = (A_\mu + iB_\mu)^A \Gamma_A$, such that $\mathcal{A}^A_\mu = A^A_\mu$ and $\tilde{\mathcal{A}}^A_\mu = iB^A_\mu$.[3] .

In this case, the $4D$ quadratic Yang-Mills-like Lagrangian is of the form $(F+iG)^A_{\mu\nu}(F-iG)^{\mu\nu}_A$ (real-valued). A theta-like Lagrangian density is of the form $\epsilon^{\mu\nu\tau\sigma}(F+iG)^{ab}_{\mu\nu}(F+iG)^{cd}_{\tau\sigma}\epsilon_{abcd}$ + complex-conjugate (also real-valued by construction). The Clifford extension of the latter Lagrangian density is $\epsilon^{\mu\nu\tau\sigma}(F+iG)^A_{\mu\nu}(F+iG)_{\tau\sigma A}$ + complex-conjugate, where the internal indices are raised and lower with $\eta^{a_1a_2\cdots a_r|b_1b_2\cdots b_r}$ and $\eta_{a_1a_2\cdots a_r|b_1b_2\cdots b_r}$ whose expressions are provided by eqs-(2.13).

Caution must be taken in this latter case when dealing with bivector indices because $\epsilon^{\mu\nu\tau\sigma}(F+iG)^{a_1a_2}_{\mu\nu}(F+iG)^{a_3a_4}_{\tau\sigma}\epsilon_{a_1a_2a_3a_4} \neq \epsilon^{\mu\nu\tau\sigma}(F+iG)^{a_1a_2}_{\mu\nu}(F+iG)^{a_3a_4}_{\tau\sigma}\eta_{a_1a_2|a_3a_4}$. This is analogous to the difference between the term $R^a_b \wedge R^b_a$ and $R^{ab} \wedge R^{cd}\epsilon_{abcd}$ in ordinary gravity.

Next we recall the relationship between a $Cl(3,1,R)$-valued gauge field theory of gravity and conformal gravity [10]. By fixing some of the gauge symmetries and imposing some constraints one recovers ordinary gravity. We shall begin by showing how the conformal algebra in four dimensions admits a Clifford algebra realization; i.e. the generators of the conformal algebra can be expressed in terms of the Clifford algebra basis generators. The conformal algebra in four dimensions $so(4,2)$ is isomorphic to $su(2,2)$.

Let $\eta_{ab} = (-,+,+,+)$ be the Minkowski spacetime (flat) metric in $D = 3+1$-dimensions. The epsilon tensors are defined as $\epsilon_{1234} = -\epsilon^{1234} = 1$. The real

[2]This can be simply verified, for example, in the case of $\Gamma_{mn} = i\; \epsilon_{mnpqr}\Gamma^{pqr}$ when $m = 1$ is a temporal-like index, and the rest of the indices are spatial-like, because Γ_{1n} is Hermitian but Γ^{pqr} is anti-Hermitian, when $n,p,q,r = 2,3,4,5$. Without the $\mathbf{i}$ factor there would be an inconsistency. Similar findings apply to the other combinations, for example, $\Gamma_{mnpqr} = i\; \epsilon_{mnpqr}\; \mathbf{1}$

[3]$\Gamma^A = \mathbf{1}, \gamma^a, \gamma^{ab}, \gamma^{abc}, \gamma^{abcd}$ leading to $2^4 = 16$ generators

Clifford $Cl(3,1,R)$ algebra associated with the tangent space of a $4D$ spacetime $\mathcal{M}$ is defined by the anticommutators $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}\ \mathbf{1}, a, b = 1,2,3,4$. Given the chosen signature, the chirality operator Γ_5 is defined as

$$\Gamma_5 \ \equiv \ -i\ \Gamma_1\ \Gamma_2\ \Gamma_3\ \Gamma_4 \ \Rightarrow\ (\Gamma_5)^2 \ = \ 1; \quad \{\Gamma_5, \Gamma_a\} \ = \ 0 \qquad (2.27a)$$

The generators $\Gamma_{ab}, \Gamma_{abc}, \Gamma_{abcd}$ are defined as usual by a signed-permutation sum of the anti-symmetrizated products of the gammas. As a result of the relations in (2.27a) one finds

$$\Gamma_{abc} \ = \ i\ \epsilon_{abcd}\ \Gamma_5\ \Gamma^d, \ \ \Gamma_{abcd} \ = \ i\ \epsilon_{abcd}\ \Gamma_5, \ \ \Gamma_{ab} \ = \ \frac{i}{2}\ \epsilon_{abcd}\ \Gamma^5\ \Gamma^{cd} \quad (2.27b)$$

We shall be using a representation of the $Cl(3,1)$ algebra where the generators (4×4 matrices)

$$\mathbf{1};\ -i\ \Gamma_1;\ \Gamma_2;\ \Gamma_3;\ \Gamma_4;\ \Gamma_5 \ = \ -i\ \Gamma_1\ \Gamma_2\ \Gamma_3\ \Gamma_4; \ \ \Gamma_{1j}; \ \ \Gamma_1\Gamma_5; \ \ j = 2,3,4 \quad (2.28)$$

are Hermitian; while $-i\ \Gamma_1\Gamma_5$; $\Gamma_k\Gamma_5$; Γ_{kl} for $k,l = 2,3,4$ are anti-Hermitian. Using eqs-(2.27a,2.27b) allows to re-write the $Cl(3,1)$ algebra-valued one-form $\mathcal{A}^A_\mu \Gamma_A dx^\mu$ in the following way

$$\mathbf{A} \ = \ \left(a_\mu\ \mathbf{1} \ + i\ b_\mu\ \Gamma_5 \ + \ e^a_\mu\ \Gamma_a \ + i\ f^a_\mu\ \Gamma_a\ \Gamma_5 \ + \ \frac{1}{4}\omega^{ab}_\mu\ \Gamma_{ab} \right)\ dx^\mu. \quad (2.29)$$

after using the following definitions $\mathcal{A}_\mu = a_\mu$; $\mathcal{A}^a_\mu = e^a_\mu$; $\mathcal{A}^{abcd}_\mu \epsilon_{abcd} \equiv b_\mu$; $\mathcal{A}^{bcd}_\mu \epsilon^a_{bcd} \equiv f^a_\mu$, and $\mathcal{A}^{ab}_\mu = \frac{1}{4}\omega^{ab}_\mu$.

As stated earlier, the 4×4 matrices $\mathbf{A}_\mu, \mu = 1,2,3,4$, will have a mixture of Hermitian and anti-Hermitian pieces. All of the gauge fields appearing in (2.29) are real. The physical significance of the real-valued field components $a_\mu, b_\mu, e^a_\mu, f^a_\mu, \omega^{ab}_\mu$ in eq-(2.29) will be explained below.

The Clifford-valued gauge field $\mathbf{A}_\mu$ transforms according to $\mathbf{A}'_\mu \ = \ \mathbf{U}^{-1}\ \mathbf{A}_\mu\ \mathbf{U} + \mathbf{U}^{-1}\partial_\mu \mathbf{U}$ under Clifford-valued gauge transformations. The Clifford-valued field strength is $\mathbf{F} = \mathbf{dA} + [\mathbf{A}, \mathbf{A}]$ so that $\mathbf{F}$ transforms covariantly $\mathbf{F}' = \mathbf{U}^{-1}\ \mathbf{F}\ \mathbf{U}$. Decomposing the field strength in terms of the Clifford algebra generators gives

$$\mathbf{F}_{\mu\nu} \ = \ F_{\mu\nu}\ \mathbf{1} \ + i\ F^5_{\mu\nu}\ \Gamma_5 \ + \ F^a_{\mu\nu}\ \Gamma_a \ + i\ F^{a5}_{\mu\nu}\ \Gamma_a\ \Gamma_5 \ + \ \frac{1}{4} F^{ab}_{\mu\nu}\ \Gamma_{ab}. \quad (2.30)$$

the Clifford-algebra-valued 2-form field strength is $\mathbf{F} = \frac{1}{2}\ \mathbf{F}_{\mu\nu}\ dx^\mu \wedge dx^\nu$ and $\mathbf{F}_{\mu\nu} \ = \ \partial_\mu \mathbf{A}_\nu \ - \ \partial_\nu \mathbf{A}_\mu \ + \ [\ \mathbf{A}_\mu, \ \mathbf{A}_\nu]$ where $\partial_\mu \mathbf{A}_\nu = \frac{\partial \mathbf{A}_\nu}{\partial x^\mu}$. The field-strength components are given by [9], [10]

$$F_{\mu\nu} \ = \partial_\mu a_\nu - \partial_\nu a_\mu \qquad (2.31a)$$

$$F^5_{\mu\nu} \ = \ \partial_\mu b_\nu - \partial_\nu b_\mu + 2 e^a_\mu f_{\nu a} - 2 e^a_\nu f_{\mu a} \qquad (2.31b)$$

$$F^a_{\mu\nu} \ = \ \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^{ab}_\mu e_{\nu b} - \omega^{ab}_\nu e_{\mu b} + 2 f^a_\mu b_\nu - 2 f^a_\nu b_\mu \qquad (2.31c)$$

$$F^{a5}_{\mu\nu} \ = \ \partial_\mu f^a_\nu - \partial_\nu f^a_\mu + \omega^{ab}_\mu f_{\nu b} - \omega^{ab}_\nu f_{\mu b} + 2 e^a_\mu b_\nu - 2 e^a_\nu b_\mu \qquad (2.31d)$$

$$F^{ab}_{\mu\nu} \;=\; \partial_\mu \omega^{ab}_\nu + \omega^{ac}_\mu \omega_{\nu c}{}^{b} + 4\left(e^a_\mu e^b_\nu - f^a_\mu f^b_\nu\right) - \mu \leftrightarrow \nu. \tag{2.31e}$$

At this stage we may provide the relation among the $Cl(3,1)$ algebra generators and the the conformal algebra $so(4,2) \sim su(2,2)$ in $4D$. Due to the key condition of the chirality matrix $(\Gamma_5)^2 = 1$, a close inspection reveals that the operators of the Conformal algebra can be written in terms of the Clifford algebra generators as [7]

$$P_a \;=\; \frac{1}{2}\Gamma_a\,(1\,-\,\Gamma_5);\quad K_a \;=\; \frac{1}{2}\Gamma_a\,(1\,+\,\Gamma_5);\quad D \;=\; -\,\frac{1}{2}\,\Gamma_5,\;\; L_{ab} \;=\; \frac{1}{2}\,\Gamma_{ab}. \tag{2.32}$$

P_a ($a = 1,2,3,4$) are the translation generators; K_a are the conformal boosts; D is the dilation generator and L_{ab} are the Lorentz generators. The total number of generators is respectively $4+4+1+6 = 15$. From the above realization of the conformal algebra generators (2.32), the explicit evaluation of the commutators yields

$$[P_a,\;D] \;=\; P_a;\quad [K_a,\;D] \;=\; -\,K_a;\quad [P_a,\;K_b] \;=\; -\,2\eta_{ab}\;D\;+\;2\;L_{ab}$$

$$[D,\;L_{ab}] \;=\; 0,\;\; [P_a,\;P_b] \;=\; 0;\quad [K_a, K_b] \;=\; 0,\;\; a,b = 1,2,3,4 \tag{2.33}$$

which is consistent with the $su(2,2) \sim so(4,2)$ commutation relations.[4]. The dilation D operator is represented by a Hermitian matrix, while the Lorentz generator L_{ab} is represented by an anti-Hermitian matrix when $a,b = 2,3,4$, and a Hermitian matrix when a or b is 1. The fact that Hermitian and anti-Hermitian matrices are both required is consistent with the fact that $U(2,2) = SU(2,2)\times U(1)$ is a pseudo-unitary group which can be obtained via the Weyl "unitary trick" from the unitary group $U(4)$ by the analog of the Wick rotation procedure [8], [10].

Having established this one can infer that the (complex) tetrad V^a_μ field (associated with translations) and its complex-conjugate partner $\bar V^a_\mu$ (associated with conformal boosts) can be defined in terms of the gauge fields e^a_μ, f^a_μ as follows

$$e^a_\mu\;\Gamma_a\;\;+i\;f^a_\mu\;\Gamma_a\Gamma_5 \;=\; V^a_\mu\;P_a\;+\;\bar V^a_\mu\;K_a \tag{2.34}$$

From eqs-(2.29,2.32) one learns that eq-(2.34) leads to a complex-conjugate pair of fields

$$e^a_\mu\;\;-i\;f^a_\mu \;=\; V^a_\mu;\quad e^a_\mu\;\;+i\;f^a_\mu \;=\; \bar V^a_\mu\;\;\Rightarrow$$

$$e^a_\mu \;=\; \frac{1}{2}\,(V^a_\mu + \bar V^a_\mu),\quad f^a_\mu \;=\; \frac{1}{2i}(\bar V^a_\mu - V^a_\mu). \tag{2.35}$$

The components of the torsion and conformal-boost curvature of conformal gravity are given respectively by the linear combinations of eqs-(2.31c, 2.31d)

[4]We should notice that the K_a, P_a generators in (2.32) are both comprised of Hermitian Γ_i, and anti-Hermitian $\Gamma_i\Gamma_5$ matrices, when $i = 2,3,4$. Whereas, one has an anti-Hermitian Γ_1 and a Hermitian $\Gamma_1\Gamma_5$ matrix. As a result, P_a, K_a are represented by 4×4 nilpotent matrices $P^2_a = K^2_a = 0$ (no sum over a)

$$F^a_{\mu\nu} \ - i \ F^{a5}_{\mu\nu} \ = \ F^a_{\mu\nu}[P]; \quad F^a_{\mu\nu} \ + i \ F^{a5}_{\mu\nu} \ = \ F^a_{\mu\nu}[K] \ \Rightarrow$$

$$F^a_{\mu\nu} \ \Gamma_a \ + i \ F^{a5}_{\mu\nu} \ \Gamma_a \ \Gamma_5 \ = \ F^a_{\mu\nu}[P] \ P_a \ + \ F^a_{\mu\nu}[K] \ K_a. \tag{2.36}$$

When $a = 1$, the left-hand side of eq-(2.36) furnishes an overall anti-Hermitian matrix; whereas, when $a = 2, 3, 4$, it yields a Hermitian matrix. Therefore, the same occurs for the right-hand side. Inserting the expressions for e^a_μ, f^a_μ in terms of the vierbein V^a_μ and $\bar{V}^a_\mu$ given by (2.35), yields the standard expressions for the torsion and conformal-boost curvature, respectively.

$$F^a_{\mu\nu}[P] \ = \ \partial_{[\mu} \ V^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ V_{\nu]b} \ - 2i \ V^a_{[\mu} \ b_{\nu]}, \tag{2.37a}$$

$$F^a_{\mu\nu}[K] \ = \ \partial_{[\mu} \ \bar{V}^a_{\nu]} \ + \ \omega^{ab}_{[\mu} \ \bar{V}_{\nu]b} \ + 2i \ \bar{V}^a_{[\mu} \ b_{\nu]}, \tag{2.37b}$$

Once again, despite the complex-valued nature of the components $F^a_{\mu\nu}[P]; F^a_{\mu\nu}[K]$, the expression in the right hand side of eq-(2.36) must yield an overall anti-Hermitian matrix when $a = 1$, and a Hermitian matrix when $a = 2, 3, 4$, as expected.

The Lorentz curvature in eq-(2.31e) can be recast in the standard form as

$$F^{ab}_{\mu\nu} = \mathcal{R}^{ab}_{\mu\nu} \ = \ \partial_{[\mu} \ \omega^{ab}_{\nu]} \ + \ \omega^{ac}_{[\mu} \ \omega^b_{\nu]c} \ + \ 2(V^a_{[\mu} \ \bar{V}^b_{\nu]} \ + \ \bar{V}^a_{[\mu} \ V^b_{\nu]}). \tag{2.37c}$$

As expected, $\mathcal{R}^{ab}_{\mu\nu}$ is *real*-valued since eq-(2.37c) is the same as eq-(2.31e) (real-valued).

The components of the (real-valued) curvature corresponding to the Weyl dilation generator given by $F^5_{\mu\nu}$ in eq-(2.31b) can be rewritten as

$$F^5_{\mu\nu} \ = \ \partial_{[\mu} \ b_{\nu]} \ + \ \frac{1}{2i} \ (\ V^a_{[\mu} \ \bar{V}_{\nu]a} \ - \ \bar{V}^a_{[\mu} \ V_{\nu]a} \ ). \tag{2.37d}$$

The Maxwell $U(1)$ curvature is given by $F_{\mu\nu}$ in eq-(2.31a). A re-scaling of the vierbein V^a_μ/l and $\bar{V}^a_\mu/l$ by a length scale parameter[5] l is necessary in order to endow the curvatures and torsion in eqs-(2.37) with the proper dimensions of $length^{-2}, length^{-1}$, respectively.

Gauge invariant actions involving Yang-Mills terms of the form $\int Tr(F \wedge^* F)$ and theta terms of the form $\int Tr(F \wedge F)$ are straightforwardly constructed. For example, a $SO(4,2)$ gauge-invariant action for conformal gravity is [9]

$$S \ = \ \int \ d^4x \ \epsilon_{abcd} \ \epsilon^{\mu\nu\rho\sigma} \ \mathcal{R}^{ab}_{\mu\nu} \ \mathcal{R}^{cd}_{\rho\sigma} \tag{2.38}$$

where the components of the Lorentz curvature 2-form $\mathcal{R}^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu$ are provided by eq-(2.37c).

The conformal boost symmetry can be fixed by choosing the gauge $b_\mu = 0$ because under infinitesimal conformal boosts transformations the field b_μ transforms as $\delta b_\mu = -2 \ \xi^a \ e_{a\mu} = -2 \ \xi_\mu$; i.e the parameter ξ_μ has the same number

[5]The length parameter l is the same as the de Sitter throat size ρ

of degrees of feedom as b_μ. After further fixing the dilational gauge symmetry by setting $F^5_{\mu\nu} = 0$, it gives $f^a_\mu = 0$, and then constraining the sum of $F^a_{\mu\nu}[P] + F^a_{\mu\nu}[K] = 0$ to zero, it furnishes the spin connection as a function of e^a_μ : $\omega^{ab}_\mu(e^a_\mu)$ and leading to the torsionless Levi-Civita connection. Finally, the action (2.38) leads to the de Sitter group $SO(4,1)$ invariant Macdowell-Mansouri-Chamseddine-West (MMCW) action [5] (suppressing spacetime indices for convenience) described earlier

$$S_{MMCW} \ = \ \int \ d^4x \ (\ R^{ab}(\omega) - \frac{1}{l^2} \ e^a \wedge e^b \ ) \wedge (\ R^{cd}(\omega) - \frac{1}{l^2} \ e^c \wedge e^d \ ) \ \epsilon_{abcd}. \quad (2.39)$$

and leading to the Gauss-Bonnet topological invariant, Einstein-Hilbert and cosmological constant terms.[6]

The familiar Einstein-Hilbert gravitational action can also be obtained from a coupling of conformal gravity to a scalar field like it occurs in a Brans-Dicke-Jordan theory of gravity. The kinetic term $\phi(D^\mu_c D^c_\mu)\phi$ in the action (after integrating by parts) and based on the conformally covariant derivative operator D^c_μ can be decomposed in the following form

$$S \ = \ \frac{1}{2} \int \ d^4x \ \sqrt{g} \ \phi \ \left(\ \frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} \ g^{\mu\nu} \ D^c_\mu \phi) \ + \ b^\mu \ (D^c_\mu \phi) \ + \ \frac{1}{6} \ R \ \phi \ \right). \quad (2.40a)$$

where the conformally covariant derivative acting on a scalar field ϕ of Weyl weight one is

$$D^c_\mu \phi \ = \ \partial_\mu \phi \ - \ b_\mu \ \phi \quad (2.40b)$$

Fixing the conformal boosts symmetry by setting $b_\mu = 0$ and the dilational symmetry by setting $\phi = constant$ leads to the Einstein-Hilbert action for ordinary gravity.

Complexified Conformal Gravity

A natural complexification of a metric can be chosen to be $\mathbf{g}_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$, and is comprised of symmetric $g_{(\mu\nu)}$ and anti-symmetric matrices $g_{[\mu\nu]}$, obeying the Hermiticity condition $\mathbf{g}^\dagger_{\mu\nu} = \mathbf{g}_{\mu\nu}$ [11], [13]. Introducing a complex vierbein $E^a_\mu = e^a_\mu + i f^a_\mu$, a Hermitian complex metric can be defined as $\mathbf{g}_{\mu\nu} = E^a_\mu (E^b_\nu)^* \eta_{ab}$, leading to the components

$$g_{(\mu\nu)} \ = \ (e^a_\mu \ e^b_\nu \ + \ f^a_\mu \ f^b_\nu) \ \eta_{ab}; \ \ i \ g_{[\mu\nu]} \ = \ - \ i \ (e^a_\mu \ f^b_\nu \ - \ f^a_\mu \ e^b_\nu) \ \eta_{ab} \quad (2.41)$$

Chamseddine formulated gravity with a complex vierbein based on $SL(2,C)$ gauge invariance and proposed an action in terms of a four-form. The resulting

[6]Gravity involves invariance under diffeomorphisms (coordinate transformations) and gravitons have spin 2, not 1. What occurs is that the torsion constraint $F^a_{\mu\nu} = 0$ allows to convert a combination of translations, Lorentz and dilation transformations of the vierbein e^a_μ into general coordinate transformations of the vierbein, see [9] for further details

theory was equivalent to bigravity. He extended the gauge group to $GL(2,C)$, constructed a star-product-deformed action and derived the Seiberg-Witten map for the complex vierbein and gauge fields. His work [13] is very *different* than the one presented here, mainly because the $Cl(4,C)$ algebra is isomorphic to $M(4,C) \sim gl(4,C)$, and not $gl(2,C)$. For further details on complex, quaternionic, and octonionic valued gravity see [12] and references therein.

Another more direct route one can take is to begin directly with the complexification of all the fields associated with a $Cl(4,C)$ gauge theory of gravity. To illustrate what the complexification of the Lagrangian (density) in eq-(2.39) looks like, one simply introduces a complex-valued spin connection and vierbein of the form $\omega_\mu^{ab} + iv_\mu^{ab}$, $e_\mu^a + ib_\mu^a$, and sets the complex-valued extension of f_μ^a to zero, leading to a complex-valued curvature two-form, whose real and imaginary components are, respectively,

$$\mathbf{F}^{ab} \ = \ d\omega^{ab} \ + \ \omega_c^a \wedge \omega^{cb} \ + \frac{1}{l^2} \ e^a \wedge e^b \ - \ v_c^a \wedge v^{cb} \ - \frac{1}{l^2} \ b^a \wedge b^b \tag{2.42}$$

$$\mathbf{G}^{ab} \ = \ dv^{ab} \ + \ \omega_c^a \wedge v^{cb} \ + \ v_c^a \wedge \omega^{cb} \ + \frac{1}{l^2} \ e^a \wedge b^b \ + \frac{1}{l^2} \ b^a \wedge e^b \tag{2.43}$$

Constraining the torsion to zero leads to $\omega^{ab} = \omega^{ab}(e^a, b^a)$, and $v^{ab} = v^{ab}(e^a, b^a)$, which is the generalization of the relation $\omega^{ab} = \omega^{ab}(e^a)$ in ordinary real-valued gravity. Consequently, the curvature-squared terms will no longer yield the Gauss-Bonnet topological invariant. Furthermore, the real part of the curvature receives an *extra* contribution given by the last two terms in eq-(2.43). Since the last term $-\frac{1}{l^2} b^a \wedge b^b$ appears with a *minus* sign, one might have a desirable *cancellation* mechanism of the cosmological constant term $\frac{1}{l^2} e^a \wedge e^b$, when $e^a = e_\mu^a dx^\mu = \pm b^a = \pm b_\mu^a dx^\mu$. While the imaginary part of the curvature (2.44) involves the imaginary part v_μ^{ab} of the spin connection plus terms involving the mixing of the real parts and imaginary parts of the spin connection and vierbein.

Given a complex-valued curvature of the form $(F + iG)_{\mu\nu}^{ab}$, where F, G are given by eqs-(2.42,2.43), a real-valued Yang-Mills-like Lagrangian is of the form $(F + iG)_{\mu\nu}^{ab} (F - iG)_{ab}^{\mu\nu}$. A theta-like MMCW Lagrangian density was provided earlier as $\epsilon^{\mu\nu\tau\sigma} (F + iG)_{\mu\nu}^{ab} (F + iG)_{\tau\sigma}^{cd} \epsilon_{abcd}$ + complex-conjugate (real-valued by construction). It is important to remark that if one wishes to obtain an overall cancellation of the cosmological constant in the latter action, it leads to an algebraic constraint between e_μ^a, b_μ^a and whose solution is *no* longer given by $e_\mu^a = \pm b_\mu^a$. Another salient feature is, besides that one has *additional* contributions to the ordinary curvature, is a key coupling, "entanglement" among the real and imaginary components of the spin connection and vierbein. Whether or not this may play a role in understanding the nature of dark energy and Quantum Gravity deserves further investigation. Finally, the modified gravitational theory involving the complex metric must *not* be confused with bi-gravity since the imaginary component $ig_{[\mu\nu]}$ is antisymmetric.

3 Conclusion

To sum up, we have shown how a $Cl(3,1,R)$ gauge theory of gravity in $4D$ can be recast as a $4D$ Conformal Gravity-Maxwell theory based on $U(2,2) = SU(2,2) \times U(1)$. By including the extra contribution of the (purely imaginary) dual fields (2.25) it leads to a complexification of the Conformal Gravity-Maxwell theory and based on the complex $Cl(4,C)$ algebra. The complexification of the $U(2,2), SU(2,2)$ groups is $U(2,2) \otimes \mathbf{C} = GL(4,C)$, and $SU(2,2) \otimes \mathbf{C} = SL(4,C)$, respectively, see [8] for more specific details. In general, the algebra $gl(N,C)$ is the complex extension of $u(p,q)$ for all p,q such that $p+q=N$ [8]. The covering of the general linear group $GL(N,R)$ does not admit $finite$ dimensional spinorial representations but infinite dimensional. For a rigorous treatment of these infinite-dim spinorial representations and the perturbative renormalization property of metric affine theories of gravity based on the semidirect product of $GL(N,R)$ with the translations T_N we refer to [4].

Therefore, to conclude, the complexified $4D$ Conformal Gravity-Maxwell theory turns out to be isomorphic to a gauge theory of gravity based on the complex group $GL(4,C)$. We also have attained the desired goal of extending the de Sitter algebra to the Clifford algebra case. The Clifford algebraic version of the de Sitter algebra $so(4,1)$ is realized via the of $Cl(4,1,R)$ algebra, and which in turn, leads to a complexification of the Conformal Gravity-Maxwell theory in $4D$ due to the isomorphism $Cl(4,C) \sim Cl(4,1,R)$. This interplay between a gauge theory of gravity based on $Cl(4,1,R)$, whose bivector-generators encode the de Sitter algebra $so(4,1)$, and conformal gravity based on $Cl(3,1,R)$ is reminiscent of the AdS_{D+1}/CFT_D correspondence between $D+1$-dim gravity in the bulk and Conformal field theory in the D-dim boundary.

The Clifford algebraic version of the de Sitter algebra was depicted by the commutators in eq-(2.10). After performing the dualization procedure of the gamma matrices of $Cl(4,1,R)$ it leads to a $4D$ complexified conformal gravity-Maxwell theory. It is interesting that this procedure via the use of Clifford algebras is the reversal to what occurs when one embeds the de Sitter algebra $so(4,1)$ into the larger conformal algebra $so(4,2)$. In this work the conformal symmetry, encoded in the $Cl(3,1,R)$ algebra, is captured from the larger $Cl(4,1,R)$ algebra which is the Clifford extension of the de Sitter algebra $so(4,1)$.

Similar results follow in the Anti de Sitter case after constructing a $Cl(3,2,R)$ gauge theory of gravity. Because $Cl(3,1,R)$ is a sub-algebra of both $Cl(4,1,R)$ *and* $Cl(3,2,R)$, after reduction of one spatial and temporal dimension, respectively, one recovers $4D$ conformal gravity in both cases. Despite that this reduction of one dimension mimics the holographic principle we have not invoked holography in this work.[7]

Related to the issue of grand-unification models with gravity we recall a model based on $Cl(5,C) = Cl(4,C) \oplus Cl(4,C)$ [10]. The gauge theory involving

[7] No boundaries of the bulk spacetime have been invoked. From the isomorphism displayed in eq-(2.26a) one learns that the bivectors of $Cl(2,3,R)$ generate the $so(2,3)$ algebra which is not the same as the Anti de Sitter algebra $so(3,2)$

the first copy $Cl(4,C)$ has been studied in this work and leads to a complexified conformal gravity-Maxwell theory in $4D$ based on $U(2,2) = SU(2,2) \times U(1)$. As mentioned earlier, in general, the unitary compact group $U(p+q)$ is related to the noncompact pseudo-unitary group $U(p,q)$ via the Weyl unitary trick [8]. Consequently, the second copy of the $Cl(4,C)$ algebra in the decomposition of $Cl(5,C) = Cl(4,C) \oplus Cl(4,C)$ has the same algebraic structure of $u(4) \oplus u(4)$ after performing the Weyl unitary trick ("analytical continuation") from $u(2,2)$ to $u(4)$. Therefore, a $Cl(4,C)$ gauge theory living in the second copy can accommodate a $U(4) \times U(4)$ gauge theory which contains the Pati-Salam and the Standard Model groups [10]. Furthermore, no violation of the Coleman-Mandula theorem takes place.

This formulation of the (pseudo) unitary groups is very different from the standard procedure to obtain the $u(N)$ generators $E_{jk} = a_j^\dagger a_k$ in terms of the complex $Cl(2N,C)$ algebra via the creation and annihilation fermionic oscillators defined as follows $a_j = \frac{1}{2}(\Gamma_{2j} + i\ \Gamma_{2j-1})$; $a_j^\dagger = \frac{1}{2}(\Gamma_{2j} - i\ \Gamma_{2j-1})$; $j = 1,2,\cdots,N$. One can verify that $\{a_j, a_k^\dagger\} = \delta_{jk}; \{a_j, a_k\} = 0; \{a_j^\dagger, a_k^\dagger\} = 0$ leading to the $u(N)$ commutation relations $[E_{jk}, E_{lm}] = \delta_{kl}E_{jm} - \delta_{jm}E_{lk}$. This construction is just a reflection of the fact that $u(N) \subset so(2N)$. In particular, $u(4) \subset so(8)$.

These results can be extended to larger gauge groups associated to higher dimensional Clifford algebras $Cl(N-1,1,R), Cl(N-2,2,R); N \geq 5$, and to higher base-spacetime dimensions $D \geq 4$. Moreover, one can even enlarge the ordinary spacetime vector coordinates x^μ to include Clifford-valued polyvector coordinates $\mathbf{X} = X^{\mathcal{M}}\Gamma_{\mathcal{M}}$, with $X^{\mathcal{M}} = x, x^\mu, x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \cdots, x^{\mu_1\mu_2\cdots\mu_D}$ belonging to the so-called C-space (Clifford space) associated to the underlying base spacetime Clifford algebra $Cl(D-1,1,R)$, which is defined by $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}\mathbf{1}; \mu,\nu = 1,2,3,\cdots,D$, with $g_{\mu\nu}$ the spacetime metric [7]. The most general Clifford-valued *polyform* is defined by $\mathcal{A}^A_{\mathcal{M}}(\mathbf{X})\Gamma_A dX^{\mathcal{M}}$, where the polyvector-valued internal index A spans the 2^N-dim Clifford algebras $Cl(N-1,1,R), Cl(N-2,2,R)$ representing the gauge symmetry. And, finally, one can proceed to construct a Clifford gauge theory of gravity in C-spaces. Extended theories of gravity in C-spaces via the conventional methods were constructed in [7].

The issue of ghosts, renormalizabilty, unitarity, $\cdots$ of the theory remains to be studied. In the meantime a detailed analysis of the two Lagrangians described above, and based on the complex-valued curvature two-form, whose real and imaginary components are, respectively, given by eqs-(2.42,2.43), warrants further investigation. We found some desirable results, like a very plausible cancellation mechanism of the cosmological constant involving an algebraic constraint between e^a_μ, b^a_μ, and the coupling of the real and imaginary components of the spin-conection and vierbein. We expect to find other novel consequences emerging from the physics behind complex conformal gravity-Maxwell theory[8] and Clifford gauge theories of gravity.

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[8]Note that one has *not* complexified spacetime as in Twistor theory

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