Absolute Rotation of the Foucault Pendulum

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Abstract

The absolute rotations of the Foucault Pendulum and its mounting fixture were derived for any latitude and Earth rotation angle. A theorem in attitude kinematics was used to obtain analytical solutions for both rotation angles and rotation angle rates. Both pendulum and mounting fixture were found to have absolute rotation rates in the counterclockwise direction in the Northern Hemisphere. The difference in rotation rates is in agreement with the observed relative clockwise precession rate of the pendulum with respect to its mounting fixture in the local Earth reference frame. The results were validated with an alternative kinematic method.

Keywords: Foucault Pendulum, Attitude kinematics, Ishlinskii's Theorem, Solid angle, Precession angle

1. Introduction

The Foucault Pendulum [1] is a long spherical pendulum with a significant mass attached to maintain its periodic motion in the presence of frictional forces. The pendulum is mounted to a nearly frictionless bearing to decouple the axial motion about the vertical axis. The motion of the pendulum is initialized such that the pendulum mass moves in a plane containing the vertical axis. The momentum associated with the pendulum's oscillation ensures that its axial rotation is decoupled from the mounting fixture. Once the pendulum begins swinging, the plane of the pendulum's motion appears to rotate or precess about its vertical axis. The precessing Foucault Pendulum was first demonstrated in Paris in 1851 and showed that the Earth rotates [1] without relying on a celestial coordinate frame.

The motion of the Foucault Pendulum is difficult to treat because the pendulum resides in the Earth's non-inertial rotating coordinate frame. Fortunately, the pendulum's motion can be analyzed using attitude kinematics. A recent work [2] showed that for a complete rotation of the Earth, the precession of the Foucault Pendulum can be explained using Ishlinskii's Theorem [3] in attitude kinematics. Results show that the rotation of the pendulum about its vertical axis is equal to the solid angle of the conical cap north of its latitude, which is $2\pi\ (1-\sin(\Theta))$ radians, where Θ is the latitude. The vertical axis of the pendulum's mounting fixture, MF, has a rotation equal to the pendulum's rotation but with an additional rotation due to the integral of the projection of the Earth's angular rate along the vertical axis, which is $2\pi \sin(\Theta)$ radians. Therefore, for one complete rotation of the Earth, the sum of both rotational contributions indicates that the MF's rotation is 2π radians for any latitude. It should be noted that the Earth's angular rate component of the MF can be measured by a rate gyro with its input axis align with the vertical axis. However, the pendulum's rotation about its axis has only a solid angle contribution, which is not measurable by a rate gyro.

A detailed analysis of the pendulum's attitude kinematics [2] was able to quantify the pendulum's motion for any angle of Earth rotation. The current work is focused on the absolute rotation of the pendulum about its vertical axis rather than its complete attitude motion. A new method to obtain the solid angle associated with the pendulum's rotation yields an analytical solution for the pendulum's rotation angle. The pendulum's rotation rate was obtained by differentiating the rotation angle equation. It was found that the pendulum rotates in the counterclockwise direction in the Northern

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Hemisphere and has a rate that is a nonlinear function of the Earth's rotation angle. The vertical axis of the MF also rotates in the counterclockwise direction but acquires a higher rotation rate than the pendulum due to a contribution from the Earth's spin rate. The difference in the pendulum's rotation rate and the MF's rotation rate accounts for the clockwise precession of the pendulum as observed from the Earth's reference frame.

Section 2 contains a derivation of the rotation angle of the Foucault Pendulum's plane of oscillation as a function of Earth rotation angle and latitude. The derivation uses a new method to compute solid angles [4], which provides an analytical solution. The results are validated with a previously published general method of computing solid angles [4, 5]. The resulting analytical equation is used to derive the associated rotation angle rate. The same method was used to calculate rotation angle and rate for the MF.

Section 3 contains numerical results that illustrate the pendulum's absolute rotation angle and associated rate for various values of latitude and Earth rotation angle. Results were also obtained for the MF's rotation angle and rate. Both sets of angles and rates indicate nonlinear dependence on the Earth's rotation angle. This feature differs from the pendulum's precession rate, which is a function of latitude but not the Earth's rotation angle.

Section 4 contains the conclusion.

2. Attitude Kinematics

The rotation of the plane of oscillation of the Foucault Pendulum, which is equivalent to the rotation of the pendulum's vertical axis, is governed by attitude kinematics. Fig. 1 illustrates the pendulum's vertical axis as it slews from orientation **A** to orientation **B** along a conical trajectory with its tip at a fixed value of latitude. It was shown in a recent publication [4] that the attitude transformation of the axis in going from orientation **A** to orientation **B** along an arbitrary trajectory has three components. The first component of the attitude transformation is the great circle arc transformation linking orientations **A** and **B**. The second component of the attitude transformation is the rotational transformation about the axis at orientation **B** that is equal to the solid angle bounded by the actual trajectory of the axis and the associated great circle arc trajectory. The third component of the attitude transformation is the time integral of the Earth's angular rate along the axis during the slewing motion. Since this work is focused on the rotation about the axis, only the second and third components of the attitude transformation are of concern. The third component of the attitude transformation causes the MF to rotate by an additional amount but does not cause the pendulum to rotate, since the pendulum is decoupled from the MF's vertical axis. Therefore, the pendulum's rotation about its vertical axis is completely determined by the second component of the attitude transformation, which is the solid angle enclosed by the actual trajectory and the great circle arc trajectory.

The general method of computing solid angles, which was presented in an earlier work [5] can be applied to the Foucault Pendulum. Referring to Fig. 1, the total transformation, **U**, is given by a rotation of λ about the **z**-axis of the Earth, as shown in eq. (1), where **R** is the rotational transformation, **z** is the axis about which the rotation is applied and λ is the magnitude of the rotation angle.

$$\mathbf{U} = \mathbf{R}(\mathbf{z}, \lambda) \tag{1}$$

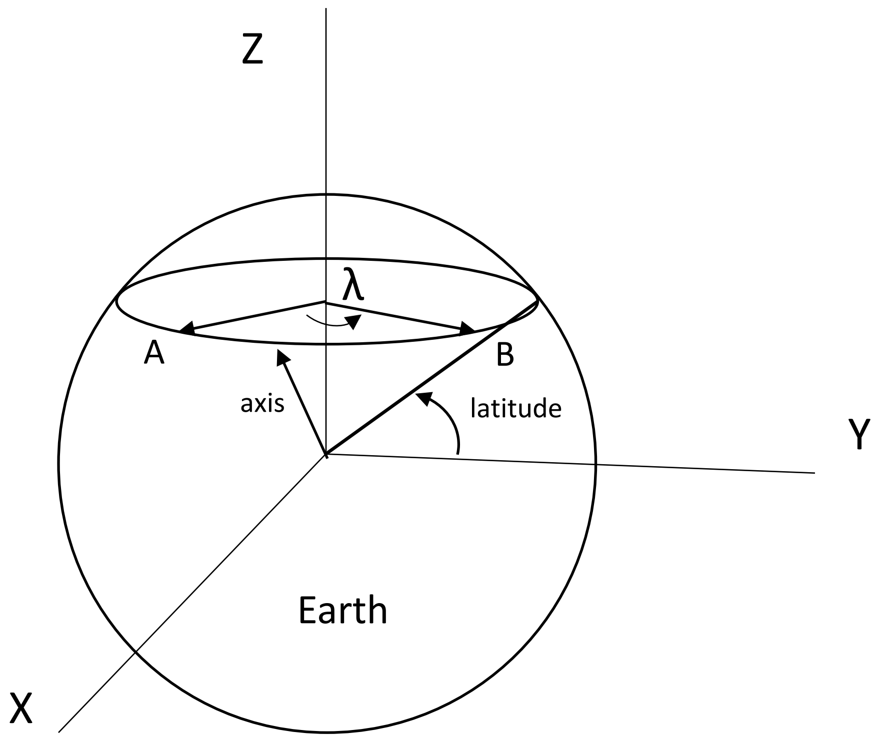

Fig. 1. The pendulum axis moves from orientation **A** to orientation **B** as the Earth rotates by angle λ.

It was shown in earlier work [2, 4] that the total transformation is the product of the slewing transformation, $\mathbf{U_S}$, and the rotational transformation, $\mathbf{R}(\mathbf{B}, \Delta)$, as shown in eq. (2). After the Earth rotates by λ, the axis will have rotated by $\lambda \sin(\Theta)$. Therefore, the angle Δ is given by the integral of the Earth's angular rate along the axis or $\lambda \sin(\Theta)$, where Θ is the latitude. Note that the slewing and rotational transformation commute, as was shown in an earlier work [4].

$$\mathbf{U} = \mathbf{R}(\mathbf{z}, \lambda) = \mathbf{U_S}\, \mathbf{R}[\mathbf{B}, \Delta] = \mathbf{R}[\mathbf{A}, \Delta]\, \mathbf{U_S} \tag{2}$$

The slewing transformation can be extracted from eq. (2) using the inverse of the respective rotational transformations, as shown in eqs. (3) and (4).

$$\mathbf{U_S} = \mathbf{R}(\mathbf{z}, \lambda)\, \mathbf{R}^{-1}[\mathbf{B}, \lambda \sin(\Theta)] = \mathbf{R}(\mathbf{z}, \lambda)\, \mathbf{R}[\mathbf{B}, -\lambda \sin(\Theta)] \tag{3}$$

$$\mathbf{U_S} = \mathbf{R^{-1}}[\mathbf{A},\ \lambda \sin(\Theta)]\, \mathbf{R}(\mathbf{z}, \lambda) = \mathbf{R}[\mathbf{A}, -\lambda \sin(\Theta)]\, \mathbf{R}(\mathbf{z}, \lambda) \tag{4}$$

The slewing transformations Eqs. (3) and (4) are equal and represent the attitude transformation associated with the slewing motion along the constant latitude arc. The other boundary of the desired solid angle is given by the great circle arc linking orientations **A** and **B**. The slewing transformation between orientations **A** and **B** is found using the Pivot Parameter method [6], which involves a sequence of two rotations of 180 degrees about two axes in the slewing plane that are separated by one half the rotation angle. The midpoint orientation of the axis along the great circle arc is needed for the Pivot Parameter method [6]. The midpoint orientation is defined by **C** in eq. (5), where **a** and **b** are unit vectors aligned with axes **A** and **B**, respectively.

$$\mathbf{C} = \frac{\mathbf{a+b}}{|\mathbf{a+b}|} \tag{5}$$

Using the Pivot Parameter method [6], the slewing transformation along the great circle arc from orientation **B** to **C**, is given in eq. (6), where the rotation angles are given in degrees.

$$\mathbf{U_G} = \mathbf{R}(\mathbf{B}, 180)\, \mathbf{R}(\mathbf{C}, 180) = \mathbf{R}(\mathbf{C}, 180)\, \mathbf{R}(\mathbf{A}, 180) \tag{6}$$

Therefore, the total slewing transformation around the solid angle region is given by eq. (7), where eqs. (3), (4) and (6) are used for the evaluation in eq. (7).

$$\mathbf{U_T} = \mathbf{U_S}\, \mathbf{U_G} \tag{7}$$

The general method to obtain the solid angle of the region from the slewing transformation about the region was developed in an earlier work [5]. It was found that the solid angle is simply the magnitude of the Euler Rotation Vector of the transformation. Therefore, the rotation angle of the pendulum, which is the solid angle of the enclosed region, is given by the magnitude of the Euler Rotation Vector for the transformation $\mathbf{U_T}$, as given in eq. (7).

Although eq. (7) can be used to obtain the rotation angle of the pendulum, a new alternative method [4] yields an analytical solution that can be used to derive the angular rate of the pendulum. It was shown in an earlier work that the solid angle of any regular spherical polygon of n sides is given by eq. (8), where ϕ_i are the exterior rotation angles about the vertices made by adjoining adjacent planes [4]. Each of the n sides is a segment of a great circle arc.

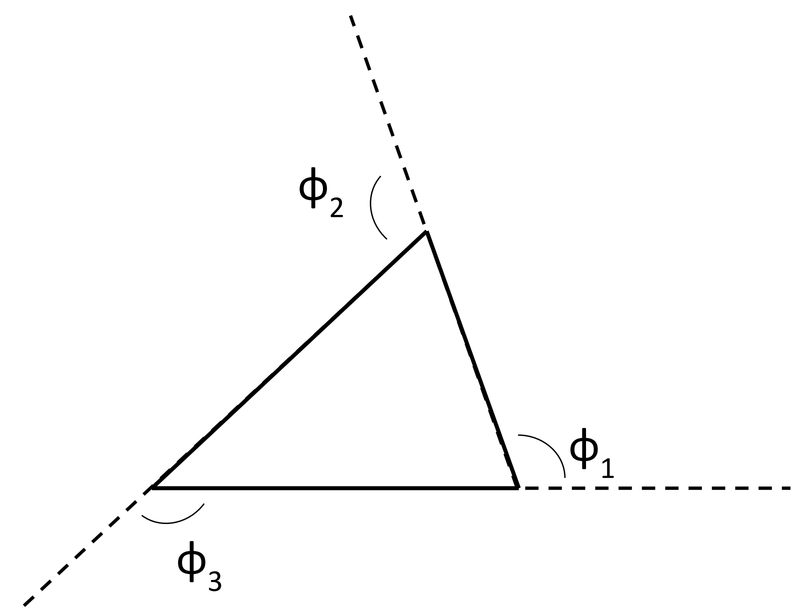

Fig. 2. Exterior angles of a spherical triangle as illustrated on a plane.

Fig. 2 illustrates the case for a spherical triangle as represented on a two dimensional plane. The sum of the exterior angles can be viewed as the rotation of an axis, as the axis moves about the spherical triangle.

$$\Omega = 2\pi - \sum_{i=1}^{n} \phi_i \tag{8}$$

Eq. (8) can be generalized to also include a region of continuous rotation of the axis as it slews about a section of the solid angle region, as shown in eq. (9), where a and b define the region of continuous rotation.

$$\Omega = 2\pi - \sum_{i=1}^{n} \phi_i - \int_a^b d\phi \tag{9}$$

Fig. 3 defines the relevant geometry that is used in eq. (9), where the true spherical geometry is represented on a flat plane for clarity. The unit vectors **a** and **b** are aligned with the axes **A** and **B**, respectively, and are defined in eq. (10), where Θ is the latitude and λ is the rotation of the Earth about its spin axis.

$$\mathbf{a} = \begin{pmatrix} \cos(\Theta) \\ 0 \\ \sin(\Theta) \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} \cos(\Theta)\cos(\lambda) \\ \cos(\Theta)\sin(\lambda) \\ \sin(\Theta) \end{pmatrix} \tag{10}$$

The unit vector **e** is normal to axis **b** and is defined by eq. (11). The unit vector **f** is also normal to axis **b** and is defined in eq. (12).

$$\mathbf{e} = \mathbf{z} \times \mathbf{b} = \begin{bmatrix} -\sin(\lambda) \\ \cos(\lambda) \\ 0 \end{bmatrix} \tag{11}$$

$$\mathbf{f} = \mathbf{e} \times \mathbf{b} = \begin{bmatrix} \sin(\Theta)\cos(\lambda) \\ \sin(\Theta)\sin(\lambda) \\ -\cos(\Theta) \end{bmatrix} \tag{12}$$

The unit vector **d** is normal to the plane that defines the great circle arc connecting **a** and **b**, and is defined by eq. (13). The magnitude of **d** is given by eq. (14).

$$\mathbf{d} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \left[\frac{\cos(\Theta)}{|\mathbf{d}|}\right] \begin{bmatrix} -\sin(\Theta)\sin(\lambda) \\ \sin(\Theta)\cos(\lambda) - \sin(\Theta) \\ \cos(\Theta)\sin(\lambda) \end{bmatrix} \tag{13}$$

$$|\mathbf{d}| = \cos(\Theta)\sqrt{sin^2(\lambda) + sin^2(\Theta)[\cos(\lambda) - 1]^2} \tag{14}$$

The rotation of the axis as it moves around the solid angle region is the sum of the rotations at orientation **b** and at orientation **a** plus the continuous rotation as the axis slews from **a** to **b** along the constant latitude trajectory. The rotation of the axis at orientation **b** is given by the angle between **f** and **d**, which is given by eq. (15), where G is given by eq. (16).

$$\Gamma = cos^{-1}(\mathbf{f} \cdot \mathbf{d}) = cos^{-1}\left[\frac{-\sin(\lambda)}{G}\right] \tag{15}$$

$$G = \sqrt{sin^2(\lambda) + sin^2(\Theta)[\cos(\lambda) - 1]^2} \tag{16}$$

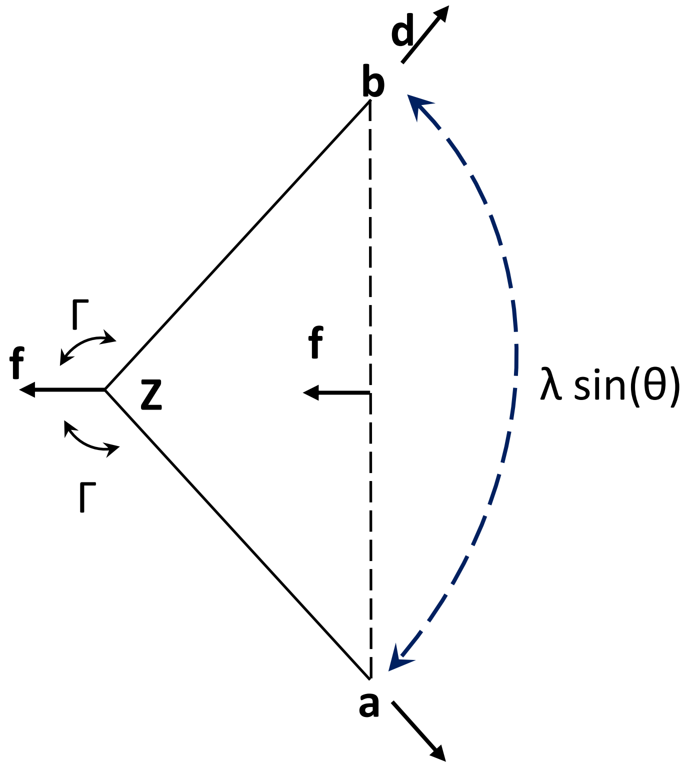

Fig. 3. Geometry related to eqs. (9 - 17) where the solid angle is bounded by the dashed lines.

Due to symmetry, the rotation of the axis at orientation **a** is the same as the rotation at orientation **b**. The rotation due to the continuous rotation as the axis moves from orientation **a** to orientation **b** is given by λ sin(Θ). Therefore using eq. (9), the solid angle of the enclosed region is given by eq. (17).

$$\Omega = 2(\pi - \Gamma) - \lambda \sin(\Theta) \tag{17}$$

The analytical solution for the solid angle given by eq. (17) is also the rotation of the pendulum's vertical axis after the Earth rotates by angle λ.

The MF rotates according to eq. (17) but has the additional rotation caused by the Earth's angular rate of λ sin(Θ), as shown in eq. (18).

$$\psi = 2(\pi - \Gamma) - \lambda \sin(\Theta) + \lambda \sin(\Theta) = 2(\pi - \Gamma) \tag{18}$$

For the special case λ = 2 π, Γ = 0 and the solid angle in eq. (17) is given by eq. (19), as expected for a complete rotation of the Earth on its axis. Eq. (20) contains the result for another special case, λ = π, Γ = π/2.

$$\Omega = 2\pi\,[1 - \sin(\Theta)] \quad (\lambda = 2\pi) \tag{19}$$

$$\Omega = \pi\,[1 - \sin(\Theta)] \quad (\lambda = \pi) \tag{20}$$

The MF rotates according to eq. (18) and has rotation angles corresponding to values of λ of 2π and π, respectively, as given in eqs. (21) and (22).

$$\psi = 2\,\pi \qquad (\lambda = 2\pi) \tag{21}$$

$$\psi = \pi \qquad (\lambda = \pi) \tag{22}$$

The rotation of the pendulum's vertical axis and that of its MF for values other than the special cases of λ = 2 π and λ = π must be evaluated using eqs. (15 – 18). For values of λ greater than 2π, one can sum the rotations for each complete Earth rotation given by eq. (19) and add the residual rotation value obtained from eq. (17) using λ mod 2π, which accounts for the remaining partial rotation. A similar method can be used for the MF using eqs. (18) & (21).

The analytical solution for the rotation angle of the pendulum given by eq. (17) permits the derivation of the rate of change of Ω by simple differentiation, as shown in eq. (23), with dΓ/dt given in eq. (24).

$$\frac{d\Omega}{dt} = -2\frac{d\Gamma}{dt} - \sin(\theta)\frac{d\lambda}{dt} = -2\sin(\theta)\,[\cos(\lambda) - 1]\;G^{-2}\frac{d\lambda}{dt} - \sin\,(\theta)\frac{d\lambda}{dt} \tag{23}$$

$$\frac{d\Gamma}{dt} = \frac{-sin^2(\theta)}{\sin\,(\Gamma)}\,[1 - \cos(\lambda)]^2\;G^{-3}\;\frac{d\lambda}{dt} = \;\sin(\theta)\,[\cos(\lambda) - 1]\;G^{-2}\;\frac{d\lambda}{dt} \tag{24}$$

Since the MF has an additional rotation rate of sin(Θ) dλ /dt due to the rotation of the Earth, its rotation rate, dψ/dt is given by eq. (25).

$$\frac{d\psi}{dt} = -2\frac{d\Gamma}{dt} = -2\sin(\theta)\,[\cos(\lambda) - 1]\;G^{-2}\;\frac{d\lambda}{dt} \tag{25}$$

The angular rate of the pendulum rotation given by eq. (23) can be used to obtain the ratio of the pendulum rotation rate and the Earth rotation rate, as given by eq. (26). Likewise using eq. (25), the ratio of the MF rotation rate and the Earth rotation rate is given by eq. (27).

$$\frac{d\Omega}{d\lambda} = -2\sin(\theta)\,[\cos(\lambda) - 1]G^{-2} - \sin\,(\theta) \tag{26}$$

$$\frac{d\psi}{d\lambda} = -2\sin(\theta)\,[\cos(\lambda) - 1]\;G^{-2} \tag{27}$$

3. Numerical Results

The rotation of the Foucault Pendulum was computed using eq. (17) for various values of latitude, θ, and Earth rotation angle, λ, as shown in Table 1. Results were validated using the general method of solid angle calculation [5] involving eq. (7) to ensure that the analytical solution was correct. For each latitude value, the rotation angle of the pendulum is not a linear function of Earth rotation angle, although the values at 180 degrees of Earth rotation are one half the values at 360 degrees.

The behavior of the pendulum rotation angle is displayed in Figs. 4, 6, 8, 10 & 12, which also include the rotation of the MF. Notice that for a complete rotation of the Earth, the rotation angle of the MF is 360 degrees for each value of latitude. The solid angle contribution to the MF's rotation angle becomes smaller and the Earth rotation rate contribution becomes larger for larger values of latitude. For 180 degrees of Earth rotation the MF rotates 180 degrees for each latitude. Each pendulum rotation angle at 180 degrees of Earth rotation is one half its value at 360 degrees of Earth rotation. This is due to symmetry of the solid angle of the spherical cap, i.e., solid angle equals π [1-sin(Θ)] for each 180 degrees of Earth rotation. The rotation angle rate in terms of the Earth rotation rate for the pendulum and MF are presented in Figs. 5, 7, 9, 11 & 13. The highest rates occur near the Earth rotation angle of 180 degrees. The nonlinear dependence of pendulum rotation angle and associated rate on the Earth rotation angle becomes larger for smaller values of latitude as shown in Figs. 10, 11, 12 & 13. For extremely small values of latitude, the pendulum and MF have rotation angles that approach 360 degrees because that is the solid angle of the Northern Hemisphere, as shown in Figs. 12 & 13. Although the pendulum's precession rate decreases as latitude becomes small, its absolute rotation angle becomes larger and approaches 360 degrees. In all cases, the difference between the pendulum rate and the MF rate agrees with the known pendulum precession rate of $-\omega \sin(\Theta)$ for all values of Earth rotation angle.

Table 1

Rotation Angle Examples for Algorithm Validation

Earth rotation angle, deg.	Pendulum angle, deg. 30 deg. Lat.	Pendulum angle, deg. 45 deg. Lat.	Pendulum angle, deg. 60 deg. Lat.
45	0.902	0.83	0.4967
90	8.13	6.89	3.84
135	33.22	23.82	11.96
180	90	52.72	24.12
225	146.78	81.62	36.27
270	171.87	98.55	44.39
315	179.10	104.61	47.73
345	179.97	105.41	48.21
360	180	105.44	48.23

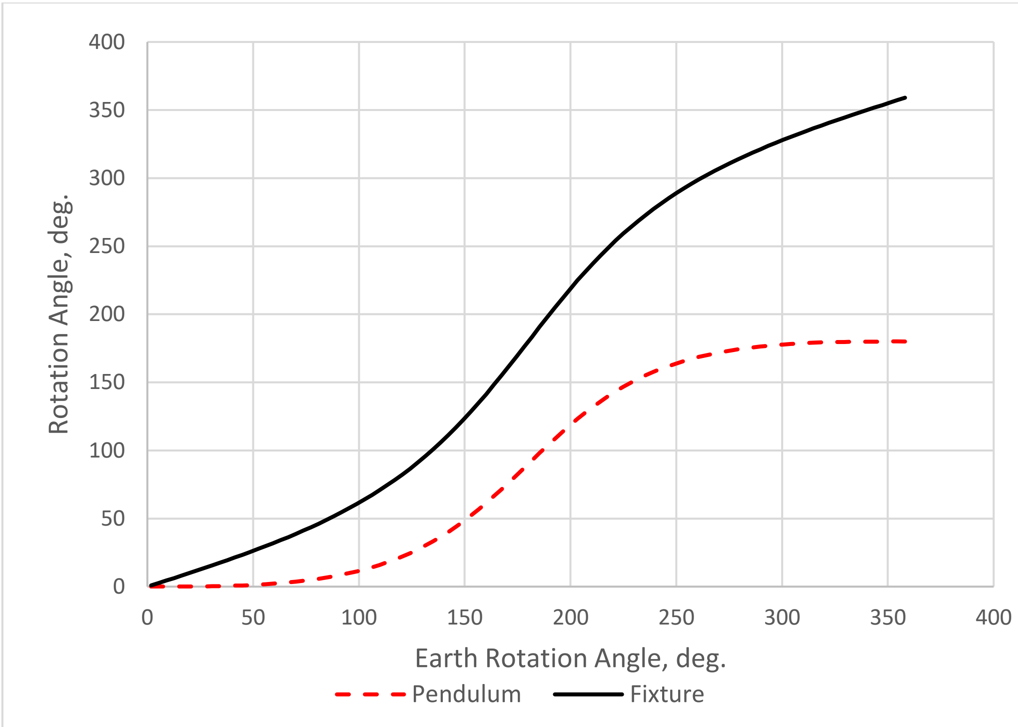

Fig. 4. Pendulum and fixture rotation angles versus the Earth rotation angle for 30 deg. latitude.

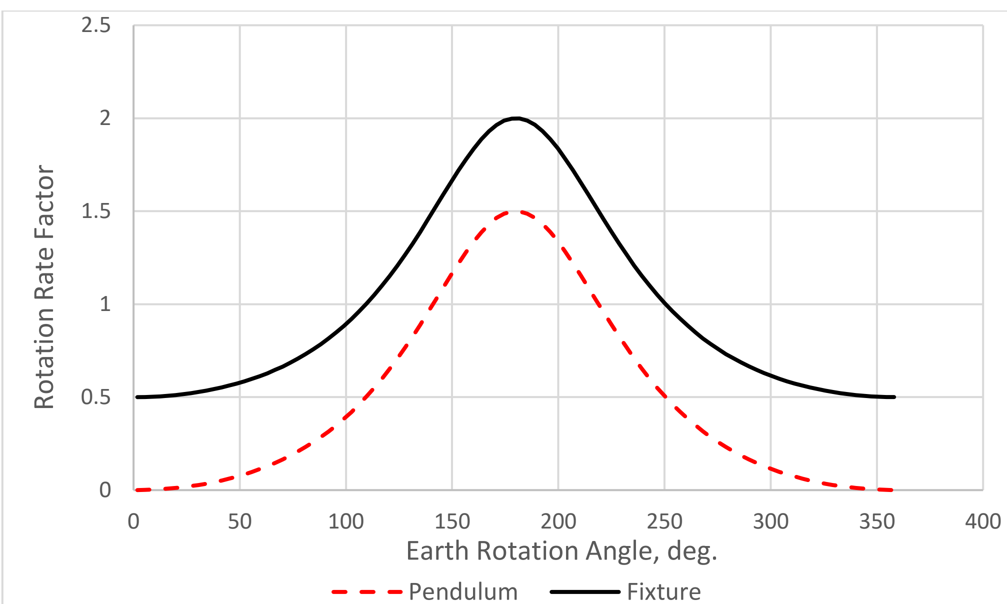

Fig. 5. Rotation rates in terms of the Earth rotation rate for 30 deg. latitude.

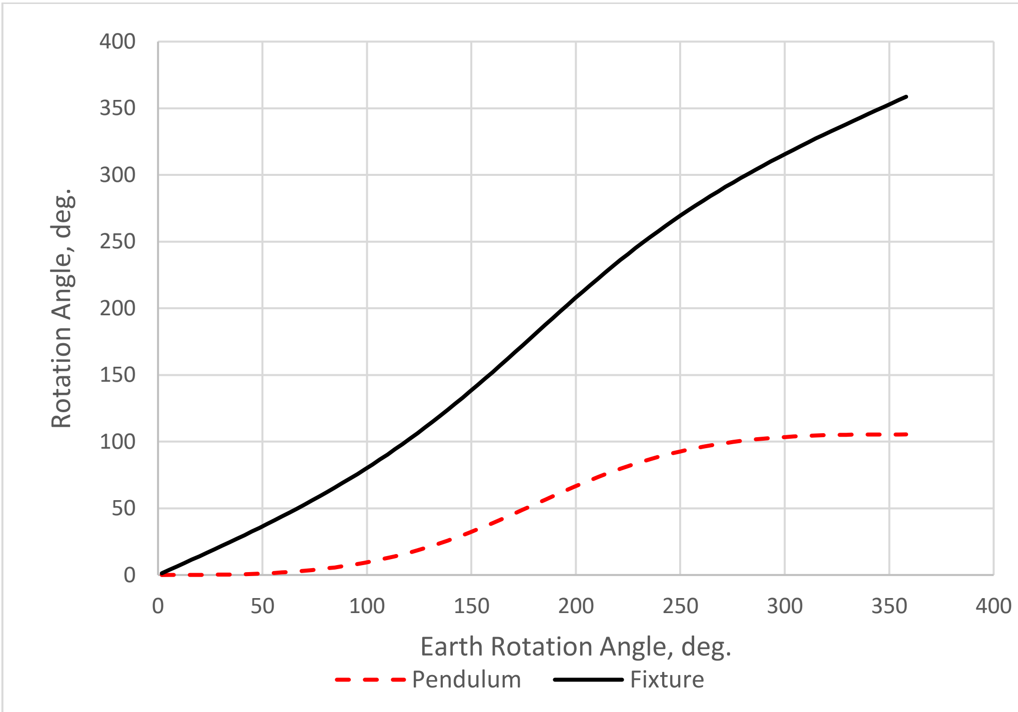

Fig. 6. Pendulum and fixture rotation angles versus the Earth rotation angle for 45 deg. latitude.

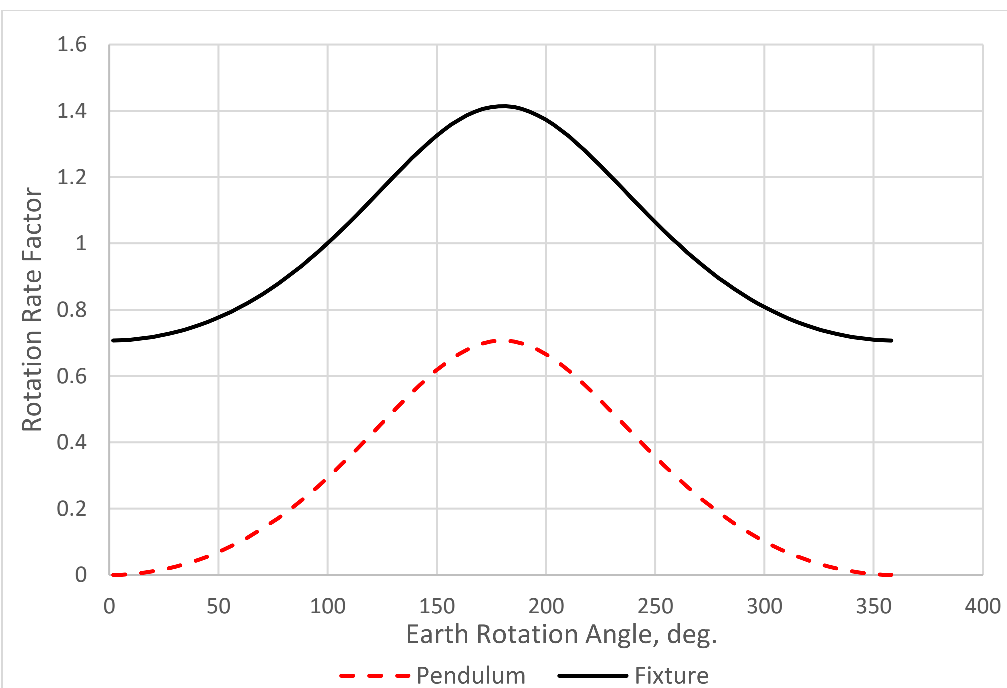

Fig. 7. Rotation rates in terms of the Earth rotation rate for 45 deg. latitude.

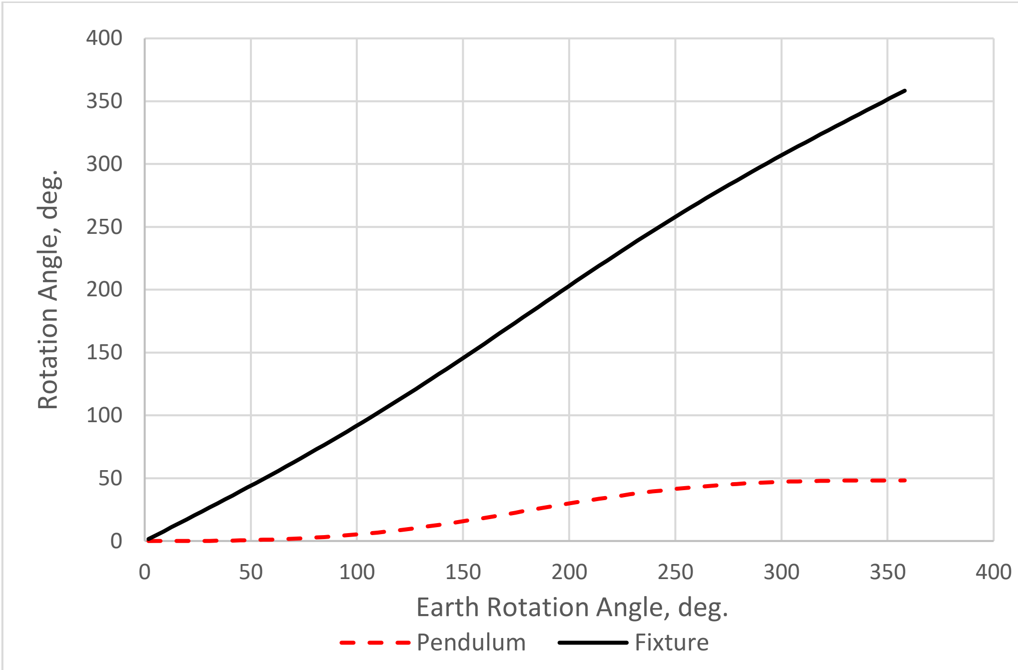

Fig. 8. Pendulum and fixture rotation angles versus the Earth rotation angle for 60 deg. latitude.

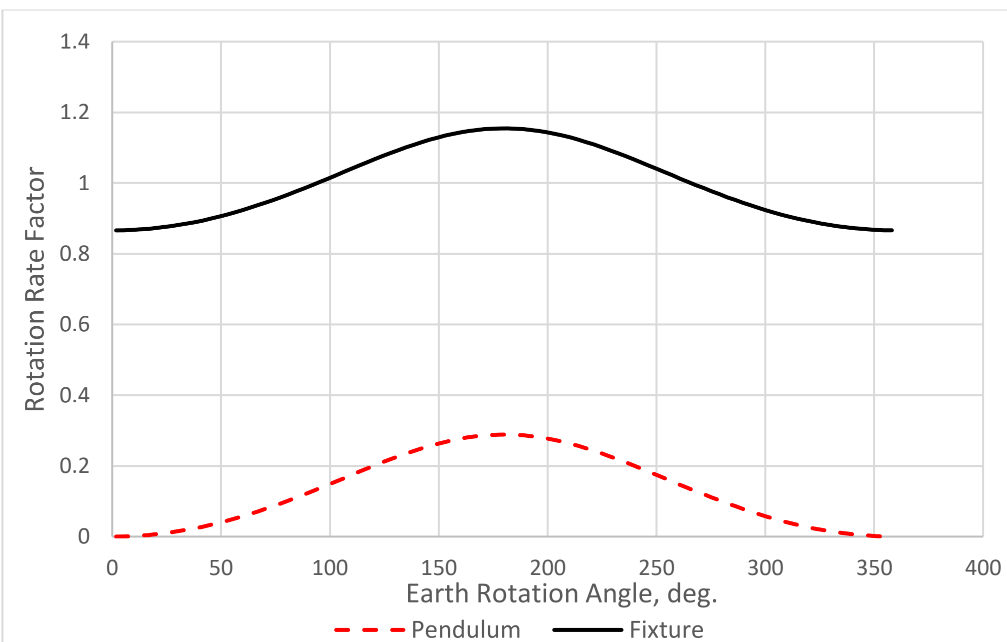

Fig. 9. Rotation rates in terms of the Earth rotation rate for 60 deg. latitude.

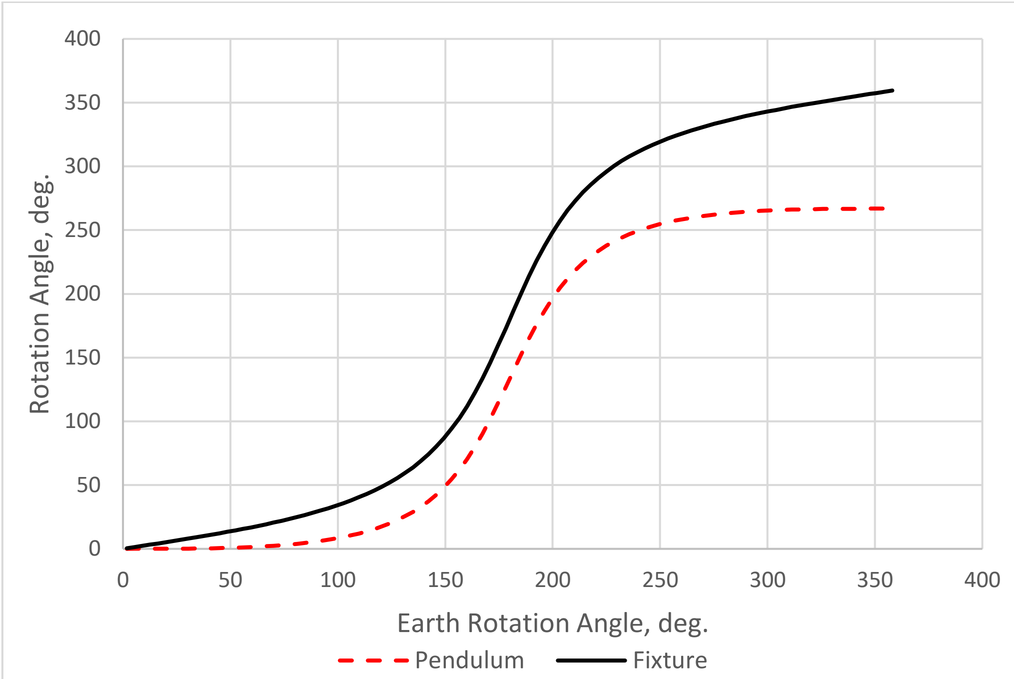

Fig. 10. Pendulum and fixture rotation angles versus the Earth rotation angle for 15 deg. latitude.

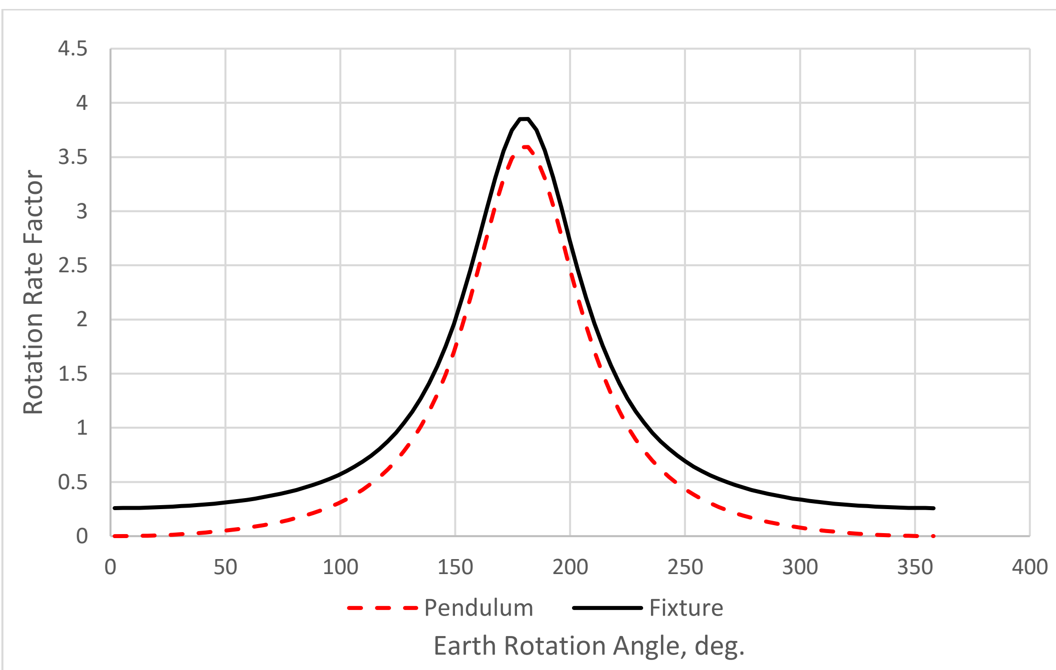

Fig. 11. Rotation rates in terms of the Earth rotation rate for 15 deg. latitude.

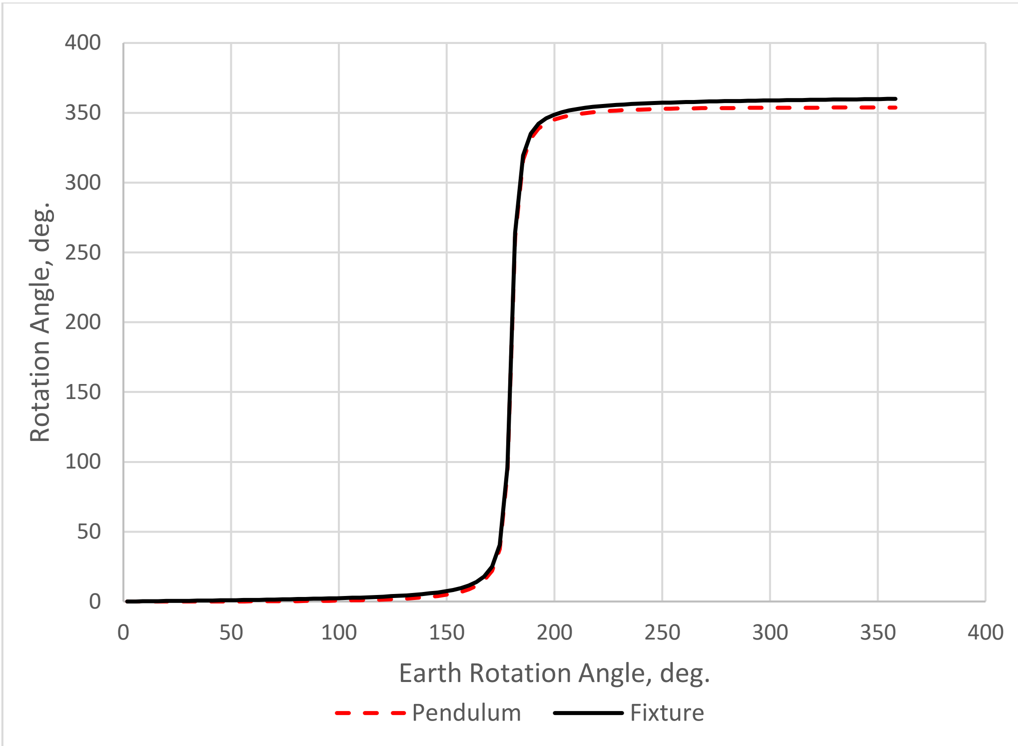

Fig. 12. Pendulum and fixture rotation angles versus the Earth rotation angle for 1 deg. latitude.

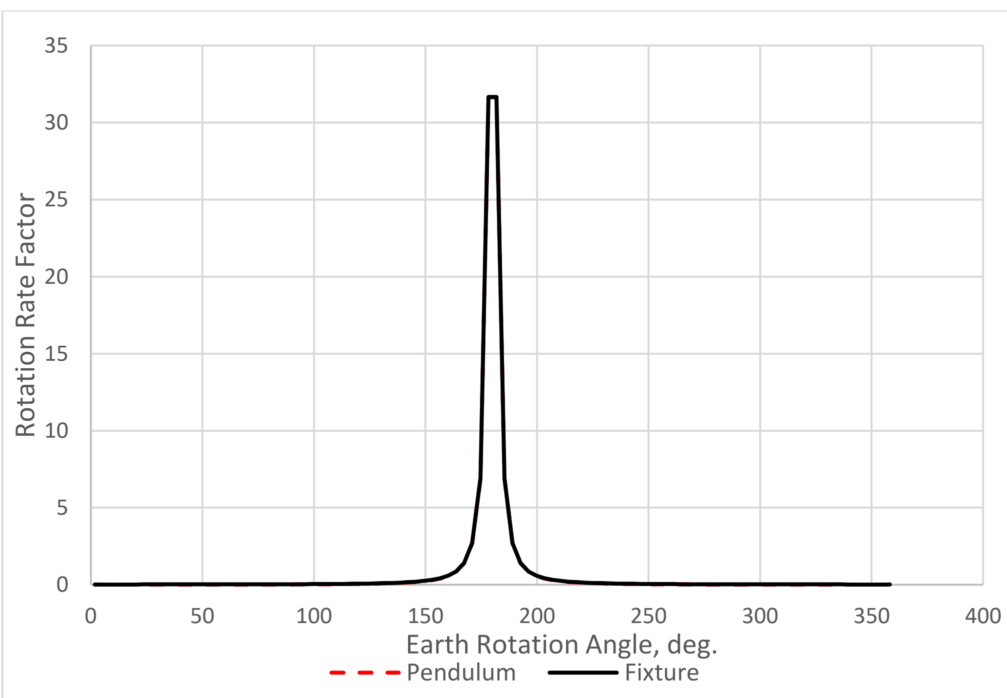

Fig. 13. Rotation rates in terms of the Earth rotation rate for 1 deg. latitude.

4. Conclusion

The analytical equation for the absolute rotation of the Foucault Pendulum about its vertical axis for arbitrary Earth rotation angle and latitude was derived. The rotation angle was found to be the solid angle enclosed by the actual trajectory of the vertical axis and the great circle arc trajectory linking the initial and final orientations of the axis. A new method to obtain the solid angle enabled the derivation of an analytical solution. A similar equation was derived for the pendulum's MF; however, the MF also includes a rotation given by the time integral of the Earth's rotation rate along the vertical axis. It was found that every complete rotation of the Earth causes the MF to rotate once about its vertical axis for any latitude location. Using the derivatives of the analytical equations, the rotation angle rate of both the pendulum and its MF were derived and found to be nonlinear functions of the Earth rotation angle. The difference in the rotation rates is in agreement with the rate of precession of the Foucault Pendulum when observed in the local reference frame. This work not only quantifies the rotation of the pendulum about its vertical axis but shows that attitude kinematics is the fundamental cause of the Foucault Pendulum's precession.

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