Critical Behavior in Continuous Dimensions and Early-Universe Cosmology

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Abstract

It is known that large-scale dynamical systems can sustain a rich variety of collective phenomena. This brief note argues that the cosmology of the early Universe can be viewed as critical behavior in continuous dimensions. We find that the self-similar properties of the metric near the Big Bang singularity are comparable to the effects produced by minimal fractality of spacetime far above the electroweak scale.

Key words: critical phenomena, metric oscillations, early Universe cosmology, gravitational singularity, minimal fractal spacetime.

According to [1], the behavior of the spatial metric $\gamma_{\alpha\beta}$ near the time singularity $t=0$ can be studied starting from
\[ \gamma_{\alpha \beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta \]  

(1)

where \( a^2, b^2, c^2 \) represent the diagonal elements of the matrix \( \gamma_{ab}(t) \) and \( l, m, n \) are unit vectors. Introducing the time-like variable \( \eta(t) \) divides the evolution of (1) into a couple of distinct regimes:

1) at large times \( \eta >> 1 \), the metric coefficients \( a \) and \( b \) oscillate, while the coefficient \( c \) varies exponentially according to

\[
\begin{align*}
a &= a_0 \sqrt{\frac{\eta}{\eta_0}} [1 + \frac{A}{\sqrt{\eta}} \sin (\eta - \eta_0)] \quad (2a) \\
b &= a_0 \sqrt{\frac{\eta}{\eta_0}} [1 - \frac{A}{\sqrt{\eta}} \sin (\eta - \eta_0)] \quad (2b) \\
c &= c_0 \exp[-A^2(\eta_0 - \eta)] \quad (2c)
\end{align*}
\]

in which \( A \) is a constant. As \( \eta \) falls off from \( \infty \) to about \( \eta \approx 1 \), the oscillations (2a) and (2b) occur with a slow reduction of their average values (\( O(\sqrt{\eta}) \)) and the functions \( a \) and \( b \) stay close in magnitude. On the other hand, the function (2c) is monotonically decreasing during all this time. Relations (2a)
– (2c) no longer apply as the parameter $\eta$ drops below 1 and shifts towards $\eta \ll 1$.

2) at ultrashort times ($\eta \ll 1$), metric coefficients and the original time variable $t$ evolve as power law functions, namely,

$$ a \propto \eta^{\frac{1+k}{2}} = \eta^{\beta_a(k)} \quad (3a) $$

$$ b \propto \eta^{\frac{1-k}{2}} = \eta^{\beta_b(k)} \quad (3b) $$

$$ c \propto \eta^{-\frac{1-k^2}{4}} = \eta^{\beta_c(k)} \quad (3c) $$

$$ t \propto \eta^{\frac{3+k^2}{4}} = \eta^{\beta_t(k)} \quad (3d) $$

where the arbitrary parameter $k$ lies in the interval $-1 < k < +1$. Using the notation

$$ h_i = (a, b, c); \quad i = 1, 2, 3 $$

renders (3a) - (3c) in the condensed form
Unlike the regime of $\eta \gg 1$ determined by (2a) - (2c), the coefficients $a$ and $b$ start to fall off while the magnitude of $c$ ramps up.

These considerations suggest that, passing from early times near the singularity ($t=0$) to far later times ($t \gg 0$), generates a transition from a Universe having a \textit{single space dimension} to a Universe with \textit{two space dimensions}. This behavior is consistent with the \textit{dimensional reduction conjecture} [2-3], according to which spacetime near the Big Bang singularity is effectively two dimensional, having one space and one time dimension only.

The power law relationships (3) and (4) bear a striking resemblance to the scaling of parameters in classical critical phenomena [4 – 5]. A textbook example of such phenomena is provided by spin systems approaching criticality in four spacetime dimensions ($d=4$), where the correlation length $\xi$ diverges with the reduced temperature $\tau$ as in

\[ h_i \propto \eta^{\beta_i(k)} \]  

(4)
Here, $\nu$ is a positive critical exponent and

$$\tau = \frac{T}{T_c} - 1$$

(6)

The overall magnetization $M$ of the system assumes the role of the order parameter and scales with $\tau$ according to

$$M \propto \tau^\beta$$

(7)

Here, the critical exponent $\beta$ also depends on the number of spacetime dimensions and on the critical exponent of the correlation function $\eta^*$, i.e.

$$\beta(d) = \frac{1}{2} \nu [d - 2 + \eta^*]$$

(8)

The perturbative treatment of the system is based on the dimensionless spin coupling constant

$$\bar{g}(d) = g \xi^{4-d}$$

(9)
It is seen from (9) that, near the phase transition at $T = T_c$, the correlation length diverges in less than four spacetime dimensions ($d < 4$) and the perturbative treatment breaks down. On the other hand, the perturbation analysis is enabled again when $d > 4$, as (9) is bounded to stay finite. Solving the tension between $d < 4$ and $d > 4$ stems from the so-called epsilon expansion method, whereby spacetime dimension flows in a continuous range of non-integer (fractal) values defined by [2]

$$d = 4 - \varepsilon$$ \hspace{1cm} (10)

These remarks indicate that there is a natural analogy between critical behavior of spin systems described by (5) - (10) and the scaling of metric coefficients described by (3) and (4). Replacing (10) in (8) yields

$$\beta(\varepsilon_i) = \frac{1}{2} \nu [2 - \varepsilon_i + \eta^*]$$ \hspace{1cm} (11)

where the dimensional deviation is taken to be coordinate dependent, that is, $\varepsilon_i = 4 - d_i$, with $i = 1, 2, 3$. 
In this context, a reasonable assumption is that the metric coefficients $h_i = (a, b, c)$ are analogs of the magnetization parameter (7), the time variable $\eta$ an analog of the reduced temperature (6), and the exponents entering (3) and (4) are analogs of (11). The side-by-side comparison is captured below,

$$h_i \Leftrightarrow M$$  \hspace{2cm} (12a)

$$\eta \Leftrightarrow \tau$$  \hspace{2cm} (12b)

$$\beta_i(k) \Leftrightarrow \beta(\varepsilon)$$  \hspace{2cm} (12c)

Furthermore, to make (11) compatible with both the $+$ and $-$ signs of (3a) - (3d), forces one to assume that, in the crossover region $\eta \rightarrow 1$, the metric oscillation regime (2a) – (2b) induces large variations of the correlation length (5) and its exponent $\nu$. A possible form of this expected behavior for $\nu$ is supplied by
\[ v(\varepsilon_i) = \begin{cases} > 0, \varepsilon_i > 0 \\ < 0, \varepsilon_i < 0 \end{cases} \]  \hfill (13)

It follows from (13) that (8) turns into

\[ \beta(\varepsilon_i) = \frac{1}{2} v(\varepsilon_i) [2 - \varepsilon_i + \eta] \]  \hfill (14)

Piecing everything together, relations (3) – (14) link the metric coefficients \( h_i \) to the dimensional deviations \( \varepsilon_i \), as summarized below

\[ \boxed{ h_i \propto \eta^{\beta_i(k)} \leftrightarrow M_i \propto \tau^{\beta(\varepsilon_i)} } \]  \hfill (15)

This is our main result. In closing, we note that the approach developed here is in alignment with the content of [6 – 10].

**References**


