On the Riemann Hypothesis

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Abstract

In this working paper I try to prove the Riemann hypothesis

let ζ the zeta function and η the diriklet function $\forall s \in \mathbb{C}$ $with\ Re(s) > 0$ $\eta(s) = \sum_{k=1}^{k=+\infty} \frac{(-1)^{k+1}}{k^s}$

We know that $\forall s \in \mathbb{C}$ $with\ \ Re(s) > 0$ $\left(1 - 2^{(1-s)}\right)\zeta(s) = \eta(s)$

Let $s = a + ib$ a complex number with $a, b \in \mathbb{R}$; $0 < a < 1$, $b \neq 0$ such that $\zeta(s) = 0$

We have also $\zeta(1 - s) = 0$

So $\eta(s) = 0$ (because $s \neq 1 + \frac{2k\pi i}{\ln 2}$, $k \in \mathbb{Z}$) and also $\eta(1 - s) = 0, \eta(\bar{s}) = 0\ and\ \eta(1 - \bar{s}) = 0$

Since $\eta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{x^{(s-1)}}{e^x + 1} = 0$ we have $\int_0^{+\infty} \frac{x^{(s-1)}}{e^x+1} = 0$ and also $\int_0^{+\infty} \frac{x^{(-s)}}{e^x+1} = 0$

an integration by substitution($x = e^t$) gives $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ and also $\int_{-\infty}^{+\infty} \frac{e^{(1-s)t}}{e^{e^t}+1} = 0$

Let the complex function f $\forall z \in \mathbb{C}$ $f(z) = \frac{e^{sz}}{e^{e^z}+1}$ f is meromorphic and poles of f are :

$z_{k,k'} = \ln(|2k + 1|\pi) + sgn(2k + 1)i\frac{\pi}{2} + i2k'\pi$ $\quad k, k' \in \mathbb{Z}$ where $sgn(2k + 1)$ is the sign of $(2k + 1)$

$z_{k,k'} = \ln((2k + 1)\pi) \pm i\frac{\pi}{2} + i2k'\pi$ $\quad k \in \mathbb{N}, k' \in \mathbb{Z}$

See that $Re\left(z_{k,k'}\right)$ is strictly positive

Let $n, m \in \mathbb{N}^*\ and\ \varepsilon \in \mathbb{R}\ with\ 0 < \varepsilon < \frac{1}{2}\ and\ A \in \mathbb{R}, A = A_n = \ln\left((2n + \varepsilon)\pi\right)$

Let $K_{(n,m)}$ the compact set in $\mathbb{C}$ (the rectangle)

$K_{(n,m)} = \{x + iy, x, y \in \mathbb{R}\ \ -m \leq x \leq A_n\ and\ 0 \leq y \leq 2\pi\}$

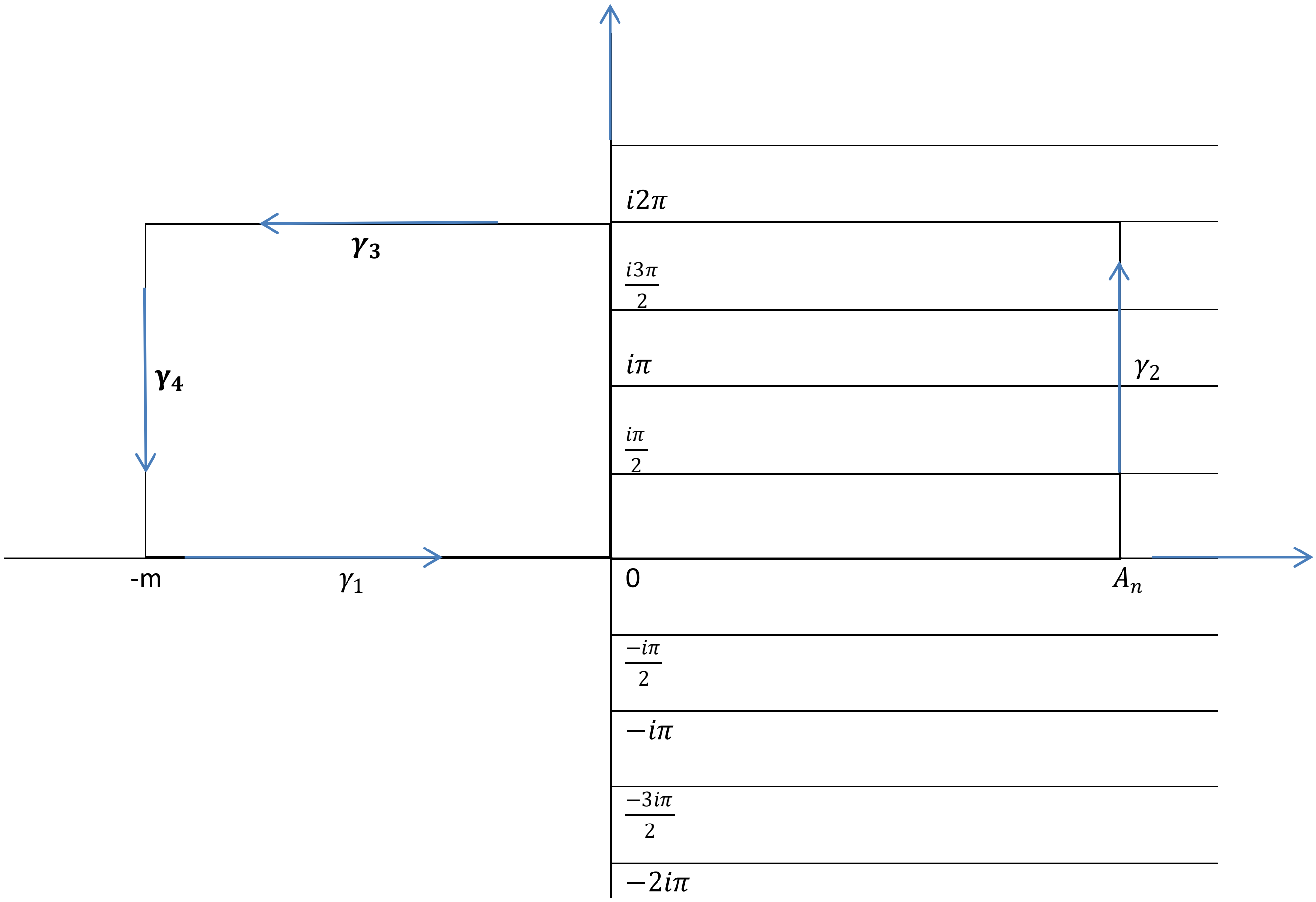
Poles of f in $K_{(n,m)}$ are

$z_k = ln((2k + 1)\pi) + i\frac{\pi}{2}$ and $z'_k = ln((2k + 1)\pi) + i\frac{3\pi}{2}$ $\quad 0 \leq k \leq (n - 1)$

(see the graph below)

The residu formula gives

$$\oint_{\partial K_{(n,m)}} f(z)dz = 2\pi i\left(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k)\right)$$


$i2\pi$
$\boldsymbol{\gamma_3}$
$\frac{i3\pi}{2}$
$i\pi$
$\boldsymbol{\gamma_4}$
γ_2
$\frac{i\pi}{2}$
-m
γ_1
0
A_n
$\frac{-i\pi}{2}$
$-i\pi$
$\frac{-3i\pi}{2}$
$-2i\pi$

$\oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz + \oint_{\gamma_4} f(z)dz = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$

$\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt - \int_{-m}^{A} \frac{e^{s(t+2\pi i)}}{e^{e^{(t+2\pi i)}}+1}dt - i\int_0^{2\pi} \frac{e^{s(it-m)}}{e^{e^{(it-m)}}+1}dt$

$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$

$$\left(1-e^{s2\pi i}\right)\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt - ie^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1}dt \quad (1)$$

$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$

Let's calculate $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1}dt$

$\forall\, z \in \mathbb{C}\, with\, |z| \le 1\; |e^z+1| \ne 0$ so the function$z \to |e^z+1|$ has aminima $p > 0$ On the cmopact

$\{z \in \mathbb{C}\, with\, |z| \le 1\}$

So $\forall\, z \in \mathbb{C}\, with\, |z| \le 1\; |e^z+1| \ge p$

$\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|e^{(it-m)}\right| = e^{(-m)} \le 1$ so $\left|e^{e^{(it-m)}}+1\right| \ge p$

So $\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|\frac{e^{sit}}{e^{e^{(it-m)}}+1}\right| \le \frac{e^{-bt}}{p}$

Since$\int_0^{2\pi} e^{-bt}du < \infty$ So $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1}dt = 0$

When m tends to$+\infty$ the equation (1) becomes

$\left(1-e^{s2\pi i}\right)\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$

Since $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ we have $\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1}dt = -\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt$

$-\left(1-e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f,z_k) + \sum_{k=0}^{(n-1)} Res(f,z'_k))$

Let's calculate $\sum_{k=0}^{(n-1)} Res(f,z_k)$

$Res(f,z_k) = \frac{e^{sz_k}}{e^{e^{z_k}} \times e^{z_k}} = \frac{e^{sz_k}}{(-1)\times e^{z_k}} = -e^{(s-1)z_k} = -e^{(s-1)\left(ln((2k+1)\pi)+i\frac{\pi}{2}\right)} = -\pi^{(s-1)}e^{(s-1)i\frac{\pi}{2}} \times \frac{1}{(2k+1)^{(1-s)}}$

$\sum_{k=0}^{(n-1)} Res(f,z_k) = -\pi^{(s-1)}e^{(s-1)i\frac{\pi}{2}}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

By the same we have

$\sum_{k=0}^{(n-1)} Res(f,z'_k) = -\pi^{(s-1)}e^{(s-1)i\frac{3\pi}{2}}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

So

$-\left(1-e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt = -2i\pi^s(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}})\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

$-\left(1-e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt = -2i\pi^s(-ie^{si\frac{\pi}{2}} + ie^{si\frac{3\pi}{2}})\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\int_0^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=-2\pi^s\left(e^{si\frac{\pi}{2}}-e^{si\frac{3\pi}{2}}\right)\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} \qquad (2)$$

$$\int_0^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_{\pi}^{2\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(i(t+\pi)+A)}}{e^{e^{(i(t+\pi)+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{-e^{(it+A)}}+1}dt=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}\frac{e^{s(it+i\pi+A)}e^{e^{(it+A)}}}{e^{e^{(it+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\int_0^{\pi}e^{s(it+i\pi+A)}dt-\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{e^{(it+A)}}+1}dt$$

$$=\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt-\int_0^{\pi}\frac{e^{s(it+i\pi+A)}}{e^{e^{(it+A)}}+1}dt+\frac{1}{si}\left[e^{s(it+i\pi+A)}\right]_0^{\pi}=\left(1-e^{si\pi}\right)\int_0^{\pi}\frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1}dt+\frac{1}{si}\left(e^{s(i2\pi+A)}-e^{s(i\pi+A)}\right)$$

$$=\left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt+\frac{1}{si}e^{sA}\left(e^{si2\pi}-e^{si\pi}\right)$$

So equality (2) becomes

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt+\frac{1}{s}e^{sA}\left(e^{si2\pi}-e^{si\pi}\right)$$

$$=-2\pi^s\left(e^{si\frac{\pi}{2}}-e^{si\frac{3\pi}{2}}\right)\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}}{e^{ie^{(it+A)}}+1}dt-\frac{1}{s}e^{sA}e^{\frac{si\pi}{2}}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}} \qquad (3)$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{sit}}{e^{ie^{(it+A)}}+1}dt=\int_{\frac{-\pi}{2}}^{0}\frac{e^{sit}}{e^{\left(e^{A}(-\sin t+i\cos t)\right)}+1}dt+\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(-\sin t+i\cos t)\right)}+1}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{\left(e^{A}(\sin t+i\cos t)\right)}+1}dt+\int_0^{\frac{\pi}{2}}\frac{e^{sit}e^{\left(e^{A}(\sin t-i\cos t)\right)}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{\left(e^{A}(\sin t+i\cos t)\right)}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt+\int_0^{\frac{\pi}{2}}e^{sit}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{\left(e^{A}(\sin t+i\cos t)\right)}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt+\frac{1}{si}\left(e^{\frac{si\pi}{2}}-1\right)$$

So equality (3) becomes

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{\left(e^{A}(\sin t+i\cos t)\right)}+1}dt-ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt+\frac{ie^{sA}}{si}\left(e^{\frac{si\pi}{2}}-1\right)-\frac{1}{s}e^{sA}e^{\frac{si\pi}{2}}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{(-sit)}}{e^{\left(e^{A}(\sin t+i\cos t)\right)}+1}dt-ie^{sA}\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt-\frac{1}{s}e^{sA}$$

$$=-2\pi^s\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

We multiply by $e^{(3-s)A}$ we get

$$-\left(1+e^{s\pi i}\right)e^{\frac{-si\pi}{2}}e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt+ie^{3A}\left(\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^A(\sin t+i\cos t))}+1}dt-\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)-\frac{1}{s}e^{3A}$$

$$=-2\pi^s e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

Let's calculate $\lim_{n\to+\infty} e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt$

$$\left|\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt\right|\le\int_A^{+\infty}\left|\frac{e^{st}}{e^{e^t}+1}\right|dt=\int_A^{+\infty}\frac{e^{at}}{e^{e^t}+1}dt\le\int_A^{+\infty}\frac{e^{at}}{e^{e^t}}dt\le\frac{1}{e^{\left(\frac{1}{2}e^A\right)}}\int_A^{+\infty}\frac{e^{at}}{e^{\left(\frac{1}{2}e^t\right)}}dt \qquad \frac{1}{e^{e^t}}=\frac{1}{e^{\left(\frac{1}{2}e^t\right)}}\times\frac{1}{e^{\left(\frac{1}{2}e^t\right)}}$$

So $\left|e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt\right|\le\frac{e^{(3-a)A}}{e^{\left(\frac{1}{2}e^A\right)}}\int_A^{+\infty}\frac{e^{at}}{e^{\left(\frac{1}{2}e^t\right)}}dt$

Clearly $\lim_{n\to+\infty} e^{(3-s)A}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt=0$

so

$$-2\pi^s e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)-\frac{1}{s}e^{3A}+o(1) \qquad (4)$$

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(\frac{1}{2}(-s^2+3s-2))t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}-e^{it}-(s-1)\left(e^{2it}-e^{it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

lemma :

$\lim_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt=0$ and $\lim_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t-i\cos t))}+1}\right|dt=0$

proof

$$e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt=e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt+e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{e^A(\sin t+i\cos t)}+1}\right|dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{\left|e^{(e^A(\sin t+i\cos t))}+1\right|}dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{\left|e^{(e^A(\sin t+i\cos t))}\right|-1}dt\le\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{t^3}{e^{(e^A\sin t)}-1}dt$$

$$\le\frac{1}{e^{\left(\frac{\sqrt{2}e^A}{2}\right)}-1}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}t^3dt$$

$$e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt\le\frac{e^{3A}}{e^{\left(\frac{\sqrt{2}e^A}{2}\right)}-1}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}t^3dt$$

So $\lim_{n\to+\infty} e^{3A}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left|\frac{t^3}{e^{(e^A(\sin t+i\cos t))}+1}\right|dt=0$

$$\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt \le \int_0^{\frac{\pi}{4}}\frac{t^3}{\left|e^{\left(e^A(\sin t+i\cos t)\right)}\right|-1}dt \le \int_0^{\frac{\pi}{4}}\frac{t^3}{e^{e^A\sin t}-1}dt = \int_0^{\frac{\sqrt{2}}{2}}\frac{(\arcsin(u))^3}{e^{(e^A u)}-1}\times\frac{1}{\sqrt{1-u^2}}du$$

By substitution $(t=\arcsin(u))$

We have $\forall u\in\left[0,\frac{\sqrt{2}}{2}\right]\ \frac{1}{\sqrt{1-u^2}}\le\sqrt{2}$ so

$$\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt \le \sqrt{2}\int_0^{\frac{\sqrt{2}}{2}}\frac{(\arcsin(u))^3}{e^{(e^A u)}-1}du$$

The taylor formula with gives

$\forall u\in\left[0,\frac{\sqrt{2}}{2}\right]\ \arcsin(u)=\arcsin(0)+u\times\frac{1}{\sqrt{1-(\xi_u)^2}}$ where $\xi_u\in\,]0,u[$

So $\forall u\in\left[0,\frac{\sqrt{2}}{2}\right]\ \arcsin(u)\le u\sqrt{2}$ (because $\frac{1}{\sqrt{1-(\xi_u)^2}}\le\sqrt{2}$)

So $\forall u\in\left[0,\frac{\sqrt{2}}{2}\right](\arcsin(u))^3\le 2\sqrt{2}u^3$

So $\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt \le 4\int_0^{\frac{\sqrt{2}}{2}}\frac{u^3}{e^{(e^A u)}-1}du = 4e^{-4A}\int_0^{\frac{e^A\sqrt{2}}{2}}\frac{v^3}{e^v-1}dv \le 4e^{-4A}\int_0^{+\infty}\frac{v^3}{e^v-1}dv$

(By substitution $e^A u=v$)

So $e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt \le 4e^{-A}\int_0^{+\infty}\frac{v^3}{e^v-1}dv$

Since $\int_0^{+\infty}\frac{v^3}{e^v-1}dv<\infty$ we get $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{4}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt=0$

So $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt=0$

We deduce that $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}\right|dt=0$

Let's prove that $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt=0$ and
$\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{sit}-e^{it}-(s-1)\left(e^{2it}-e^{it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt=0$

The taylor formula with integral gives

$\exists M\in\mathbb{R}^{*+}\ \forall t\in\left[0,\frac{\pi}{2}\right]\quad\left|e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2\right|\le Mt^3$

So $\left|\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt\right|\le M\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt$

So $\left|e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt\right|\le Me^{3A}\int_0^{\frac{\pi}{2}}\left|\frac{t^3}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}\right|dt$

Using the lemma we get $\lim_{n\to+\infty}\left|e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt\right|=0$

So $\lim_{n\to+\infty}e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-sit}-e^{-it}-(s-1)\left(e^{-2it}-e^{-it}\right)-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^A(\sin t+i\cos t)\right)}+1}dt=0$

By the same we get $\lim_{n\to+\infty} e^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{sit}-e^{it}-(s-1)(e^{2it}-e^{it})-\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt=0$

We deduce that

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(\frac{1}{2}(-s^2+3s-2))t^2}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+o(1) \qquad (5)$$

We deduce from equality (4)

$$-2\pi^{s}e^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}=e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{-2it}-e^{-it})}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(s-1)(e^{2it}-e^{it})}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)+e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(\frac{1}{2}(-s^2+3s-2))t^2}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}(-s^2+3s-2)t^2}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$+o(1)$$

Let the complex function h $\quad \forall z\in\mathbb{C}\ \ h(z)=\frac{e^{qz}}{e^{e^{z}}+1}\quad q\in\mathbb{N}^*$

The residu formula on $K_{(n,m)}$ gives

$$\left(1-e^{q2\pi i}\right)\int_{-m}^{A}\frac{e^{qt}}{e^{e^{t}}+1}dt+i\int_0^{2\pi}\frac{e^{q(it+A)}}{e^{e^{(it+A)}}+1}dt-ie^{-sm}\int_0^{2\pi}\frac{e^{qit}}{e^{e^{(it-m)}}+1}dt$$

$$=2\pi i\left(\sum_{k=0}^{(n-1)}Res(h,z_k)+\sum_{k=0}^{(n-1)}Res(h,z'_k)\right)$$

When m tends to$+\infty$ we get

$$i\int_0^{2\pi}\frac{e^{q(it+A)}}{e^{e^{(it+A)}}+1}dt==-2\pi^{q}\left(e^{qi\frac{\pi}{2}}-e^{qi\frac{3\pi}{2}}\right)\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

By the same as above we have

$$i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{q(it+A)}}{e^{ie^{(it+A)}}+1}dt-\frac{1}{q}e^{qA}e^{\frac{qi\pi}{2}}=-2\pi^{q}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

$$i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{q(it+A)}}{e^{ie^{(it+A)}}+1}dt=-2\pi^{q}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}+\frac{1}{q}e^{qA}e^{\frac{qi\pi}{2}}$$

$$ie^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{-qit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-ie^{3A}\int_0^{\frac{\pi}{2}}\frac{e^{qit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt-\frac{1}{q}e^{3A}$$

$$=-2\pi^{q}e^{(3-q)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}$$

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-qit}}{e^{(e^{A}(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{qit}}{e^{\left(e^{A}(\sin t-i\cos t)\right)}+1}dt\right)$$

$$=-2\pi^{q}e^{(3-q)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-q)}}+\frac{1}{q}e^{3A} \qquad (6)$$

Let's calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$

Using equality (6) for $q=1$ we get

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=-2\pi e^{2A}n+e^{3A}=-\pi e^{2A}(2n+\varepsilon-\varepsilon)+e^{3A}$$

$$=-\pi e^{2A}(2n+\varepsilon)+\pi\varepsilon e^{2A}+e^{3A}=-e^{3A}+\pi\varepsilon e^{2A}+e^{3A}=\pi\varepsilon e^{2A}$$

So $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=\pi\varepsilon e^{2A}$

Let's calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(e^{-2it}-e^{-it})}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$

Using equality (6) for $q=2$ we get

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-2it}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{2it}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$$

$$=-2\pi^2e^A\sum_{k=0}^{(n-1)}(2k+1)+\frac{1}{2}e^{3A}$$

$$=-2\pi^2e^An^2+\frac{1}{2}e^{3A}$$

So

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{(e^{-2it}-e^{-it})}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{(e^{2it}-e^{it})}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=-2\pi^2e^An^2+\frac{1}{2}e^{3A}-\pi\varepsilon e^{2A}$$

$$=\frac{1}{2}e^A(-4\pi^2n^2+e^{2A}-2\pi\varepsilon e^A\ )=-\frac{1}{2}e^A\left(-4\pi^2n^2+\left((2n+\varepsilon)\pi\right)^2-2\pi\varepsilon(2n+\varepsilon)\pi\right)$$

$$=\frac{1}{2}e^A\pi^2\left(-4n^2+(2n+\varepsilon)^2-2\varepsilon(2n+\varepsilon)\right)=\frac{1}{2}e^A\pi^2(-4n^2+4n^2+4\varepsilon n+\varepsilon^2-4\varepsilon n-2\varepsilon^2)=-\frac{1}{2}\pi^2\varepsilon^2e^A$$

By the same we can calculate $e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)$ we find

$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{t^2}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=C+o(1)$ where C is constante depending only on ε

(By using the equation (5) wich is also true for $q\in\mathbb{N}^*$ we can take for example $q=3$ there is a lot of calculus)

So equality (5) becomes

$$e^{3A}\left(i\int_0^{\frac{\pi}{2}}\frac{e^{-sit}}{e^{(e^A(\sin t+i\cos t))}+1}dt-i\int_0^{\frac{\pi}{2}}\frac{e^{sit}}{e^{\left(e^A(\sin t-i\cos t)\right)}+1}dt\right)=\pi\varepsilon e^{2A}-\frac{1}{2}(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}(-s^2+3s-2)C+o(1)$$

Thus equality (4) gives

$$-2\pi^se^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}$$

$$=\pi\varepsilon e^{2A}-\frac{1}{2}(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}(-s^2+3s-2)C-\frac{1}{s}e^{3A}+o(1)$$

$$2\pi^sse^{(3-s)A}\sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}=e^{3A}-\pi\varepsilon se^{2A}+\frac{1}{2}s(s-1)\pi^2\varepsilon^2e^A+\frac{1}{2}s(s^2-3s+2)C+o(1)$$

Let $C'(s) = \frac{1}{2}s(s^2 - 3s + 2)C$ so

$$2\pi^s se^{(3-s)A} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} = e^{3A} - \pi\varepsilon se^{2A} + \frac{1}{2}s(s-1)\pi^2\varepsilon^2 e^A + C'(s) + o(1)$$

$$2\pi^s s\big((2n+\varepsilon)\pi\big)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

$$= \big((2n+\varepsilon)\pi\big)^3 - \pi\varepsilon s\big((2n+\varepsilon)\pi\big)^2 + \frac{1}{2}s(s-1)\pi^2\varepsilon^2(2n+\varepsilon)\pi + C'(s) + o(1)$$

$$2s(2n+\varepsilon)^{(3-s)} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}} = (2n+\varepsilon)^3 - \varepsilon s(2n+\varepsilon)^2 + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon) + C'(s) + o(1) \qquad (7)$$

We have also

$$2s \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$$

$$= (2n+\varepsilon)^s - \varepsilon s(2n+\varepsilon)^{(s-1)} + \frac{1}{2}s(s-1)\varepsilon^2(2n+\varepsilon)^{(s-2)} + C'(s)(2n+\varepsilon)^{(s-3)} + o\big((2n+\varepsilon)^{(s-3)}\big) \qquad (8)$$