On the Riemann Hypothesis

Abdelkrim ben Mohamed

Abstract

In this paper we try to disprove the Riemann hypothesis

Part1

let ζ the zeta function and η the diriklet function $\forall s \in \mathbb{C}$ $with\ Re(s) > 0$ $\eta(s) = \sum_{k=1}^{k=+\infty} \frac{(-1)^{k+1}}{k^s}$

We know that $\forall s \in \mathbb{C}$ $with\ Re(s) > 0$ $\left(1 - 2^{(1-s)}\right)\zeta(s) = \eta(s)$

Let $s = a + ib$ a complex number with $a, b \in \mathbb{R}$; $0 < a < 1$, $b \neq 0$ such that $\zeta(s) = 0$

We have also $\zeta(1-s) = 0$

So $\eta(s) = 0$ (because $s \neq 1 + \frac{2k\pi i}{\ln 2}$, $k \in \mathbb{Z}$) and also $\eta(1-s) = 0, \eta(\bar{s}) = 0\ and\ \eta(1-\bar{s}) = 0$

Since $\eta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{x^{(s-1)}}{e^x+1} = 0$ we have $\int_0^{+\infty} \frac{x^{(s-1)}}{e^x+1} = 0$ and also $\int_0^{+\infty} \frac{x^{(-s)}}{e^x+1} = 0$

an integration by substitution($x = e^t$) gives $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ and also $\int_{-\infty}^{+\infty} \frac{e^{(1-s)t}}{e^{e^t}+1} = 0$

Let the complex function f $\forall z \in \mathbb{C}$ $f(z) = \frac{e^{sz}}{e^{e^z}+1}$ f is meromorphic and poles of f are :

$z_{k,k'} = \ln(|2k+1|\pi) + sgn(2k+1)i\frac{\pi}{2} + i2k'\pi$ $\quad k, k' \in \mathbb{Z}$ where $sgn(2k+1)$ is the sign of $(2k+1)$

$z_{k,k'} = \ln((2k+1)\pi) \pm i\frac{\pi}{2} + i2k'\pi$ $\quad k \in \mathbb{N}, k' \in \mathbb{Z}$

See that $Re\left(z_{k,k'}\right)$ is strictly positive

Let $n, m \in \mathbb{N}^*$ $and\ \varepsilon \in \mathbb{R}\ with\ 0 < \varepsilon < \frac{1}{2}\ and\ A \in \mathbb{R}, A = A_n = \ln\left((2n+\varepsilon)\pi\right)$

Let $K_{(n,m)}$ the compact set in $\mathbb{C}$ (the rectangle)

$K_{(n,m)} = \{x + iy, x, y \in \mathbb{R}\ -m \leq x \leq A_n\ and\ 0 \leq y \leq 2\pi\}$

Poles of f in $K_{(n,m)}$ are

$z_k = ln((2k+1)\pi) + i\frac{\pi}{2}$ and $z'_k = ln((2k+1)\pi) + i\frac{3\pi}{2}$ $\quad 0 \leq k \leq (n-1)$

(see the graph below)

The residu formula gives

$\oint_{\partial K_{(n,m)}} f(z)dz = 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

$\oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \oint_{\gamma_3} f(z)dz + \oint_{\gamma_4} f(z)dz = 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

$\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt - \int_{-m}^{A} \frac{e^{s(t+2\pi i)}}{e^{e^{(t+2\pi i)}}+1} dt - i\int_0^{2\pi} \frac{e^{s(it-m)}}{e^{e^{(it-m)}}+1} dt$

$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

$\left(1 - e^{s2\pi i}\right)\int_{-m}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt - ie^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt$ (1)

$= 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

Let's calculate $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt$

$\forall\, z \in \mathbb{C}\, with\, |z| \le 1\; |e^z + 1| \ne 0$ so the function$z \to |e^z + 1|$ has aminima $p > 0$ On the cmopact

$\{z \in \mathbb{C}\, with\, |z| \le 1\}$

So $\forall\, z \in \mathbb{C}\, with\, |z| \le 1\; |e^z + 1| \ge p$

$\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|e^{(it-m)}\right| = e^{(-m)} \le 1$ so $\left|e^{e^{(it-m)}} + 1\right| \ge p$

So $\forall m \in \mathbb{N}^*\;\; \forall\, t \in [0,2\pi]\;\; \left|\frac{e^{sit}}{e^{e^{(it-m)}}+1}\right| \le \frac{e^{-bt}}{p}$

Since$\int_0^{2\pi} e^{-bt} du < \infty$ So $\lim_{m\to+\infty} e^{-sm}\int_0^{2\pi} \frac{e^{sit}}{e^{e^{(it-m)}}+1} dt = 0$

When m tends to$+\infty$ the equation (1) becomes

$\left(1 - e^{s2\pi i}\right)\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

Since $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ we have $\int_{-\infty}^{A} \frac{e^{st}}{e^{e^t}+1} dt = -\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt$

$-\left(1 - e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = 2\pi i(\sum_{k=0}^{(n-1)} Res(f, z_k) + \sum_{k=0}^{(n-1)} Res(f, z'_k))$

Let's calculate $\sum_{k=0}^{(n-1)} Res(f, z_k)$

$Res(f, z_k) = \frac{e^{sz_k}}{e^{e^{z_k}} \times e^{z_k}} = \frac{e^{sz_k}}{(-1)\times e^{z_k}} = -e^{(s-1)z_k} = -e^{(s-1)\left(ln\left((2k+1)\pi\right)+i\frac{\pi}{2}\right)} = -\pi^{(s-1)}e^{(s-1)i\frac{\pi}{2}} \times \frac{1}{(2k+1)^{(1-s)}}$

$\sum_{k=0}^{(n-1)} Res(f, z_k) = -\pi^{(s-1)}e^{(s-1)i\frac{\pi}{2}}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

By the same we have

$\sum_{k=0}^{(n-1)} Res(f, z'_k) = -\pi^{(s-1)}e^{(s-1)i\frac{3\pi}{2}}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$

So

$-\left(1 - e^{s2\pi i}\right)\int_{A}^{+\infty} \frac{e^{st}}{e^{e^t}+1} dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}+1} dt = -2i\pi^s(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}})\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)^{(1-s)}}$ (2)

Let the complex function g $\forall z \in \mathbb{C}$ $g(z) = \frac{e^{sz}}{e^{e^z}-1}$ g is meromorphic and poles of g are :

$$z_{k,k'} = \ln(|2k|\pi) + sgn(k)i\frac{\pi}{2} + i2k'\pi \qquad k,k' \in \mathbb{Z}\,,\ k \neq 0$$

$$z_{k,k'} = \ln(2k\pi) \pm i\frac{\pi}{2} + i2k'\pi \qquad k \in \mathbb{N}, k' \in \mathbb{Z}\ , k \neq 0$$

Let H_{A_n} the compact set in $\mathbb{C}$ (the rectangle) $H_{A_n} = \{x+iy, x,y \in \mathbb{R} \quad 0 \leq x \leq A_n \ and\ 0 \leq y \leq 2\pi\}$

Poles of g in H_{A_n} are

$z_k = ln((2k)\pi) + i\frac{\pi}{2}$ and $z'_k = ln((2k)\pi) + i\frac{3\pi}{2}$ $\quad 1 \leq k \leq n$

By the same way the residu formula on H_{A_n} gives

$$\left(1-e^{s2\pi i}\right)\int_0^A \frac{e^{st}}{e^{e^t}-1}dt + i\int_0^{2\pi} \frac{e^{s(it+A)}}{e^{e^{(it+A)}}-1}dt - i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt \qquad (3)$$

$$= 2i\pi^s\left(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}\right)\sum_{k=1}^{n}\frac{1}{(2k)^{(1-s)}}$$

Adding the equalities (2) and (3) we get

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt + \left(1-e^{s2\pi i}\right)\int_0^A \frac{e^{st}}{e^{e^t}-1}dt + 2i\int_0^{2\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt - i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt$$

$$= 2i\pi^s\left(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}\right)\left(\sum_{k=1}^{n}\frac{1}{(2k)^{(1-s)}} - \sum_{k=0}^{(n-1)}\frac{1}{(2k+1)^{(1-s)}}\right)$$

$$= 2i\pi^s\left(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}\right)\left(\sum_{k=1}^{n}\frac{1}{(2k)^{(1-s)}} - \sum_{k=1}^{n}\frac{1}{(2k-1)^{(1-s)}}\right)$$

$$= 2i\pi^s\left(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}\right)\left(\sum_{k=1}^{n}\frac{(-1)^{2k}}{(2k)^{(1-s)}} + \sum_{k=1}^{n}\frac{(-1)^{(2k-1)}}{(2k-1)^{(1-s)}}\right)$$

$$= 2i\pi^s\left(e^{(s-1)i\frac{\pi}{2}} + e^{(s-1)i\frac{3\pi}{2}}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$$

$$= 2i\pi^s\left(-ie^{si\frac{\pi}{2}} + ie^{si\frac{3\pi}{2}}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$$

$$= 2\pi^s\left(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$$

So

$$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt + \left(1-e^{s2\pi i}\right)\int_0^A \frac{e^{st}}{e^{e^t}-1}dt + 2i\int_0^{2\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt - i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt \qquad (4)$$

$$= 2\pi^s\left(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$$

We have $\int_0^A \frac{e^{st}}{e^{e^t}-1}dt - \int_0^A \frac{e^{st}}{e^{e^t}+1}dt = 2\int_0^A \frac{e^{st}}{e^{2e^t}-1}dt = 2\int_0^A \frac{e^{st}}{e^{e^{(t+\ln 2)}}-1}dt = 2e^{(-s\ln 2)}\int_{\ln 2}^{(A+\ln 2)}\frac{e^{su}}{e^{e^u}-1}du$

$= \frac{1}{2^{(s-1)}}\int_{\ln 2}^{(A+\ln 2)}\frac{e^{su}}{e^{e^u}-1}du$ (by substitution $t+\ln 2 = u$)

So $2^{(s-1)}\int_0^A \frac{e^{st}}{e^{e^t}-1}dt - 2^{(s-1)}\int_0^A \frac{e^{st}}{e^{e^t}+1}dt = \int_{ln\,2}^{(A+ln\,2)}\frac{e^{su}}{e^{e^u}-1}du$

$$2^{(s-1)}\int_0^A \frac{e^{st}}{e^{e^t}-1}dt - 2^{(s-1)}\int_0^A \frac{e^{st}}{e^{e^t}+1}dt = \int_{ln\,2}^{0}\frac{e^{su}}{e^{e^u}-1}du + \int_0^A \frac{e^{su}}{e^{e^u}-1}du + \int_A^{(A+ln\,2)}\frac{e^{su}}{e^{e^u}-1}du$$

So $(2^{(s-1)}-1)\int_0^A \frac{e^{st}}{e^{e^t}-1}dt = 2^{(s-1)}\int_0^A \frac{e^{st}}{e^{e^t}+1}dt + \int_{\ln 2}^0 \frac{e^{su}}{e^{e^u}-1}du + \int_A^{(A+ln\,2)} \frac{e^{su}}{e^{e^u}-1}du$

So $\int_0^A \frac{e^{st}}{e^{e^t}-1}dt = \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\int_0^A \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du + \frac{1}{(2^{(s-1)}-1)}\int_A^{(A+ln\,2)} \frac{e^{st}}{e^{e^t}-1}dt$

Since $\int_{-\infty}^{+\infty} \frac{e^{st}}{e^{e^t}+1} = 0$ we have $\int_0^A \frac{e^{st}}{e^{e^t}+1}dt = -\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt - \int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt$

So $\int_0^A \frac{e^{st}}{e^{e^t}-1}dt = \frac{-2^{(s-1)}}{(2^{(s-1)}-1)}\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du + \frac{1}{(2^{(s-1)}-1)}\int_A^{(A+ln\,2)} \frac{e^{st}}{e^{e^t}-1}dt$

Equality (4) gives

$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + 2i\int_0^{2\pi} \frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt - i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}-1}dt$

$+\left(1-e^{s2\pi i}\right)\left[\frac{-2^{(s-1)}}{(2^{(s-1)}-1)}\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du + \frac{1}{(2^{(s-1)}-1)}\int_A^{(A+ln\,2)} \frac{e^{st}}{e^{e^t}-1}dt\right]$

$= 2\pi^s(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}})\sum_{k=1}^{(2n)} \frac{(-1)^k}{k^{(1-s)}}$

So

$-i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}-1}dt + \frac{-2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du + 2i\int_0^{2\pi} \frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt$

$-\left(1-e^{s2\pi i}\right)\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{(A+ln\,2)} \frac{e^{st}}{e^{e^t}-1}dt$

$= 2\pi^s(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}})\sum_{k=1}^{(2n)} \frac{(-1)^k}{k^{(1-s)}}$

Let $C(s) and\ D(s,A)$ such that

$C(s) = -i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}-1}dt + \frac{-2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du$

$D(s,A) = -\left(1-e^{s2\pi i}\right)\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{+\infty} \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{(A+ln\,2)} \frac{e^{st}}{e^{e^t}-1}dt$

So $C(s) + D(s,A) + 2i\int_0^{2\pi} \frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt = 2\pi^s(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}})\sum_{k=1}^{(2n)} \frac{(-1)^k}{k^{(1-s)}}$ (5)

Let's prove that $C(s) = 0$

$C(s) = -i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}-1}dt + \frac{-2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_{ln\,2}^0 \frac{e^{su}}{e^{e^u}-1}du$

The residu formula on the compact (rectangle) $\{x+iy, x, y \in \mathbb{R}\ -m \le x \le 0 and\ 0 \le y \le 2\pi\}$

When m tends to$+\infty$ we get

$\left(1-e^{s2\pi i}\right)\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt + i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}+1}dt = 0$

So $\left(1-e^{s2\pi i}\right)\int_{-\infty}^0 \frac{e^{st}}{e^{e^t}+1}dt = -i\int_0^{2\pi} \frac{e^{sit}}{e^{e^{it}}+1}dt$

On the compact (rectangle) $\{x+iy, x, y \in \mathbb{R} \quad 0 \le x \le \ln 2 \ and \ 0 \le y \le 2\pi\}$ the residu formula gives

$$\left(1-e^{s2\pi i}\right)\int_0^{\ln 2}\frac{e^{st}}{e^{e^t}-1}dt + i\int_0^{2\pi}\frac{e^{s(it+\ln 2)}}{e^{e^{(it+\ln 2)}}-1}dt - i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt = 0$$

So $\left(1-e^{s2\pi i}\right)\int_{\ln 2}^{0}\frac{e^{st}}{e^{e^t}-1}dt = i\int_0^{2\pi}\frac{e^{s(it+\ln 2)}}{e^{e^{(it+\ln 2)}}-1}dt - i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt$

$$C(s) = -i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt + \frac{2^{(s-1)}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}+1}dt + \frac{1}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{s(it+\ln 2)}}{e^{e^{(it+\ln 2)}}-1}dt - \frac{1}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt$$

$$C(s) = -\frac{2^{(s-1)}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt + \frac{2^{(s-1)}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}+1}dt + \frac{e^{s\ln 2}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{2e^{it}}-1}dt$$

$$C(s) = -\frac{2^{(s-1)}}{(2^{(s-1)}-1)}i\left(\int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}-1}dt - \int_0^{2\pi}\frac{e^{sit}}{e^{e^{it}}+1}dt\right) + \frac{2^{s}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{2e^{it}}-1}dt$$

$$C(s) = -\frac{2^{(s-1)}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{2e^{sit}}{e^{2e^{it}}-1}dt + \frac{2^{s}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{2e^{it}}-1}dt$$

$$C(s) = -\frac{2^{s}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{2e^{it}}-1}dt + \frac{2^{s}}{(2^{(s-1)}-1)}i\int_0^{2\pi}\frac{e^{sit}}{e^{2e^{it}}-1}dt$$

$$C(s) = 0$$

So equality (5) becomes

$$D(s,A) + 2i\int_0^{2\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt = 2\pi^s\left(e^{si\frac{\pi}{2}} - e^{si\frac{3\pi}{2}}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}} = 2\pi^s e^{\frac{si\pi}{2}}\left(1-e^{si\pi}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$$

$$\int_0^{2\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt = \int_0^{\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt + \int_{\pi}^{2\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt = \int_0^{\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt + \int_0^{\pi}\frac{e^{s(i(u+\pi)+A)}e^{e^{(i(u+\pi)+A)}}}{e^{2e^{(i(u+\pi)+A)}}-1}du$$

$$= \int_0^{\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt + e^{si\pi}\int_0^{\pi}\frac{e^{s(iu+A)}e^{-e^{(iu+A)}}}{e^{-2e^{(iu+A)}}-1}du = \int_0^{\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt + e^{si\pi}\int_0^{\pi}\frac{e^{s(iu+A)}e^{e^{(iu+A)}}}{1-e^{2e^{(iu+A)}}}du$$

$$\left(1-e^{si\pi}\right)\int_0^{\pi}\frac{e^{s(it+A)}e^{e^{(it+A)}}}{e^{2e^{(it+A)}}-1}dt$$

$$= \left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt \qquad \text{(integral by substitution)}$$

So $D(s,A) + 2i\left(1-e^{si\pi}\right)e^{\frac{si\pi}{2}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = 2\pi^s e^{\frac{si\pi}{2}}\left(1-e^{si\pi}\right)\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$

So $\frac{1}{(1-e^{si\pi})e^{\frac{si\pi}{2}}}D(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = 2\pi^s\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$ (6)

Let $T(s,A) = \frac{1}{(1-e^{si\pi})e^{\frac{si\pi}{2}}}D(s,A)$

So $T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = 2\pi^s\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$

Let's calculate $\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}}$

We know that $\frac{1}{k^{(1-s)}} = \frac{1}{\Gamma(1-s)}\int_0^{+\infty}x^{(-s)}\,e^{-kx}dx$ so

$$\frac{(-1)^k}{k^{(1-s)}} = \frac{1}{\Gamma(1-s)}\int_0^{+\infty}x^{(-s)}\,(-e^{-x})^k dx$$

$$\sum_{k=1}^{(2n)}\frac{(-1)^k}{k^{(1-s)}} = \frac{1}{\Gamma(1-s)}\int_0^{+\infty} x^{(-s)}\sum_{k=1}^{(2n)}(-e^{-x})^k\,dx = \frac{1}{\Gamma(1-s)}\int_0^{+\infty} x^{(-s)}\frac{(1-e^{(-2nx)})\times(-e^{-x})}{1+e^{-x}}dx$$

$$= \frac{1}{\Gamma(1-s)}\int_0^{+\infty} x^{(-s)}\frac{(e^{(-2nx)}-1)}{e^x+1}dx = \frac{1}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{e^x+1}dx - \frac{1}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}}{e^x+1}dx = \frac{1}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{e^x+1}dx$$

So $T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{e^x+1}dx$

Where $T(s,A) = \frac{1}{(1-e^{si\pi})e^{\frac{si\pi}{2}}}D(s,A)$ and

$$D(s,A) = -\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\left(1-e^{s2\pi i}\right)\int_A^{(A+\ln 2)}\frac{e^{st}}{e^{e^t}-1}dt$$

$$= \left(1-e^{s2\pi i}\right)\left[-\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\int_A^{(A+\ln 2)}\frac{e^{st}}{e^{e^t}-1}dt\right]$$

So $T(s,A) = \left(1+e^{si\pi}\right)e^{\frac{-si\pi}{2}}\left[-\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt - \frac{2^{(s-1)}}{(2^{(s-1)}-1)}\int_A^{+\infty}\frac{e^{st}}{e^{e^t}+1}dt + \frac{1}{(2^{(s-1)}-1)}\int_A^{(A+\ln 2)}\frac{e^{st}}{e^{e^t}-1}dt\right]$

We have

$$T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{e^x+1}dx$$

So $T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{2}dx + \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}x^{(-s)}e^{(-2nx)}\left(\frac{1}{e^x+1}-\frac{1}{2}\right)dx$

So $T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\frac{x^{(-s)}e^{(-2nx)}}{2}dx - \frac{2\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\left(x^{(-s)}e^{(-2nx)}\right)\left(\frac{e^x-1}{2(e^x+1)}\right)dx$

So $T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \frac{\pi^s}{\Gamma(1-s)}\int_0^{+\infty}x^{(-s)}e^{(-2nx)})\,dx - \frac{\pi^s}{\Gamma(1-s)}\int_0^{+\infty}\left(x^{(-s+1)}e^{(-2nx)}\right)\left(\frac{e^x-1}{x(e^x+1)}\right)dx$

By substutition($u = 2nx$)

$$T(s,A) + 2i\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{s(it+A)}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt = \pi^s(2n)^{(s-1)} - \frac{\pi^s}{\Gamma(1-s)}(2n)^{(s-2)}\int_0^{+\infty}\left(u^{(-s+1)}e^{(-u)}\right)\left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)}\right)du$$

We multiply by $e^{(2-s)A}$ we get

$$e^{(2-s)A}T(s,A) + 2ie^{2A}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{e^{sit}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1}dt$$

$$= e^{(2-s)A}\left[\pi^s(2n)^{(s-1)} - \frac{\pi^s}{\Gamma(1-s)}(2n)^{(s-2)}\int_0^{+\infty}\left(u^{(-s+1)}e^{(-u)}\right)\left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)}\right)du\right]$$

$$= (\pi(2n+\varepsilon))^{(2-s)}\left[\pi^s(2n)^{(s-1)} - \frac{\pi^s}{\Gamma(1-s)}(2n)^{(s-2)}\int_0^{+\infty}\left(u^{(-s+1)}e^{(-u)}\right)\left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)}\right)du\right]$$

$$= \pi^2(2n+\varepsilon)^{(2-s)}\left[(2n)^{(s-1)} - \frac{1}{\Gamma(1-s)}(2n)^{(s-2)}\int_0^{+\infty}\left(u^{(-s+1)}e^{(-u)}\right)\left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)}\right)du\right]$$

$$= \pi^2 \left[(2n+\varepsilon)^{(2-s)}(2n)^{(s-1)} - \frac{1}{\Gamma(1-s)}(2n+\varepsilon)^{(2-s)}(2n)^{(s-2)} \int_0^{+\infty} \left(u^{(-s+1)}e^{(-u)}\right) \left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)} \right) du \right]$$

$$(2n)^{(s-1)} = (2n+\varepsilon-\varepsilon)^{(s-1)} = (2n+\varepsilon)^{(s-1)} \left(1 - \frac{\varepsilon}{2n+\varepsilon}\right)^{(s-1)} = (2n+\varepsilon)^{(s-1)} \left(1 - \frac{(s-1)\varepsilon}{2n+\varepsilon} + O(\frac{1}{(2n+\varepsilon)^2})\right)$$

$$(2n+\varepsilon)^{(2-s)}(2n)^{(s-1)} = (2n+\varepsilon)\left(1 + \frac{(1-s)\varepsilon}{2n+\varepsilon} + O(\frac{1}{(2n+\varepsilon)^2})\right) = (2n+\varepsilon) + (1-s)\varepsilon + O(\frac{1}{2n+\varepsilon})$$

$$e^{(2-s)A}T(s,A) + 2ie^{2A} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{e^{sit}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1} dt$$

$$= \pi^2 \left[(2n+\varepsilon) + (1-s)\varepsilon + O(\frac{1}{2n+\varepsilon}) - \frac{1}{\Gamma(1-s)} \frac{(2n)^{(s-2)}}{(2n+\varepsilon)^{(s-2)}} \int_0^{+\infty} \left(u^{(-s+1)}e^{(-u)}\right) \left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)} \right) du \right] \quad (7)$$

Since $\eta(\bar{s}) = 0$ We have also

$$e^{(2-\bar{s})A}T(\bar{s},A) + 2ie^{2A} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\bar{s}it}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1} dt$$

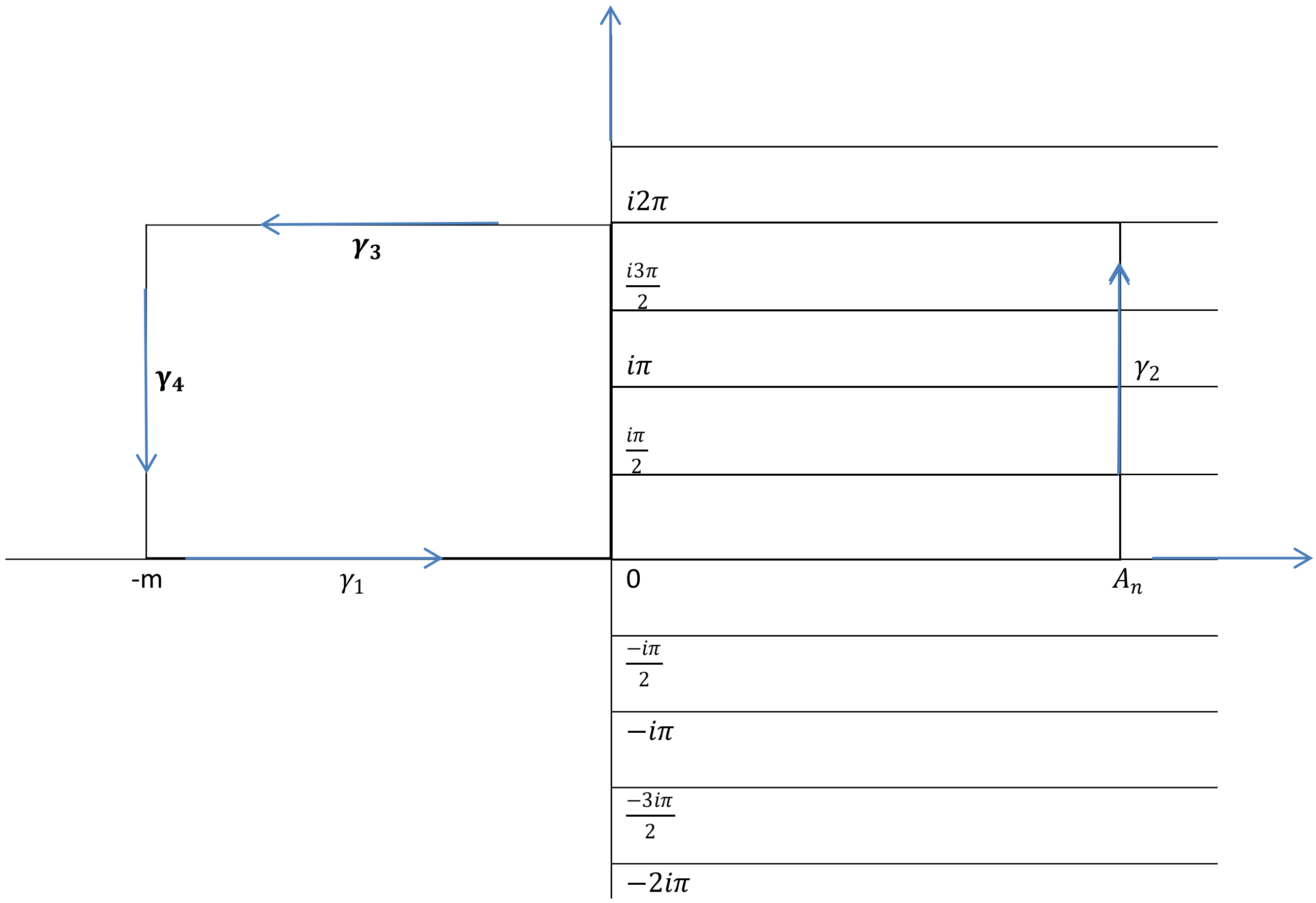
$$= \pi^2 \left[(2n+\varepsilon) + (1-\bar{s})\varepsilon + O(\frac{1}{2n+\varepsilon}) - \frac{1}{\Gamma(1-\bar{s})} \frac{(2n)^{(\bar{s}-2)}}{(2n+\varepsilon)^{(\bar{s}-2)}} \int_0^{+\infty} \left(u^{(-\bar{s}+1)}e^{(-u)}\right) \left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)} \right) du \right] \quad (8)$$

By substraction (7)-(8) we get

$$e^{(2-s)A}T(s,A) - e^{(2-\bar{s})A}T(\bar{s},A) + 2ie^{2A} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{e^{sit}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1} dt - 2ie^{2A} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\bar{s}it}e^{ie^{(it+A)}}}{e^{2ie^{(it+A)}}-1} dt$$

$$= \pi^2 \left[\begin{array}{l} (1-s)\varepsilon - (1-\bar{s})\varepsilon - \frac{1}{\Gamma(1-s)} \frac{(2n)^{(s-2)}}{(2n+\varepsilon)^{(s-2)}} \int_0^{+\infty} \left(u^{(-s+1)}e^{(-u)}\right) \left(\frac{e^{\left(\frac{u}{2n}\right)}-1}{\frac{u}{2n}\left(e^{\left(\frac{u}{2n}\right)}+1\right)} \right) du \\ \quad + \frac{1}{\Gamma(1-\bar{s})} \frac{(2n)^{(\bar{s}-2)}}{(2n+\varepsilon)^{(\bar{s}-2)}} \int_0^{+\infty} \left(u^{(-\bar{s}+1)}e^{(-u)}\right) \left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)} \right) du + O(\frac{1}{2n+\varepsilon}) \end{array} \right]$$

$$= \pi^2 \left[\begin{array}{l} (\bar{s}-s)\varepsilon - \frac{1}{\Gamma(1-s)} \frac{(2n)^{(s-2)}}{(2n+\varepsilon)^{(s-2)}} \int_0^{+\infty} \left(u^{(-s+1)}e^{(-u)}\right) \left(\frac{e^{\left(\frac{u}{2n}\right)}-1}{\frac{u}{2n}\left(e^{\left(\frac{u}{2n}\right)}+1\right)} \right) du \\ + \frac{1}{\Gamma(1-\bar{s})} \frac{(2n)^{(\bar{s}-2)}}{(2n+\varepsilon)^{(\bar{s}-2)}} \int_0^{+\infty} \left(u^{(-\bar{s}+1)}e^{(-u)}\right) \left(\frac{e^{(\frac{u}{2n})}-1}{\frac{u}{2n}\left(e^{(\frac{u}{2n})}+1\right)} \right) du + O(\frac{1}{2n+\varepsilon}) \end{array} \right]$$


$i2\pi$
γ_3
$\frac{i3\pi}{2}$
$i\pi$
γ_4
γ_2
$\frac{i\pi}{2}$
-m
γ_1
0
A_n
$\frac{-i\pi}{2}$
$-i\pi$
$\frac{-3i\pi}{2}$
$-2i\pi$