The Electromagnetic Field Strength and the Lorentz Force in Geometric Algebra Cl$_{3,0}$

Jesús Sánchez

Independent Researcher, Bilbao, Spain
Email: jesus.sanchez.bilbao@gmail.com
https://www.researchgate.net/profile/Jesus_Sanchez64
ORCID 0000-0002-5631-8195

Abstract

In this paper, we will calculate the electromagnetic field strength and the Lorentz force in Geometric Algebra Cl$_{3,0}$. And we will compare it with their equivalent in the tensor covariant formalism. What in covariant formalism is (Lorentz force):

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

We will convert it in Geometric Algebra as:

$$\frac{dp}{d\tau} = qFU \quad (21)$$

Being:

$$\frac{dp}{d\tau} = \frac{dp_0}{d\tau} + \frac{dp_{yz}}{d\tau}\hat{x} + \frac{dp_{zx}}{d\tau}\hat{y} + \frac{dp_{xy}}{d\tau}\hat{z} + \frac{dp_x}{d\tau}\hat{y}\hat{z} + \frac{dp_y}{d\tau}\hat{z}\hat{x} + \frac{dp_z}{d\tau}\hat{x}\hat{y} + \frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \quad (20)$$

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} \qquad (19)$$

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} \qquad (18)$$

Leading to the following equations:

$$\frac{dp_x}{d\tau} = q\left(E_xU_{xyz} - B_yU_z + B_zU_y\right) \qquad (24)$$

$$\frac{dp_y}{d\tau} = q\left(E_yU_{xyz} + B_xU_z - B_zU_x\right) \qquad (25)$$

$$\frac{dp_z}{d\tau} = q\left(E_zU_{xyz} - B_xU_y + B_yU_x\right) \qquad (26)$$

$$\frac{dp_{xyz}}{d\tau} = q\left(E_xU_x + E_yU_y + E_zU_z\right) \qquad (30)$$

That corresponds one to one with the Covariant formalism equivalent:

$$\frac{dp_4}{d\tau} = q\left(E_xu^1 + E_yu^2 + E_zu^3\right) \qquad (13)$$

$$\frac{dp_1}{d\tau} = q\left(-E_xu^4 - B_zu^2 + B_yu^3\right) \qquad (14)$$

$$\frac{dp_2}{d\tau} = q\left(-E_yu^4 + B_zu^1 - B_xu^3\right) \qquad (15)$$

$$\frac{dp_3}{d\tau} = q\left(-E_z u^4 - B_y u^1 + B_x u^2\right) \qquad (16)$$

with the following equivalences:

$$u^4 = U_{xyz} \qquad u^1 = U_x \qquad u^2 = U_y \qquad u^3 = U_z \qquad (18.1)$$

$$\frac{dp_{xyz}}{d\tau} = \frac{dp_4}{d\tau} \qquad \frac{dp_x}{d\tau} = -\frac{dp_1}{d\tau} \qquad \frac{dp_y}{d\tau} = -\frac{dp_2}{d\tau} \qquad \frac{dp_z}{d\tau} = -\frac{dp_3}{d\tau} \quad (32)$$

Also, we will obtain four extra equations not appearing in the classical formalism and will explain its meaning. In the same way, we will expand the electromagnetic Field strength elements and the velocity multivector of the particle:

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} + \boldsymbol{E_0} \qquad (36)$$

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + \boldsymbol{U_{yz}\hat{x}} + \boldsymbol{U_{zx}\hat{y}} + \boldsymbol{U_{xy}\hat{z}} + \boldsymbol{U_0} \quad (35)$$

New equations and new elements appearing will be explained, being the most important one, the Electromagnetic trivector $\boldsymbol{B_{xyz}}$.

Lastly, an insight of the possible implications of these learnings in the Dirac Equation will be commented.

Keywords

Geometric Algebra, Covariant formulation of classical Electromagnetism, Electromagnetic field strength, Lorentz Force, Electromagnetic trivector

1. Introduction

In this paper, we will calculate the electromagnetic field strength and the Lorentz force in Geometric Algebra $Cl_{3,0}$, and we will compare it with their equivalent in the tensor covariant formalism, explaining the different conclusions.

Afterwards an expanded version of the formulation will be commented with all its insights, including possible implications in the Dirac Equation.

If you are not new to GA, probably you can skip chapters 2 to 6.

2. Introduction to Geometric Algebra

If you do not know anything regarding geometric algebra, I strongly recommend you [1]. You have a complete study of Geometric Algebra in [3].

We will use Geometric Algebra $Cl_{3,0}$. This means, it has three basis vectors with positive signature and zero basis vectors with negative signature. We will explain this in a minute. In Geometric Algebra $Cl_{3,0}$, we have the vectors:

$$\hat{x} \quad \hat{y} \quad \hat{z}$$

In this case, we will consider them orthonormal, and, in the appendix A1, I will explain which would be the difference in the calculations if they were not orthonormal.

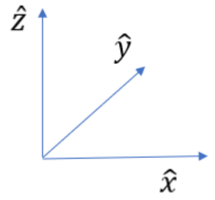

Fig.1 Orthonormal basis vectors in $Cl_{3,0}$

The square of these vectors in Geometric Algebra is its norm to the square. The norm of a vector is a scalar (not a vector anymore). As we have considered the basis as orthonormal, its square is the scalar 1.

$$\hat{x}^2 = \hat{x}\hat{x} = \|\hat{x}\|^2 = 1 \qquad (1)$$
$$\hat{y}^2 = \hat{y}\hat{y} = \|\hat{y}\|^2 = 1 \qquad (2)$$
$$\hat{z}^2 = \hat{z}\hat{z} = \|\hat{z}\|^2 = 1 \qquad (3)$$

In the nomenclature $Cl_{3,0}$, the 3 stands for the number of vectors which square is positive and the 0 for the number of basis vectors which squares is negative. In this case, no basis vectors have a negative square (also known as negative signature), so all of them have a positive square (positive signature), that equals +1 in an orthonormal basis.

The basis vectors can be multiplied by each other (this operation is called Geometric Product). For orthonormal or orthogonal bases, this product follows the anticommutative property, this is:

$$\hat{x}\hat{y} = -\hat{y}\hat{x} \qquad (4)$$
$$\hat{y}\hat{z} = -\hat{z}\hat{y} \qquad (5)$$
$$\hat{z}\hat{x} = -\hat{x}\hat{z} \qquad (6)$$

This combination of two vectors via this product is called a bivector. The bivector instead of representing a vector (an oriented segment), it represents an oriented plane. So $\hat{x}\hat{y}$ represents the plane xy with its normal in a certain direction. And $\hat{y}\hat{x}$ represents the same plane xy but with its normal in the opposite direction.

In Geometric Algebra we do not talk about normal vectors anymore. Instead, we talk about the orientation of a theoretical rotation in that plane. See Fig.2 for a visual explanation. Also, in [1][2] and [3], you can find more information about the meaning or interpretation of the bivectors.

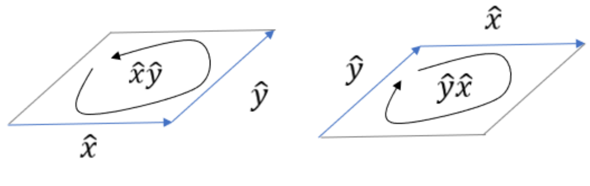

Fig.2 Representation of the bivectors $\hat{x}\hat{y}$ and $\hat{y}\hat{x}$. They represent the same plane with opposite orientation. In fact, $\hat{x}\hat{y} = -\hat{y}\hat{x}$.

If we multiply the three vectors, we obtain the trivector (also called pseudoscalar in the literature [1][3]):

$$\hat{x}\hat{y}\hat{z} = -\hat{y}\hat{x}\hat{z} = \hat{y}\hat{z}\hat{x} = -\hat{z}\hat{y}\hat{x} = \hat{z}\hat{x}\hat{y} = -\hat{x}\hat{z}\hat{y} \quad (7)$$

You can check that the same relations as in equations (4)(5)(6) apply. So, every time you swap the position of two vectors you have to put a minus sign (or multiply by -1, as you prefer).
The meaning of the trivector is an oriented volume. The same as the bivector is a plane with two possible orientations. The trivector is a volume with two possible orientations. You can see visual representation in Fig. 3. Again in [1] [2] and [3] you can find a more information regarding trivectors.

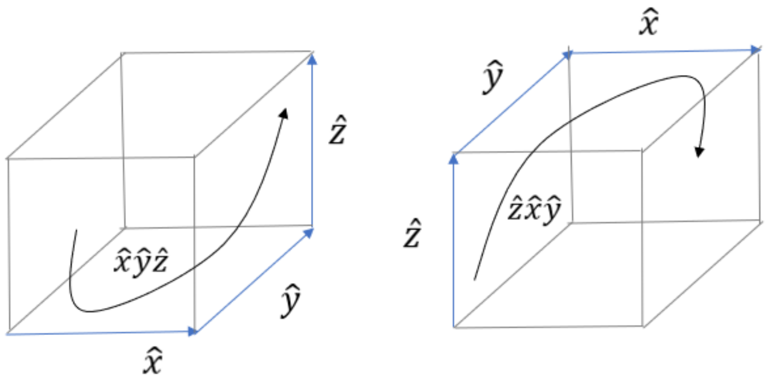

Fig.3 Representation of the two possible orientations of the trivector.
We can check that $\hat{x}\hat{y}\hat{z} = -\hat{y}\hat{x}\hat{z}$.

3. Operations in Geometric Algebra

One of the most surprising characteristics of Geometric Algebra is that you can mix scalars with vectors, bivectors and trivectors. You represent this a sum. For example, a typical element in Geometric Algebra could have the form:

$$A = 3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z}$$

The same that is done with polynomials or complex numbers, that is to leave the sum among different components indicated, it is done in Geometric Algebra. This type of element in Geometric Algebra that has different components as scalars, vectors, bivectors etc. is called a multivector. So, the A element in the example above is a multivector.

In a multivector, the vectors and the trivector are called odd-grade elements. The reason is because they are composed by one vector or by three vectors (odd grade number).

In a multivector, the scalars and the bivectors are called even grade elements. The reason is because the elements have 0 vectors (the scalars) or 2 vectors (the bivectors). We consider the 0 and 2 even for this purpose.

And if you want to make a product between two multivectors in Geometric Algebra you just have to follow the laws (1) to (6). For example:

$$(2 + 3\hat{x})(5\hat{y} + 7\hat{x} + \hat{y}\hat{z} + \hat{z}\hat{x})$$

The first thing we have to do is to multiply component by component as we would do in a polynomial for example. But the very important thing is that you have to keep the order of the product as we have seen that it is not commutative, so:

$$(2 + 3\hat{x})(5\hat{y} + 7\hat{x} + \hat{y}\hat{z} + \hat{z}\hat{x}) \\ = 10\hat{y} + 14\hat{x} + 2\hat{y}\hat{z} + 2\hat{z}\hat{x} + 15\hat{x}\hat{y} + 21\hat{x}\hat{x} + 3\hat{x}\hat{y}\hat{z} + 3\hat{x}\hat{z}\hat{x} =$$

Now, with the relations (1) to (6) we will operate the square of $\hat{x}$ and we will swap the $\hat{x}$ and the $\hat{z}$ in the last component:

$$= 10\hat{y} + 14\hat{x} + 2\hat{y}\hat{z} + 2\hat{z}\hat{x} + 15\hat{x}\hat{y} + 21(+1) + 3\hat{x}\hat{y}\hat{z} - 3\hat{x}\hat{x}\hat{z} =$$

Now, we have again a square of $\hat{x}$ in the last component, so we can operate:

$$= 10\hat{y} + 14\hat{x} + 2\hat{y}\hat{z} + 2\hat{z}\hat{x} + 15\hat{x}\hat{y} + 21 + 3\hat{x}\hat{y}\hat{z} - 3(+1)\hat{z} =$$
$$= 10\hat{y} + 14\hat{x} + 2\hat{y}\hat{z} + 2\hat{z}\hat{x} + 15\hat{x}\hat{y} + 21 + 3\hat{x}\hat{y}\hat{z} - 3\hat{z} =$$

If we order the terms, starting by the scalar, vectors, bivectors and finally the trivector we have:

$$= 21 + 14\hat{x} + 10\hat{y} - 3\hat{z} + 15\hat{x}\hat{y} + 2\hat{y}\hat{z} + 2\hat{z}\hat{x} + 3\hat{x}\hat{y}\hat{z}$$

Let's see another example:

$$(3 + \hat{x} + 2\hat{y})(5\hat{x} + 7\hat{y}) =$$

We start multiplying the components but keeping always the order of the vectors.

$$(3 + \hat{x} + 2\hat{y})(5\hat{x} + 7\hat{y}) = 15\hat{x} + 21\hat{y} + 5\hat{x}\hat{x} + 7\hat{x}\hat{y} + 10\hat{y}\hat{x} + 14\hat{y}\hat{y} =$$

Now we apply the (1) to (6) to the squares:

$$15\hat{x} + 21\hat{y} + 5(+1) + 7\hat{x}\hat{y} + 10\hat{y}\hat{x} + 14(+1) = 15\hat{x} + 21\hat{y} + 5 + 7\hat{x}\hat{y} + 10\hat{y}\hat{x} + 14 =$$

We can see that now we have two scalars (5 and 14) that have appeared coming from vector products that can be summed, so:

$$= 15\hat{x} + 21\hat{y} + 19 + 7\hat{x}\hat{y} + 10\hat{y}\hat{x} =$$

Also, we see that we have the same bivector $\hat{x}\hat{y}$ in two different forms, so we apply (1) to (6) to get:

$$= 15\hat{x} + 21\hat{y} + 19 + 7\hat{x}\hat{y} - 10\hat{x}\hat{y} = 15\hat{x} + 21\hat{y} + 19 - 3\hat{x}\hat{y} =$$

Ordering the terms:

$$= 19 + 15\hat{x} + 21\hat{y} - 3\hat{x}\hat{y}$$

To sum up, we can say that the geometric product keeps the associative and the distributive properties but not the commutative property. In an orthonormal basis the commutative property is substituted by the anticommutative property as can be seen in (4) to (6). For n on orthonormal basis, the thing is not so simple, but we will not treat this case in this paper. You can see a hint about it in Appendix A1.

4. Square of the bivectors and the trivector

If we multiply a bivector by itself (applying (1) to (6)):

$$\hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y}\hat{x} = -\hat{x}(1)\hat{x} = -\hat{x}\hat{x} = -1$$
$$\hat{y}\hat{z}\hat{y}\hat{z} = -1$$
$$\hat{z}\hat{x}\hat{z}\hat{x} = -1$$

We see that the result is -1. The same happens with the trivector:

$$\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = -\hat{x}\hat{y}\hat{z}\hat{x}\hat{z}\hat{y} = \hat{x}\hat{y}\hat{z}\hat{z}\hat{x}\hat{y} = \hat{x}\hat{y}(1)\hat{x}\hat{y} = \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{y}\hat{y}\hat{x} = -\hat{x}(1)\hat{x} = -1$$

In Geometric Algebra the imaginary or complex numbers are not used and are not necessary. The reason is that there are elements that are already in fact the square root of -1, as the bivectors or the trivector. Instead of using imaginary numbers, in Geometric Algebra $Cl_{3,0}$ these will be substituted by bivectors and trivectors with geometric meaning.

The imaginary unit *i* was defined as "something unknown" (whatever it is) that is the square root of -1. Now, that we have elements that are in fact, known, and are the square root of -1 (the bivectors and the trivector), we can be more specific and use these elements to play this role. We can use this conversion from the *i* imaginary unit into bivectors and trivector, mainly in Quantum Mechanics, not used in this paper, but yes in [5][6].

When the *i* does not have any preferred spatial direction will be related to the trivector $\hat{x}\hat{y}\hat{z}$.This happens for example when the *i* appears related to mass, energy or time.

If the *i* is related to something with a preferred direction like speed or momentum, normally the *i* is related to a bivector. We will not use it in this paper, but yes in [5][6].

Summing up, even if we are in $Cl_{3,0}$ with the three basis vectors with positive signature (positive square), the algebra itself has created two type of elements more (bivectors and trivector) which square is negative.

In a multivector we will have these two types of elements depending on its square, the scalars and the vectors which square is +1 (positive signature) and the bivectors and the trivector which square is -1 (negative signature).

5. Inverse of a vector in Geometric Algebra

Another interesting property in Geometric Algebra is that you can take the inverse a vector. We can calculate its value for a basis vector the following way. We start with equation (1):

$$\hat{x}\hat{x} = 1$$

We premultiply both equations by the inverse of $\hat{x}$:

$$\hat{x}^{-1}\hat{x}\hat{x} = \hat{x}^{-1}(1)$$

By definition, the product of the inverse of an element by itself is equal to 1.

$$(1)\hat{x} = \hat{x}^{-1}$$

So,

$$\hat{x} = \hat{x}^{-1}$$
$$\hat{x}^{-1} = \hat{x}$$

The inverse of a basis vector in an orthonormal basis is the vector itself. So:

$$\hat{x}^{-1} = \hat{x}$$
$$\hat{y}^{-1} = \hat{y}$$
$$\hat{z}^{-1} = \hat{z}$$

When we have to take the inverse a product of vectors (bivectors or trivectors) you can check in [3] that apart from inverting each element you have to reverse the order of them, this way:

$$(\hat{x}\hat{y})^{-1} = \hat{y}^{-1}\hat{x}^{-1} = \hat{y}\hat{x} = -\hat{x}\hat{y}$$

or

$$(\hat{x}\hat{y}\hat{z})^{-1} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z}$$

Remember that every time you swap two vectors, you add a minus sign (4) to (6). To convert $\hat{z}\hat{y}\hat{x}$ into $-\hat{x}\hat{y}\hat{z}$ you have two make three swaps, that is the reason of the final negative sign.

We will use the convention that the division by a vector is to postmultiply by the inverse of that vector. This means for example, if we want to do the following operation, this will be the result:

$$\frac{\hat{x}}{\hat{y}} = \hat{x}(\hat{y})^{-1} = \hat{x}\hat{y}$$

Remind that we are always talking about orthonormal bases. To have a hint about not orthonormal bases, you check Annex A1.

6. Reverse operation and reverse product

There is another operation we can make in Geometric Algebra that is the reversion of a multivector. I will represent this with a line above the multivector. This operation reverses all the internal order of bivectors and trivectors. As an example:

$$A = 3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z}$$
$$\bar{A} = \overline{(3 + 5\hat{x} + 3\hat{y} + 4\hat{y}\hat{z} - 2\hat{x}\hat{y}\hat{z})} = 3 + 5\hat{x} + 3\hat{y} + 4\hat{z}\hat{y} - 2\hat{z}\hat{y}\hat{x}$$
$$= 3 + 5\hat{x} + 3\hat{y} - 4\hat{y}\hat{z} + 2\hat{x}\hat{y}\hat{z}$$

You can see that it is similar to a conjugate in complex numbers. It changes the sign of the elements which square is -1 (in this case, are the bivectors and the trivector).

With this, we can define the reverse product. It consists of the product of a multivector by the reverse of itself.

The main characteristic of the reverse product is that if the multivector only has one type of elements with positive square and only one type of elements of negative square, the result of this product is a scalar.

This means, if the multivector only has scalars (positive square) and bivectors (negative square) the reverse product of the multivector will be a scalar. The same if it only has vectors (positive square) and bivectors (negative square). Or vectors (positive square) and trivector (negative square).

But when the multivector has scalars and vectors (both positive square) and a bivector for example, the result could be not scalar. The same if it has scalars and both bivectors and the trivector (both negative square).

This is, the multivector has to have only scalars or vectors (not both) mixed with only bivectors or trivectors (not both).

Let's see some examples. B only has vectors (positive square) and the trivector (negative square), the result must be scalar:

$$B = 5\hat{x} + 3\hat{y} + 4x\hat{y}\hat{z}$$
$$\bar{B} = \overline{(5\hat{x} + 3\hat{y} + 4x\hat{y}\hat{z})} = 5\hat{x} + 3\hat{y} - 4x\hat{y}\hat{z}$$

$$\begin{aligned} B\bar{B} = (5\hat{x} + 3\hat{y} + 4\hat{x}\hat{y}\hat{z})(5\hat{x} + 3\hat{y} - 4\hat{x}\hat{y}\hat{z}) \\ &= 25 + 15\hat{x}\hat{y} - 20\hat{x}\hat{x}\hat{y}\hat{z} + 15\hat{y}\hat{x} + 9 - 12\hat{y}\hat{x}\hat{y}\hat{z} + 20\hat{x}\hat{y}\hat{z}\hat{x} + 12\hat{x}\hat{y}\hat{z}\hat{y} \\ &- 16\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} \\ &= 25 + 15\hat{x}\hat{y} - 20\hat{y}\hat{z} - 15\hat{x}\hat{y} + 9 + 12\hat{x}\hat{y}\hat{y}\hat{z} - 20\hat{x}\hat{y}\hat{x}\hat{z} - 12\hat{x}\hat{y}\hat{y}\hat{z} \\ &+ 16\hat{x}\hat{y}\hat{x}\hat{z}\hat{y}\hat{z} = \end{aligned}$$

We sum the scalars, we see that the elements in xy sum zero, we square to +1 the vectors that are the same and consecutive and we continue swapping vectors to try to simplify:

$$= 34 - 20\hat{y}\hat{z} + 12\hat{x}\hat{z} + 20\hat{x}\hat{x}\hat{y}\hat{z} - 12\hat{x}\hat{z} - 16\hat{x}\hat{x}\hat{y}\hat{z}\hat{y}\hat{z}$$
$$= 34 - 20\hat{y}\hat{z} + 12\hat{x}\hat{z} + 20\hat{y}\hat{z} - 12\hat{x}\hat{z} + 16\hat{x}\hat{x}\hat{y}\hat{y}\hat{z}\hat{z} = 34 + 16 = 50$$

The result, 50, is a scalar as we expected.

Another example. C has only scalars and the trivector, the result should be scalar:

$$C = 5 + 4\hat{x}\hat{y}\hat{z}$$
$$\bar{C} = \overline{(5 + 4\hat{x}\hat{y}\hat{z})} = 5 - 4\hat{x}\hat{y}\hat{z}$$
$$C\bar{C} = (5 + 4\hat{x}\hat{y}\hat{z})(5 - 4\hat{x}\hat{y}\hat{z}) = 25 - 20\hat{x}\hat{y}\hat{z} + 20\hat{x}\hat{y}\hat{z} - 16\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} =$$

The elements in xyz sum zero. Swapping vectors in the last element we get:

$$= 25 + 16\hat{x}\hat{y}\hat{x}\hat{z}\hat{y}\hat{z} = 25 - 16\hat{x}\hat{x}\hat{y}\hat{z}\hat{y}\hat{z} = 25 + 16\hat{x}\hat{x}\hat{y}\hat{y}\hat{z}\hat{z} = 25 + 16 = 31$$

31 is a scalar as expected.

New example. D has scalars but has a mix of bivectors and trivectors (the result could be not a scalar):

$$D = 5 + 2\hat{x}\hat{y} + 3\hat{x}\hat{y}\hat{z}$$
$$\bar{D} = \overline{(5 + 2\hat{x}\hat{y} + 4\hat{x}\hat{y}\hat{z})} = 5 - 2\hat{x}\hat{y} - 3\hat{x}\hat{y}\hat{z}$$

$$D\bar{D} = (5 + 2\hat{x}\hat{y} + 3\hat{x}\hat{y}\hat{z})(5 - 2\hat{x}\hat{y} - 3\hat{x}\hat{y}\hat{z})$$
$$= 25 - 10\hat{x}\hat{y} - 15\hat{x}\hat{y}\hat{z} + 10\hat{x}\hat{y} - 4\hat{x}\hat{y}\hat{x}\hat{y} - 6\hat{x}\hat{y}\hat{x}\hat{y}\hat{z} + 15\hat{x}\hat{y}\hat{z}$$
$$- 6\hat{x}\hat{y}\hat{z}\hat{x}\hat{y} - 9\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} =$$

We see that the terms in $\hat{x}\hat{y}$ and $\hat{x}\hat{y}\hat{z}$ vanish. Also, we swap some vectors:

$$= 25 - 4\hat{x}\hat{y}\hat{x}\hat{y} - 6\hat{x}\hat{y}\hat{x}\hat{y}\hat{z} - 6\hat{x}\hat{y}\hat{z}\hat{x}\hat{y} - 9\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = 25 + 4\hat{x}\hat{x}\hat{y}\hat{y} + 6\hat{x}\hat{x}\hat{y}\hat{y}\hat{z} + 6\hat{x}\hat{y}\hat{z}\hat{y}\hat{x}$$
$$= 25 + 4 + 6\hat{z} - 6\hat{x}\hat{y}\hat{y}\hat{z}\hat{x} = 29 + 6\hat{z} - 6\hat{x}\hat{z}\hat{x} = 29 + 6\hat{z} + 6\hat{x}\hat{x}\hat{z}$$
$$= 29 + 6\hat{z} + 6\hat{z} = 29 + 12\hat{z}$$

We see that the result is not a scalar as we had both bivectors and the trivector (both of negative signature) in the same multivector.

One important thing to comment about the reverse product is that acts very similar to the scalar product of an element with himself (the square) in the bra-ket notation of Dirac Algebra[6].

In the bra-ket notation of Dirac algebra, when you want to calculate the square of a complex function or vector, you multiply this function or vector by the conjugate of itself, so you always get a real scalar result. This reverse product makes the same, you multiply a multivector by a version of itself where the sign of different elements of this multivector have changed with the aim of obtaining a real scalar as a result.

7. Summary of Geometric Algebra $Cl_{3,0}$

We have seen that the Geometric Algebra have some elements called multivectors that are composed by scalars, vectors, bivectors and a trivector. In fact, although the Geometric Algebra $Cl_{3,0}$ has only three basis vectors, it has really 8 degrees of freedom. A general multivector in Geometric Algebra could have the form (being all the coefficients α_i real scalars):

$$A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} + \alpha_{xyz}\hat{x}\hat{y}\hat{z}$$

This means, although we have only three special dimensions (x, y and z) we have really 8 degrees of freedom (or 8 expanded dimensions in a meta sense) coming from this original three special dimensions.

These eight degrees of freedom are represented by these 8 α_i scalars. These scalars are always real. As commented, we do not need imaginary numbers in Geometric Algebra as we have two type of elements (the bivectors and the trivector) which square is -1 and fulfills this necessity.

One comment for the people that has some experience in Geometric Algebra used in Physics. If you are new in Geometric Algebra, please do not read it, so you do not start running.

In most of the literature regarding the use of Geometric Algebra both in Quantum Mechanics and General Relativity the $Cl_{1,3}$ or $Cl_{3,1}$ is used. This means, there are three basis vectors (the spatial dimensions) with one signature and another one (the time) with opposite signature.

The issue is that these 4 dimensions expand to 16 degrees of freedom. But in reality, only the sub-even algebra of these 16 degrees of freedom is used (only 8 degrees of the 16 possible are used). So why is this $Cl_{1,3}$ or $Cl_{3,1}$ used in the first place? We know that with $Cl_{3,0}$ we already have the 8 degrees of freedom we need.

The need of $Cl_{1,3}$ and $Cl_{3,1}$ is to accommodate the time dimension in Geometric Algebra. But we will explain in the next chapter why this is not necessary anymore.

8. So, where is the time?

Probably you might be asking where the time is.

We have $\hat{x}$, $\hat{y}$ and $\hat{z}$. But where is the $\hat{t}$? As I have commented in some papers already [2][5][6] we can use the trivector as the basis vector of the dimension of time. Does this mean that the dimension of time does not exist? No, the dimension of time has its own freedom (its own scalar coefficient t) but the basis vector $\hat{t}$ that accompanies this coefficient is a combination of the space vectors.

I know, it is very difficult to believe but if you continue reading the next chapters, you will see that this works perfectly.

In fact, we will work with the following definition:

$$\hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \qquad (7)$$

The reason of why we define the inverse of the basis vector instead of the basis vector itself, we will see later. Anyhow, following the rules in chapter 5 you can see that for an orthonormal basis (not in general for other bases):

$$\hat{t} = (\hat{x}\hat{y}\hat{z})^{-1} = \hat{z}^{-1}\hat{y}^{-1}x^{-1} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z} \quad (7.1)$$

So:

$$\hat{x}\hat{y}\hat{z} = -\hat{t}$$

So, a general multivector will be of the type:

$$\begin{aligned} A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} + \alpha_{xyz}\hat{x}\hat{y}\hat{z} \\ = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} - \alpha_{xyz}\hat{t} \end{aligned}$$

Even, we reorder putting the time consecutive to the spatial dimensions we would have:

$$A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} - \alpha_{xyz}\hat{t} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x}$$

We can even recall the α_{xyz} as α_t:

$$\alpha_t = \alpha_{xyz}$$

This leads to,

$$\begin{aligned} A &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} - \alpha_t\hat{t} \\ &= \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} - \alpha_t\hat{t} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} \end{aligned}$$

You can see that the α_t assures that the time has its own freedom compared to the spatial dimensions. But we do not need an original dimension more to accommodate it, it appears naturally in Geometric Algebra. In fact, we will not use it is it is above, we will use the more convenient definition we put in the beginning for a multivector:

$$A = \alpha_0 + \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z} + \alpha_{xy}\hat{x}\hat{y} + \alpha_{yz}\hat{y}\hat{z} + \alpha_{zx}\hat{z}\hat{x} + \alpha_{xyz}\hat{x}\hat{y}\hat{z}$$

And we will explain how to work with these multivectors when time is involved.

If you have worked with Geometric Algebra before in $Cl_{1,3}$ or $Cl_{3,1}$, I give you in advance the following relations we will use. If you do not what we are talking about, just skip the following equations and continue reading:

$$i = I = \sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = \hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \qquad (7.2)$$

$$\sigma_1 = \gamma_1\gamma_0 = \hat{x} \qquad (7.3)$$

$$\sigma_2 = \gamma_2\gamma_0 = \hat{y} \qquad (7.4)$$

$$\sigma_3 = \gamma_3\gamma_0 = \hat{z} \qquad (7.5)$$

$$\gamma_0 \to \hat{t}^{-1} = \hat{x}\hat{y}\hat{z} \qquad (7.6)$$

As commented before, not always the *i* will be equal to the trivector, but sometimes to the bivectors also [6]. But in this paper, it will not be necessary to make the distinction. You can check more things regarding $\hat{t}$ as a composition of special vectors in Annex A2.

It seems that the odd-grade elements of the multivector, the vectors and the trivector, are the ones that for whatever reason we perceive as dimensions, three dimensions of space and one of time. And the bivectors and the scalars probably we perceive them in another form like forces, scalation of metrics (GR?) etc.

Regarding non-orthonormal bases or non-Euclidean metric, you can find more information in Annex A1.

9. The Covariant formulation of the Classical Electromagnetism

Now, we will start with the work. In this chapter, we will not use Geometric Algebra. We will just use the covariant formulation of the Classical Electromagnetism. We will obtain the applicable equations in that formulation so we can compare them with the ones we will obtain with Geometric Algebra $Cl_{3,0}$. In [4] we can see that Electromagnetic tensor is defined as:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

As it is normally done in these cases, we can normalize the units so c=1 and $\hbar$=1, leading to:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (8)$$

And the Lorentz Force is defined for this case as:

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

Where u^β is the four velocity and τ is the proper time. And q is the electric charge of the particle subject to the force. p_α is the momentum of the particle, so $\frac{dp_\alpha}{d\tau}$ is the variation of the momentum with respect to proper time.

For the covariant formulation, we will use the indexes 4,1,2,3 for time, x, y, and z dimensions respectively. We will use for time 4 instead of 0 to avoid a misunderstanding with another element that will appear later. So:

$$\frac{dp_\alpha}{d\tau} = \begin{pmatrix} \frac{dp_4}{d\tau} \\ \frac{dp_1}{d\tau} \\ \frac{dp_2}{d\tau} \\ \frac{dp_3}{d\tau} \end{pmatrix} \qquad (10)$$

$$u^\beta = \begin{pmatrix} u^4 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} \qquad (11)$$

So, the equation (9) gets the following form:

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

$$\begin{pmatrix} \frac{dp_4}{d\tau} \\ \frac{dp_1}{d\tau} \\ \frac{dp_2}{d\tau} \\ \frac{dp_3}{d\tau} \end{pmatrix} = q \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} u^4 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} \qquad (12)$$

Making the operations, we get the following result:

$$\frac{dp_4}{d\tau} = q\left(E_x u^1 + E_y u^2 + E_z u^3\right) \qquad (13)$$

$$\frac{dp_1}{d\tau} = q\left(-E_x u^4 - B_z u^2 + B_y u^3\right) \qquad (14)$$

$$\frac{dp_2}{d\tau} = q\left(-E_y u^4 + B_z u^1 - B_x u^3\right) \qquad (15)$$

$$\frac{dp_3}{d\tau} = q\left(-E_z u^4 - B_y u^1 + B_x u^2\right) \qquad (16)$$

These will be the equation we will use for comparison with the Geometric Algebra $Cl_{3,0}$ result.

10. Electromagnetic Field Strength and Lorentz force in Geometric Algebra $Cl_{3,0}$

Now, we will try to replicate the same result but with Geometric Algebra $Cl_{3,0}$. We will convert all the elements in equation (9) to a Geometric Algebra $Cl_{3,0}$ form.

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

We start with the 4-velocity u. Normally the vector associated with velocity is a special one ($\hat{x}$, $\hat{y}$ or $\hat{z}$) or a combination of them. But, as we saw in [6] the vectors of a momentum (that is the same as velocity multiplied by a scalar, the mass), were bivectors instead of vectors.

Let's recheck it here why using velocity instead.

The units of velocity are space divided by time. In SI units:

$$v_{units} = \frac{space}{time} = \frac{m}{s}$$

If we consider a velocity in the direction of the x axis the vectors would be:

$$v_{x\_vectors} = \frac{\hat{x}}{\hat{t}} = \hat{x}(\hat{t})^{-1} = \hat{x}\hat{x}\hat{y}\hat{z} = (1)\hat{y}\hat{z} = \hat{y}\hat{z} \qquad (17)$$

Where we have used the division by a vector convention commented in chapter 5, the equivalence of time with trivector commented in (7) and the square of a basis vector commented in (1).

You can see that the vectors of a velocity are a bivector instead of a vector. In fact, they are the complementary bivector of the vector that we would normally use. In this case, if the direction is x, the bivector is $\hat{y}\hat{z}$.

So, we have the bivectors for the three spatial directions. We need the vector for the time direction. In this case, for that we will use the trivector as we have considered in equation (7.6), where γ_0 is the time as considered in Space Time Algebra. You can see in Annex 2 more info about this.

So, the four-velocity vector in Geometric Algebra $Cl_{3,0}$ would be:

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} \qquad (18)$$

Where the U_{xyz} is the component through time as already commented.

So, the relation of U with u^β would be:

$$u^4 = U_{xyz} \qquad u^1 = U_x \qquad u^2 = U_y \qquad u^3 = U_z \qquad (18.1)$$

Coming back to equation (9):

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

Now, we will convert the electromagnetic Field strength $F_{\alpha\beta}$ to Geometric Algebra.
This has already been studied (for example in [1] and [3]) with the following result:

$$F = E + IB = E + \hat{x}\hat{y}\hat{z}B = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} \qquad (19)$$

As this has already been studied and validated, I will not enter in detail here. Anyhow, I will comment that if you want to follow a process like we did in (17), it won't exactly work. The units of the electric field are acceleration (factored by the scalars mass and charge).

If we check acceleration:

$$a_{units} = \frac{m}{s^2}$$

$$a_{x\_vectors} = \frac{\hat{x}}{\hat{t}^2} = \hat{x}(\hat{t}^2)^{-1} = \hat{x}(\hat{t}^{-1})^2 = \hat{x}(-1) = -\hat{x}$$

So, considering that the sign is a convention, we can say that yes, the electric field has the vector is its direction as vector.
But if we try to do the same with the magnetic field, it will not work. The magnetic field units are acceleration divided by velocity (factored by scalars as mass and charge).

$$\frac{a_{units}}{v_{units}} = \frac{\frac{m}{s^2}}{\frac{m}{s}} = \frac{1}{s} = s^{-1}$$

We see that the unit would be the trivector or three units of space vectors if you want. But in any case, would be the trivector or a space vector (if the other two cancel acc (1) to (3)) but never a bivector.

It is clear that the vector units of the magnetic field are the bivector. In fact, the magnetic field is the most "bivector" field I can think about, but this cannot be obtained using the "trick" of the measurement units.

Anyhow, as commented equation (19) has been validated already in the literature [1][3].

The only pending point in equation (9) is the left side:

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

The units of $\frac{dp_\alpha}{d\tau}$ would be momentum/time. Momentum is a velocity factored by the mass scalar, so its units would be the same as the four-velocity (the bivectors). But, if they are divided by time (multiplied by the trivector), the result should be vectors.

But as we have commented before with the magnetic field, this "trick" does not seem to work all the time, and this is one of the cases. We will see that $\frac{dp_\alpha}{d\tau}$ keeps the same units as the momentum (the bivectors) and the division by time is not considered. Probably because the proper time is considered scalar in opposition to the real time coordinate that would be the trivector? It is not clear. Anyhow, we will demonstrate with calculations that this is like that.

Anyhow, we will consider that we do not know anything of this, and we will define the multivector $\frac{dp_\alpha}{d\tau}$ with all its possible components. But as knowing, the final result, I will exchange the nomenclature of some elements. But the calculations would be the same, I will just exchange some names for convenience with the result we will obtain:

$$\frac{dp}{d\tau}=\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \quad (20)$$

So now, we have all the elements to reproduce equation (9) in Geometric Algebra $Cl_{3,0}$ using (18)(19) and (20)

$$\frac{dp_\alpha}{d\tau}=qF_{\alpha\beta}u^\beta \qquad (9)$$

$$\frac{dp}{d\tau}=qFU \quad (21)$$

Extending all the elements in (18)(19) and (20) in (21) we have:

$$\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} = q\left(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}+B_x\hat{y}\hat{z}+B_y\hat{z}\hat{x}+B_z\hat{x}\hat{y}\right)\left(U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y}\right) \quad (22)$$

Making the calculations we have:

$$\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z}=$$
$$q\left(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}+B_x\hat{x}\hat{y}+B_y\hat{z}\hat{x}+B_z\hat{x}\hat{y}\right)\left(U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y}\right)=$$
$$q\begin{pmatrix} E_xU_{xyz}\hat{y}\hat{z}+E_xU_x\hat{x}\hat{y}\hat{z}+E_xU_y\hat{x}\hat{z}\hat{x}+E_xU_z\hat{y}+ \\ +E_yU_{xyz}\hat{y}\hat{x}\hat{y}\hat{z}+E_yU_x\hat{z}+E_yU_y\hat{y}\hat{z}\hat{x}+E_yU_z\hat{y}\hat{x}\hat{y}+ \\ +E_zU_{xyz}\hat{z}\hat{x}\hat{y}\hat{z}+E_zU_x\hat{z}\hat{y}\hat{z}+E_zU_y\hat{x}+E_zU_z\hat{z}\hat{x}\hat{y}+ \\ +B_xU_{xyz}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z}+B_xU_x\hat{y}\hat{z}\hat{y}\hat{z}+B_xU_y\hat{y}\hat{z}\hat{z}\hat{x}+B_xU_z\hat{y}\hat{z}\hat{x}\hat{y}+ \\ +B_yU_{xyz}\hat{z}\hat{x}\hat{x}\hat{y}\hat{z}+B_yU_x\hat{z}\hat{x}\hat{y}\hat{z}+B_yU_y\hat{z}\hat{x}\hat{z}\hat{x}+B_yU_z\hat{z}\hat{x}\hat{x}\hat{y}+ \\ +B_zU_{xyz}\hat{x}\hat{y}\hat{x}\hat{y}\hat{z}+B_zU_x\hat{x}\hat{y}\hat{y}\hat{z}+B_zU_y\hat{x}\hat{y}\hat{z}\hat{x}+B_zU_z\hat{x}\hat{y}\hat{x}\hat{y} \end{pmatrix}=$$

$$=q\begin{pmatrix} E_xU_{xyz}\hat{y}\hat{z}+E_xU_x\hat{x}\hat{y}\hat{z}-E_xU_y\hat{z}+E_xU_z\hat{y}+ \\ +E_yU_{xyz}\hat{z}\hat{x}+E_yU_x\hat{z}+E_yU_y\hat{x}\hat{y}\hat{z}-E_yU_z\hat{x}+ \\ +E_zU_{xyz}\hat{x}\hat{y}-E_zU_x\hat{y}+E_zU_y\hat{x}+E_zU_z\hat{x}\hat{y}\hat{z}+ \\ -B_xU_{xyz}\hat{x}-B_xU_x-B_xU_y\hat{x}\hat{y}+B_xU_z\hat{z}\hat{x}+ \\ -B_yU_{xyz}\hat{y}+B_yU_x\hat{x}\hat{y}-B_yU_y-B_yU_z\hat{y}\hat{z}+ \\ -B_zU_{xyz}\hat{z}-B_zU_x\hat{z}\hat{x}+B_zU_y\hat{y}\hat{z}-B_zU_z \end{pmatrix} \qquad (23)$$

Now, we can get the components that multiply each of the vectors, bivectors or trivector and create separate equations. For example, we get all the elements that multiply by $\hat{y}\hat{z}$ both in the left side and in the right side of the equation, leading to:

$$\frac{dp_x}{d\tau}=q\left(E_xU_{xyz}-B_yU_z+B_zU_y\right) \qquad (24)$$

If we get now the elements that multiply by $\hat{z}\hat{x}$ we obtain:

$$\frac{dp_y}{d\tau}=q\left(E_yU_{xyz}+B_xU_z-B_zU_x\right) \qquad (25)$$

And so on. Putting all together we have:

$$\frac{dp_x}{d\tau}=q\left(E_xU_{xyz}-B_yU_z+B_zU_y\right) \qquad (24)$$

$$\frac{dp_y}{d\tau} = q\left(E_y U_{xyz} + B_x U_z - B_z U_x\right) \qquad (25)$$

$$\frac{dp_z}{d\tau} = q\left(E_z U_{xyz} - B_x U_y + B_y U_x\right) \qquad (26)$$

$$\frac{dp_{yz}}{d\tau} = q\left(-E_y U_z + E_z U_y - B_x U_{xyz}\right) \qquad (27)$$

$$\frac{dp_{zx}}{d\tau} = q\left(E_x U_z - E_z U_x - B_y U_{xyz}\right) \qquad (28)$$

$$\frac{dp_{xy}}{d\tau} = q\left(E_y U_x + E_x U_y - B_z U_{xyz}\right) \qquad (29)$$

$$\frac{dp_{xyz}}{d\tau} = q\left(E_x U_x + E_y U_y + E_z U_z\right) \qquad (30)$$

$$\frac{dp_o}{d\tau} = q\left(-B_x U_x - B_y U_y - B_z U_z\right) \qquad (31)$$

Now, if we compare for example equation (30) obtained with Geometric Algebra $Cl_{3,0}$ with equation (13) obtained with covariant formulation:

$$\frac{dp_{xyz}}{d\tau} = q\left(E_x U_x + E_y U_y + E_z U_z\right) \qquad (30)$$

$$\frac{dp_4}{d\tau} = q\left(E_x u^1 + E_y u^2 + E_z u^3\right) \qquad (13)$$

We see that they are the same equation using the following relations:

$$u^4 = U_{xyz} \qquad u^1 = U_x \qquad u^2 = U_y \qquad u^3 = U_z \qquad (18.1)$$

And:

$$\frac{dp_{xyz}}{d\tau} = \frac{dp_4}{d\tau}$$

Now, let us compare equation (24) with (14) reordering the position of two terms:

$$\frac{dp_x}{d\tau} = q\left(E_x U_{xyz} - B_y U_z + B_z U_y\right) \qquad (24)$$

$$\frac{dp_1}{d\tau} = q\left(-E_x u^4 + B_y u^3 - B_z u^2\right) \qquad (14)$$

If we apply 18.1:

$$u^4 = U_{xyz} \qquad u^1 = U_x \qquad u^2 = U_y \qquad u^3 = U_z \qquad (18.1)$$

We see that they are the same equation but with signs changed.
This means, they are the same if we consider:

$$\frac{dp_x}{d\tau} = -\frac{dp_1}{d\tau} \qquad (18.2)$$

We could say that we define the momentum in the other direction and that would work. It is not exactly true, as the four-velocity we have considered is in the same direction in both formulations. So, it would not be coherent. Another reason could be the Minkowski metric should be multiplying somewhere and change the signs. In GA I try not to use this trick, as I consider that the metric is implicit in in the basis vectors (see Annex A1 of this paper and papers [2][5][6]).

The third option would be that is the variation of the momentum through time (not the momentum itself) which has another sign as convention in both formulations. And it could be related to the case I have commented regarding the time or the inverse of time (which changes sign). And sometimes, it is difficult to know which to use.

Anyhow, if we consider the sign as a convention (or as an error regarding directions through time that should be corrected) the equation is correct if we define:

$$\frac{dp_x}{d\tau} = -\frac{dp_1}{d\tau} \qquad (18.2)$$

This is, we consider that the variation of momentum has a different sign convention in both formulations.

This way we have that these four equations are the same in both formulations, so the conversion to Geometric Algebra $Cl_{3,0}$ has worked (except this issue with the sign to be studied, that is solved using (18.2)):

$$\frac{dp_{xyz}}{d\tau} = q\left(E_x U_x + E_y U_y + E_z U_z\right) \qquad (30)$$

$$\frac{dp_x}{d\tau} = q\left(E_x U_{xyz} - B_y U_z + B_z U_y\right) \qquad (24)$$

$$\frac{dp_y}{d\tau} = q\left(E_y U_{xyz} + B_x U_z - B_z U_x\right) \qquad (25)$$

$$\frac{dp_z}{d\tau} = q\left(E_z U_{xyz} - B_x U_y + B_y U_x\right) \qquad (26)$$

That are equivalent to these in covariant formulation:

$$\frac{dp_4}{d\tau} = q\left(E_x u^1 + E_y u^2 + E_z u^3\right) \qquad (13)$$

$$\frac{dp_1}{d\tau} = q\left(-E_x u^4 - B_z u^2 + B_y u^3\right) \qquad (14)$$

$$\frac{dp_2}{d\tau} = q\left(-E_y u^4 + B_z u^1 - B_x u^3\right) \qquad (15)$$

$$\frac{dp_3}{d\tau} = q\left(-E_z u^4 - B_y u^1 + B_x u^2\right) \qquad (16)$$

Considering:

$$u^4 = U_{xyz} \qquad u^1 = U_x \qquad u^2 = U_y \qquad u^3 = U_z \qquad (18.1)$$

And:

$$\frac{dp_{xyz}}{d\tau} = \frac{dp_4}{d\tau} \qquad \frac{dp_x}{d\tau} = -\frac{dp_1}{d\tau} \qquad \frac{dp_y}{d\tau} = -\frac{dp_2}{d\tau} \qquad \frac{dp_z}{d\tau} = -\frac{dp_3}{d\tau} \qquad (32)$$

So, the above ones are (somehow) ok, but what about the rest?

$$\frac{dp_{yz}}{d\tau} = q\left(-E_y U_z + E_z U_y - B_x U_{xyz}\right) \qquad (27)$$

$$\frac{dp_{zx}}{d\tau} = q\left(E_x U_z - E_z U_x - B_y U_{xyz}\right) \qquad (28)$$

$$\frac{dp_{xy}}{d\tau} = q\left(E_y U_x + E_x U_y - B_z U_{xyz}\right) \qquad (29)$$

$$\frac{dp_o}{d\tau} = q\left(-B_x U_x - B_y U_y - B_z U_z\right) \qquad (31)$$

These ones do not appear in the covariant formulation. If we recall the definition we have used for $\frac{dp}{d\tau}$:

$$\frac{dp}{d\tau}=\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \quad (20)$$

The linear momentum has units of bivector, as we can see in (33) for example using the direction in x. We have used all the equivalences commented in (17)

$$p_{x\_vectors}=kg(scalar)\cdot v_{x\_vectors}=\frac{\hat{x}}{\hat{t}}=\hat{x}(\hat{t})^{-1}=\hat{x}\hat{x}\hat{y}\hat{z}=(1)\hat{y}\hat{z}=\hat{y}\hat{z} \quad (33)$$

But the angular momentum is the linear momentum multiplied by a direction of space so its dimensions are vectors not bivectors, as we can see in (34). We consider for example a linear momentum in x, multiplied by a distance in y.

$$L_{y\_px}=m(distance\ in\ y)\cdot p_{x\_vectors}=\hat{y}\hat{y}\hat{z}=(1)\hat{z}=\hat{z} \quad (34)$$

So, in equation (20) all the coefficients ($\frac{dp_{yz}}{d\tau}$, $\frac{dp_{zx}}{d\tau}$ and $\frac{dp_{xy}}{d\tau}$) that multiply the single vectors ($\hat{x}$, $\hat{y}$ or $\hat{z}$) could be acting modifying the angular momentum.

The issue is that if it is really acting to the real angular momentum of the particle, this effect should have been already manifested in experiments. So, the only possible explanation would be that this effect is oscillatory by its nature, the forces are changing during time, so the average trajectory of the particle is not affected, only locally in a kind of zitterbewegung (rapid oscillatory movement of the particle) that does not change the “macroscopic” trajectory. The average trajectory is not modified.

Another explanation would be that the angular momentum affected is not the one related to trajectory but the internal one (its own rotation status). This means that the force is affecting the orientation of the axes or the velocity of rotation etc.

Another possibility is that those elements that multiply the single vectors are not the linear momentum, they could be position, over acceleration or others. Anyhow, we always have to take into account that in reality should be very near to zero or oscillatory not changing the average trajectory as they have not been considered until now. They seem something implicit in the oscillatory movements or in the probabilistic nature of certain measurements.

Here, what we see is that the Geometric Algebra, shows us that there are other equations that are affecting parameters that have not been considered in the past. Hidden variables? Let’s go to the next chapter.

11. Expanding the equations of the Electromagnetic Field Strength and the Lorentz Force in Geometric Algebra Cl3,0

Following the philosophy of the previous chapter, let’s extend the equation (22):

$$\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z}$$
$$=q\big(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}+B_x\hat{y}\hat{z}+B_y\hat{z}\hat{x}+B_z\hat{x}\hat{y}\big)\big(U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y}\big) \quad (22)$$

But considering the complete multivector U (I put in bold the addings compared with equation (18):

$$U=U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y}+\boldsymbol{U_{yz}\hat{x}}+\boldsymbol{U_{zx}\hat{y}}+\boldsymbol{U_{xy}\hat{z}}+\boldsymbol{U_0} \quad (35)$$

So here we have added the components $\mathbf{U_{ij}}$ of the initial angular momentum (internal as a rotation? Or oscillatory during the trajectory?) in the initial values of U. Also, we have introduced the scalar $\mathbf{U_0}$ which will be commented later. As commented the $\mathbf{U_{ij}}$ could represent other things (position?) that should be checked about. I will consider the angular momentum.

And in the electromagnetic Field I will add the two remaining components in equation (19):

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} + \boldsymbol{E_0} \quad (36)$$

I have introduced the Electromagnetic trivector $\mathbf{B_{xyz}}$ and the Electromagnetic scalar $\mathbf{E_0}$: Now, the equation (19) should read:

$$\begin{aligned}\frac{dp_0}{d\tau} + \frac{dp_{yz}}{d\tau}\hat{x} + \frac{dp_{zx}}{d\tau}\hat{y} + \frac{dp_{xy}}{d\tau}\hat{z} + \frac{dp_x}{d\tau}\hat{y}\hat{z} + \frac{dp_y}{d\tau}\hat{z}\hat{x} + \frac{dp_z}{d\tau}\hat{x}\hat{y} + \frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \\ = q\big(E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} \\ + \boldsymbol{E_0}\big)\big(U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + \boldsymbol{U_{yz}\hat{x}} + \boldsymbol{U_{zx}\hat{y}} + \boldsymbol{U_{xy}\hat{z}} \\ + \boldsymbol{U_0}\big) \quad (37)\end{aligned}$$

If we operate, we get:

$$\begin{aligned}\frac{dp_0}{d\tau} + \frac{dp_{yz}}{d\tau}\hat{x} + \frac{dp_{zx}}{d\tau}\hat{y} + \frac{dp_{xy}}{d\tau}\hat{z} + \frac{dp_x}{d\tau}\hat{y}\hat{z} + \frac{dp_y}{d\tau}\hat{z}\hat{x} + \frac{dp_z}{d\tau}\hat{x}\hat{y} + \frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \\ = q\big(E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} \\ + \boldsymbol{E_0}\big)\big(U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + \boldsymbol{U_{yz}\hat{x}} + \boldsymbol{U_{zx}\hat{y}} + \boldsymbol{U_{xy}\hat{z}} \\ + \boldsymbol{U_0}\big) =\end{aligned}$$

$$= q\begin{pmatrix} E_xU_{xyz}\hat{y}\hat{z} + E_xU_x\hat{x}\hat{y}\hat{z} - E_xU_y\hat{z} + E_xU_z\hat{y} + E_x\boldsymbol{U_{yz}} + E_x\boldsymbol{U_{zx}}\hat{x}\hat{y} - E_x\boldsymbol{U_{xy}}\hat{z}\hat{x} + E_x\boldsymbol{U_0}\hat{x} \\ +E_yU_{xyz}\hat{z}\hat{x} + E_yU_x\hat{z} + E_yU_y\hat{x}\hat{y}\hat{z} - E_yU_z\hat{x} - E_y\boldsymbol{U_{yz}}\hat{x}\hat{y} + E_y\boldsymbol{U_{zx}} + E_y\boldsymbol{U_{xy}}\hat{y}\hat{z} + E_y\boldsymbol{U_0}\hat{y} \\ +E_zU_{xyz}\hat{x}\hat{y} - E_zU_x\hat{y} + E_zU_y\hat{x} + E_zU_z\hat{x}\hat{y}\hat{z} + E_z\boldsymbol{U_{yz}}\hat{z}\hat{x} - E_z\boldsymbol{U_{zx}}\hat{y}\hat{z} + E_z\boldsymbol{U_{xy}} + E_z\boldsymbol{U_0}\hat{z} \\ -B_xU_{xyz}\hat{x} - B_xU_x - B_xU_y\hat{x}\hat{y} + B_xU_z\hat{z}\hat{x} + B_x\boldsymbol{U_{yz}}\hat{x}\hat{y}\hat{z} - B_x\boldsymbol{U_{zx}}\hat{z} + B_x\boldsymbol{U_{xy}}\hat{y} + B_x\boldsymbol{U_0}\hat{y}\hat{z} \\ -B_yU_{xyz}\hat{y} + B_yU_x\hat{x}\hat{y} - B_yU_y - B_yU_z\hat{y}\hat{z} + B_y\boldsymbol{U_{yz}}\hat{z} + B_y\boldsymbol{U_{zx}}\hat{x}\hat{y}\hat{z} - B_y\boldsymbol{U_{xy}}\hat{x} + B_y\boldsymbol{U_0}\hat{z}\hat{x} \\ -B_zU_{xyz}\hat{z} - B_zU_x\hat{z}\hat{x} + B_zU_y\hat{y}\hat{z} - B_zU_z - B_z\boldsymbol{U_{yz}}\hat{y} + B_z\boldsymbol{U_{zx}}\hat{x} + B_z\boldsymbol{U_{xy}}\hat{x}\hat{y}\hat{z} + B_z\boldsymbol{U_0}\hat{x}\hat{y} \\ -\boldsymbol{B_{xyz}}U_{xyz} - \boldsymbol{B_{xyz}}U_x\hat{x} - \boldsymbol{B_{xyz}}U_y\hat{y} - \boldsymbol{B_{xyz}}U_z\hat{z} + \boldsymbol{B_{xyz}U_{yz}}\hat{y}\hat{z} + \boldsymbol{B_{xyz}U_{zx}}\hat{z}\hat{x} + \boldsymbol{B_{xyz}U_{xy}}\hat{x}\hat{y} + \boldsymbol{B_{xyz}U_0}\hat{x}\hat{y}\hat{z} \\ \boldsymbol{E_0}U_{xyz}\hat{x}\hat{y}\hat{z} + \boldsymbol{E_0}U_x\hat{y}\hat{z} + \boldsymbol{E_0}U_y\hat{z}\hat{x} + \boldsymbol{E_0}U_z\hat{x}\hat{y} + \boldsymbol{E_0U_{yz}}\hat{x} + \boldsymbol{E_0U_{zx}}\hat{y} + \boldsymbol{E_0U_{xy}}\hat{z} + \boldsymbol{E_0U_0} \end{pmatrix} \quad (38)$$

We can see that a lot of new elements appear, leaving the equations as:

$$\frac{dp_x}{d\tau} = q\big(E_xU_{xyz} - B_yU_z + B_zU_y + E_y\boldsymbol{U_{xy}} - E_z\boldsymbol{U_{zx}} + B_x\boldsymbol{U_0} + \boldsymbol{B_{xyz}U_{yz}} + \boldsymbol{E_0}U_x\big) \quad (39)$$

$$\frac{dp_y}{d\tau} = q\big(E_yU_{xyz} + B_xU_z - B_zU_x - E_x\boldsymbol{U_{xy}} + E_z\boldsymbol{U_{yz}} + B_y\boldsymbol{U_0} + \boldsymbol{B_{xyz}U_{zx}} + \boldsymbol{E_0}U_y\big) \quad (40)$$

$$\frac{dp_z}{d\tau} = q\big(E_zU_{xyz} - B_xU_y + B_yU_x + E_x\boldsymbol{U_{zx}} - E_y\boldsymbol{U_{yz}} + B_z\boldsymbol{U_0} + \boldsymbol{B_{xyz}U_{xy}} + \boldsymbol{E_0}U_z\big) \quad (41)$$

$$\frac{dp_{yz}}{d\tau} = q\big(-E_yU_z + E_zU_y - B_xU_{xyz} + E_x\boldsymbol{U_0} - B_y\boldsymbol{U_{xy}} + B_z\boldsymbol{U_{zx}} - \boldsymbol{B_{xyz}}U_x + \boldsymbol{E_0U_{yz}}\big) \ (42)$$

$$\frac{dp_{zx}}{d\tau} = q\big(E_xU_z - E_zU_x - B_yU_{xyz} + E_y\boldsymbol{U_0} + B_x\boldsymbol{U_{xy}} - B_z\boldsymbol{U_{yz}} - \boldsymbol{B_{xyz}} + \boldsymbol{E_0U_{zx}}\big) \ (43)$$

$$\frac{dp_{xy}}{d\tau} = q\big(E_y U_x + E_x U_y - B_z U_{xyz} + E_z \boldsymbol{U_0} - B_x \boldsymbol{U_{zx}} + B_y \boldsymbol{U_{yz}} - \boldsymbol{B_{xyz}} U_z + \boldsymbol{E_0} \boldsymbol{U_{xy}}\big) \quad (44)$$

$$\frac{dp_{xyz}}{d\tau} = q\big(E_x U_x + E_y U_y + E_z U_z + B_x \boldsymbol{U_{yz}} + B_y \boldsymbol{U_{zx}} + B_z \boldsymbol{U_{xy}} + \boldsymbol{B_{xyz}} \boldsymbol{U_0} + \boldsymbol{E_0} U_{xyz}\big) \quad (45)$$

$$\frac{dp_o}{d\tau} = q\big(-B_x U_x - B_y U_y - B_z U_z + E_x \boldsymbol{U_{yz}} + E_y \boldsymbol{U_{zx}} + E_z \boldsymbol{U_{xy}} - \boldsymbol{B_{xyz}} U_{xyz} + \boldsymbol{E_0} \boldsymbol{U_0}\big) \quad (46)$$

As commented the elements in bold are new compared to the covariant formalism. So to be correct, they should be very small (almost neglectable to be not detected) or oscillatory provoking local effects but not in the average trajectory measured in the particle.

12. Explaining the expanded equations of the Electromagnetic strength and the Lorentz Force Law in Geometric Algebra $Cl_{3,0}$

Let's start comparing the original equations that had a map relation with the covariant formalism (24) to (26) and (30) with the new ones obtained for them (39) to (41) and (45):

$$\frac{dp_x}{d\tau} = q\big(E_x U_{xyz} - B_y U_z + B_z U_y\big) \quad (24)$$

$$\frac{dp_y}{d\tau} = q\big(E_y U_{xyz} + B_x U_z - B_z U_x\big) \quad (25)$$

$$\frac{dp_z}{d\tau} = q\big(E_z U_{xyz} - B_x U_y + B_y U_x\big) \quad (26)$$

$$\frac{dp_{xyz}}{d\tau} = q\big(E_x U_x + E_y U_y + E_z U_z\big) \quad (30)$$

$$\frac{dp_x}{d\tau} = q\big(E_x U_{xyz} - B_y U_z + B_z U_y + E_y \boldsymbol{U_{xy}} - E_z \boldsymbol{U_{zx}} + B_x \boldsymbol{U_0} + \boldsymbol{B_{xyz}} \boldsymbol{U_{yz}} + \boldsymbol{E_0} U_x\big) \quad (39)$$

$$\frac{dp_y}{d\tau} = q\big(E_y U_{xyz} + B_x U_z - B_z U_x - E_x \boldsymbol{U_{xy}} + E_z \boldsymbol{U_{yz}} + B_y \boldsymbol{U_0} + \boldsymbol{B_{xyz}} \boldsymbol{U_{zx}} + \boldsymbol{E_0} U_y\big) \quad (40)$$

$$\frac{dp_z}{d\tau} = q\big(E_z U_{xyz} - B_x U_y + B_y U_x + E_x \boldsymbol{U_{zx}} - E_y \boldsymbol{U_{yz}} + B_z \boldsymbol{U_0} + \boldsymbol{B_{xyz}} \boldsymbol{U_{xy}} + \boldsymbol{E_0} U_z\big) \quad (41)$$

$$\frac{dp_{xyz}}{d\tau} = q\big(E_x U_x + E_y U_y + E_z U_z + B_x \boldsymbol{U_{yz}} + B_y \boldsymbol{U_{zx}} + B_z \boldsymbol{U_{xy}} + \boldsymbol{B_{xyz}} \boldsymbol{U_0} + \boldsymbol{E_0} U_{xyz}\big) \quad (45)$$

We see that the new elements that appear in (39) compared to (24) mainly depends in the current angular momentum represented by the elements $\boldsymbol{U_{ij}}$. As commented before, this angular momentum could be internal (rotation) or external (something regarding the trajectory). We do not know. What it is clear is that it should change over time so the mean value of $\frac{dp_x}{d\tau}$ is not different in general (talking about "average" value) than in equation (24). The other option clearly is that the values of $\boldsymbol{U_{ij}}$ are directly 0 from the beginning.

Another element appearing is the $\boldsymbol{E_0}$, the electromagnetic scalar. The solution for this, could be directly that it does not exist, it is always zero. Also, that is so small that cannot be detected (in practice is zero). That it is oscillatory so does not change the average value.

And the last option is that as it escalates all the values, its effect is not measured. As the measurement devices (rods, clocks) will be escalated in the same proportion than the rest of the events. So, in a local frame you will not see this escalation. Only from a distant frame where this effect is not having place. It would be a kind of escalation metric number (Ricci scalar, trace of the metric, product of the metric diagonal, determinant of the metric?) that is affecting everything in the local frame. And it could only be seen form a distant frame. Yes, something regarding GR? Who knows.

In fact, seeing the equation, its effect would be a continuous escalation of the values during time. This is not possible if it is really happening in a relative sense towards the elements in the same frame. But if it is happening everywhere in the same frame with no relative changes, in fact the speed and accelerations measured will not suffer this escalation within its own frame.
So, in a practical sense talking about a certain frame, the most practical thing is to consider $\boldsymbol{E_0}$ equals to zero until a general study of what this escalation factors could mean. Very probably related to metric in certain frames (GR).

For $\boldsymbol{U_0}$ we can tell the same story. It is a scalar that appears in the velocity multivector. What can be the meaning of a scalar in a velocity multivector? Again, it is a kind of scalation factor, for all the magnitudes that will multiply the velocity. So, it will escalate all the effects, but the meaning as itself as a velocity component is not known. The same as commented for $\boldsymbol{E_0}$**,** considering that $\boldsymbol{U_0}$ equals to zero should work inside the frame we are working on. Only when more studies are done regarding these escalation factors, we can start introducing values for it.

And the last element is the jewel of the crown: $\boldsymbol{B_{xyz}}$**,** the electromagnetic trivector. In equations (39) to (41) and (45) its effect only appears if there is current angular momentum $\boldsymbol{U_{ij}}$ in place. As commented internal (rotation) or external (something regarding trajectory like an helicoidal for example) to be defined.

As long as the $\boldsymbol{U_{ij}}$ values of angular momentum are changing during time the effect of $\boldsymbol{B_{xyz}}$ will just be seen as an erratic movement along the trajectory but not changing the average of it. The "macroscopic" trajectory will be the same, but the $\boldsymbol{B_{xyz}}$ will create accelerations in different directions that will appear and probabilistically will cancel each other, so the total mean value will be the same but as local effects, you will see these movements.

In fact, if we can check the rest of the equations that have appeared in GA $Cl_{3,0}$ but are not included in the covariant classical formalism:

$$\frac{dp_{yz}}{d\tau} = q\left(-E_yU_z + E_zU_y - B_xU_{xyz} + E_x\boldsymbol{U_0} - B_y\boldsymbol{U_{xy}} + B_z\boldsymbol{U_{zx}} - \boldsymbol{B_{xyz}}U_x + \boldsymbol{E_0}\boldsymbol{U_{yz}}\right) \quad (42)$$

$$\frac{dp_{zx}}{d\tau} = q\left(E_xU_z - E_zU_x - B_yU_{xyz} + E_y\boldsymbol{U_0} + B_x\boldsymbol{U_{xy}} - B_z\boldsymbol{U_{yz}} - \boldsymbol{B_{xyz}} + \boldsymbol{E_0}\boldsymbol{U_{zx}}\right) \quad (43)$$

$$\frac{dp_{xy}}{d\tau} = q\left(E_yU_x + E_xU_y - B_zU_{xyz} + E_z\boldsymbol{U_0} - B_x\boldsymbol{U_{zx}} + B_y\boldsymbol{U_{yz}} - \boldsymbol{B_{xyz}}U_z + \boldsymbol{E_0}\boldsymbol{U_{xy}}\right) \quad (44)$$

$$\frac{dp_o}{d\tau} = q\left(-B_xU_x - B_yU_y - B_zU_z + E_x\boldsymbol{U_{yz}} + E_y\boldsymbol{U_{zx}} + E_z\boldsymbol{U_{xy}} - \boldsymbol{B_{xyz}}U_{xyz} + \boldsymbol{E_0}\boldsymbol{U_0}\right) \quad (46)$$

If we consider the $\frac{dp_{ij}}{d\tau}$ the variation of the angular momentum during time, we see that they are continuously affected, by the linear speed U_i and the angular momentum $\boldsymbol{U_{ij}}$ **.** So, they are varying continuously creating this oscillatory movement in trajectory and changes in angular momentum in rotation or in trajectory also (to be confirmed).

We see again the electromagnetic trivector $\boldsymbol{B_{xyz}}$ acting but this time, its action depends on the linear speed U_i**.**

It is clear that there is a lot to study here. But the important thing is that some effects not considered until now could exist and could be affecting the local trajectory or the rotation of the particle depending on its current angular momentum and in the electromagnetic trivector value $\boldsymbol{B_{xyz}}$.

This is, some experiments that were considered probabilistic or even with an action though distance should be rethought considering the existence of the $\boldsymbol{B_{xyz}}$ field and that the internal (or external) angular momentum could influence also locally the trajectory because the electromagnetic effects could vary depending on its values. As said, not the average trajectory but yes local, probably oscillatory movements.

Even measurements as spin, could be influenced by the value $\boldsymbol{B_{xyz}}$. As I commented in an old paper [7] without equations, the hidden variables could just be a field which effects affect the measurables as the spin but are hidden in all other interactions, so we consider its effect as "magic". So, when we measure the spin of a particle and the one of its entangled particle, this electromagnetic trivector $\boldsymbol{B_{xyz}}$ and the other variables as the angular momentum $\boldsymbol{U_{ij}}$ should be taken into account.

The electromagnetic trivector $\boldsymbol{B_{xyz}}$ in particular, could be oscillatory and provoke also a change in the internal angular momentum, the orientation of axes etc. So, its effects would not be important in the typical measurements we could take as speed, trajectory… but yes for other measurements as spin… Its effects should be taken into account in whatever theory that wants to explain the results.

13. The Dirac Equation

In [6] I created a one-to-one map between the Dirac Equation in Matrix Algebra and the Dirac Equation in Geometric Algebra $Cl_{3,0}$. I am not going to repeat what was already commented there. But yes, what was not commented. We obtained the equation:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t}-\hat{y}\hat{z}\frac{\partial}{\partial x}-\hat{z}\hat{x}\frac{\partial}{\partial y}-\hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi-m\psi_{even}\hat{z}+m\psi_{odd}\hat{z}=0 \quad (47)$$

Where:

$$\begin{aligned}\psi_{even}&=\psi_0+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}\\ \psi_{odd}&=\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\hat{z}\psi_{xyz}\end{aligned}$$

$$\psi=\psi_{even}+\psi_{odd}=\psi_0+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\hat{x}\hat{y}\psi_{xy}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\hat{z}\psi_{xyz} \quad (48)$$

Making a parallelism of what we have got in this paper regarding the values of the multivector velocity U and the wavefunction ψ (reordering terms for easy comparison).

$$U=U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y}+\boldsymbol{U_{yz}\hat{x}}+\boldsymbol{U_{zx}\hat{y}}+\boldsymbol{U_{xy}\hat{z}}+\boldsymbol{U_0} \quad (35)$$

$$\psi=\hat{x}\hat{y}\hat{z}\psi_{xyz}+\hat{y}\hat{z}\psi_{yz}+\hat{z}\hat{x}\psi_{zx}+\hat{x}\hat{y}\psi_{xy}+\hat{x}\psi_x+\hat{y}\psi_y+\hat{z}\psi_z+\psi_0 \quad (49)$$

We can see that some elements could be related to the linear momentum and others to the angular momentum (whether internal or external we do not know). So, there could be a relation between the wavefunction and the multivector velocity in this sense:

$$\begin{gathered}\psi_{xyz}\to U_{xyz}\\ \psi_{yz}\to U_x\\ \psi_{zx}\to U_y\\ \psi_{xy}\to U_z\\ \psi_x\to U_{yz}\\ \psi_y\to U_{zx}\\ \psi_z\to U_{xy}\end{gathered}$$

$$\psi_0 \to U_0$$

This means that the wavefunction has a explicit relation with the current velocity multivector of the particle (the linear momentum parameters U_i and the angular momentum ones $\boldsymbol{U_{ij}}$).

Another comment is in the equation itself. We see that in the left side some elements are missing. In fact, they are the cross elements that Dirac wanted to get rid of. But could be that in certain situations they are necessary to be taken into account? Something like:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} - \hat{x}\frac{\partial^2}{\partial y \partial z} - \hat{y}\frac{\partial^2}{\partial z \partial x} - \hat{z}\frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial ?}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0$$

We should operate to see the result and to check in fact which elements are vanishing due to cross geometric products or just because the wavefunction vanishes depending on which cross partial derivatives are taken.

Also, remind that the Dirac equation was just the half of the total equation that included a reverse product:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z} - m\right)\psi\left(-\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} + \hat{y}\hat{z}\frac{\partial}{\partial x} + \hat{z}\hat{x}\frac{\partial}{\partial y} + \hat{x}\hat{y}\frac{\partial}{\partial z} - m\right) = 0$$

So probably, the total equation above should be taken into account when adding elements as some could vanish in the total sum.

Besides that, the electromagnetic potential should be added when using the Dirac Equation to calculate the Hydrogen atom for example. To calculate the electromagnetic potential, should we consider the other elements (the ones in bold) appearing in the electromagnetic field strength in (36)?

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}\hat{x}\hat{y}\hat{z}} + \boldsymbol{E_0} \qquad (36)$$

Anyhow, I will leave all this for another paper, as it was not the idea for this one.

14. Conclusions

In this paper, we have calculated the electromagnetic field strength and the Lorentz force in Geometric Algebra $Cl_{3,0.}$ And we have compared it with their equivalent in the tensor covariant formalism.

What in covariant formalism is (Lorentz force):

$$\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta \qquad (9)$$

We have converted in Geometric Algebra in:

$$\frac{dp}{d\tau} = qFU \quad (21)$$

Where:

$$\frac{dp}{d\tau}=\frac{dp_0}{d\tau}+\frac{dp_{yz}}{d\tau}\hat{x}+\frac{dp_{zx}}{d\tau}\hat{y}+\frac{dp_{xy}}{d\tau}\hat{z}+\frac{dp_x}{d\tau}\hat{y}\hat{z}+\frac{dp_y}{d\tau}\hat{z}\hat{x}+\frac{dp_z}{d\tau}\hat{x}\hat{y}+\frac{dp_{xyz}}{d\tau}\hat{x}\hat{y}\hat{z} \quad (20)$$

$$F=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}+B_x\hat{y}\hat{z}+B_y\hat{z}\hat{x}+B_z\hat{x}\hat{y} \quad (19)$$

$$U=U_{xyz}\hat{x}\hat{y}\hat{z}+U_x\hat{y}\hat{z}+U_y\hat{z}\hat{x}+U_z\hat{x}\hat{y} \quad (18)$$

Getting the following equations:

$$\frac{dp_x}{d\tau}=q\left(E_xU_{xyz}-B_yU_z+B_zU_y\right) \quad (24)$$

$$\frac{dp_y}{d\tau}=q\left(E_yU_{xyz}+B_xU_z-B_zU_x\right) \quad (25)$$

$$\frac{dp_z}{d\tau}=q\left(E_zU_{xyz}-B_xU_y+B_yU_x\right) \quad (26)$$

$$\frac{dp_{xyz}}{d\tau}=q\left(E_xU_x+E_yU_y+E_zU_z\right) \quad (30)$$

That corresponds one to one with the Covariant formalism equivalent:

$$\frac{dp_4}{d\tau}=q\left(E_xu^1+E_yu^2+E_zu^3\right) \quad (13)$$

$$\frac{dp_1}{d\tau}=q\left(-E_xu^4-B_zu^2+B_yu^3\right) \quad (14)$$

$$\frac{dp_2}{d\tau}=q\left(-E_yu^4+B_zu^1-B_xu^3\right) \quad (15)$$

$$\frac{dp_3}{d\tau}=q\left(-E_zu^4-B_yu^1+B_xu^2\right) \quad (16)$$

Taking the following equivalences:

$$u^4=U_{xyz} \qquad u^1=U_x \qquad u^2=U_y \qquad u^3=U_z \quad (18.1)$$

$$\frac{dp_{xyz}}{d\tau}=\frac{dp_4}{d\tau} \qquad \frac{dp_x}{d\tau}=-\frac{dp_1}{d\tau} \qquad \frac{dp_y}{d\tau}=-\frac{dp_2}{d\tau} \qquad \frac{dp_z}{d\tau}=-\frac{dp_3}{d\tau} \quad (32)$$

Also, we have obtained four extra equations not appearing in the classical formalism.

$$\frac{dp_{yz}}{d\tau}=q\left(-E_yU_z+E_zU_y-B_xU_{xyz}\right) \quad (27)$$

$$\frac{dp_{zx}}{d\tau}=q\left(E_xU_z-E_zU_x-B_yU_{xyz}\right) \quad (28)$$

$$\frac{dp_{xy}}{d\tau}=q\left(E_yU_x+E_xU_y-B_zU_{xyz}\right) \quad (29)$$

$$\frac{dp_o}{d\tau}=q\left(-B_xU_x-B_yU_y-B_zU_z\right) \quad (31)$$

An explanation of them is done in the body of the paper, chapter 10.

Even more, we have continued expanding the electromagnetic field strength to the eight components of $GA_{3,0.}$

$$F = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + B_x\hat{y}\hat{z} + B_y\hat{z}\hat{x} + B_z\hat{x}\hat{y} + \boldsymbol{B_{xyz}}\boldsymbol{\hat{x}\hat{y}\hat{z}} + \boldsymbol{E_0} \quad (36)$$

Where the component Electromagnetic trivector $\boldsymbol{B_{xyz}}$ is the most important of the new ones.

The same for the velocity multivector, where some angular momentum $\boldsymbol{U_{ij}}$ elements appear:

$$U = U_{xyz}\hat{x}\hat{y}\hat{z} + U_x\hat{y}\hat{z} + U_y\hat{z}\hat{x} + U_z\hat{x}\hat{y} + \boldsymbol{U_{yz}\hat{x}} + \boldsymbol{U_{zx}\hat{y}} + \boldsymbol{U_{xy}\hat{z}} + \boldsymbol{U_0} \quad (35)$$

The explanation of all the elements and the equation obtained is done in chapters 11 and 12.

Lastly, in chapter 13, an application of all the learnings of the paper to a hypothetical expanded Dirac Equation compared to the one appearing in [6] is done:

$$\left(\hat{x}\hat{y}\hat{z}\frac{\partial}{\partial t} - \hat{y}\hat{z}\frac{\partial}{\partial x} - \hat{z}\hat{x}\frac{\partial}{\partial y} - \hat{x}\hat{y}\frac{\partial}{\partial z}\right)\psi - m\psi_{even}\hat{z} + m\psi_{odd}\hat{z} = 0 \quad (47)$$

All the calculations in the paper could seem complicated. And in fact, they are burdensome but there is one important thing with Geometric Algebra $Cl_{3,0}$. There is a limit in the equations and in the number of unknowns. The limit is 8.

As we consider only 3 spatial dimensions, the total expanded degrees of freedom in whatever discipline we are working, should always be 2^3=8. This is, the scalar, the three vectors, the three bivectors and the trivector.

In fact, it is a small victory compared to the 16 unknowns you can get in $Cl_{1,3}$ (2^4) or the different number of dimensions considered in other models used to study physics.

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If you consider this helpful, do not hesitate to drop your BTC here:

bc1q0qce9tqykrm6gzzhemn836cnkp6hmel5lmz36f

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A1. Annex A1. Considering non-orthonormal basis

In the paper we have considered all the time an orthonormal basis in Euclidean metric. I will give here some hints of what we should do in case we do not have an orthonormal basis or even if we work in a non-Euclidean metric in geometric algebra.

You will find more information in the papers [2] and [5].

If the basis is orthogonal but not orthonormal, the difference is in equations (1) to (3) that now, would read:

$$\hat{x}^2 = \hat{x}\hat{x} = ||\hat{x}||^2 \qquad (A1.1)$$
$$\hat{y}^2 = \hat{y}\hat{y} = ||\hat{y}||^2 \qquad (A2.2)$$
$$\hat{z}^2 = \hat{z}\hat{z} = ||\hat{z}||^2 \qquad (A3.3)$$

Where in general the norm is different to 1. And depending on the signature of the metric could the square of the norm could be positive or negative.

So, for example, imagine a basis where:

$$\hat{x}^2 = \hat{x}\hat{x} = ||\hat{x}||^2 = g_{xx} = 3^2 \qquad (A1.4)$$
$$\hat{y}^2 = \hat{y}\hat{y} = ||\hat{y}||^2 = g_{yy} = -5^2 \qquad (A1.5)$$
$$\hat{z}^2 = \hat{z}\hat{z} = ||\hat{z}||^2 = g_{zz} = 2^2 \qquad (A1.6)$$

You can see that we have added the nomenclature g_{ii} typical for a diagonal element of the metric tensor in a non-Euclidean metric, typically in general relativity for example. In Geometric Algebra these g_{ii} are the same as the square of the norm of the basis vectors. Check [2] for more information.

Imagine we have to perform the following operation that represents whatever physics calculation in that basis/metric:

$$(2 + \hat{x})(5\hat{x}\hat{y} + 7\hat{x})$$

We will perform the product as usual:

$$(2 + \hat{x})(5\hat{x}\hat{y} + 7\hat{x}) = 10\hat{x}\hat{y} + 14\hat{x} + 5\hat{x}\hat{x}\hat{y} + 7\hat{x}\hat{x} =$$

Now, we have to apply (A1.1) to perform the calculation of the square of $\hat{x}$.

$$= 10\hat{x}\hat{y} + 14\hat{x} + 5(3^2)\hat{y} + 7(3^2) = 10\hat{x}\hat{y} + 14\hat{x} + 45\hat{y} + 63$$

As the basis is still orthogonal (but not orthonormal), if we would need to make a reversion of vectors, we would have used the equations (4) to (6) as we have done all along the paper.

But if the basis is not orthogonal? Here is where the things get more complicated. In that case, we cannot use the reverse equations (4) to (6). Instead, we have to use the following equations, to make a reversion [2]:

$$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x} \quad (A1.7)$$
$$\hat{y}\hat{z} = 2g_{yz} - \hat{z}\hat{y} \quad (A1.8)$$
$$\hat{z}\hat{x} = 2g_{zx} - \hat{x}\hat{z} \quad (A1.9)$$

Where the g_{ij} correspond to the cross component of the metric tensor between x and y in a nin-Euclidean metric. These components g_{ij} can be considered also as the scalar product of the two basis vectors $\hat{x}$ and $\hat{y}$.

In fact, an easy to demonstrate relations (A1.7) to (A1.9) is via the definition of the scalar product in Geometric Algebra. You can find this definition in [1] and [3] (2.3).

$$\hat{x} \cdot \hat{y} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

Considering the element g_{xy} of the metric tensor ans the scalar product of the two basis vectors:

$$\hat{x} \cdot \hat{y} = g_{xy} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

And now, operating:

$$g_{xy} = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}$$

$$2g_{xy} = \hat{x}\hat{y} + \hat{y}\hat{x}$$
$$2g_{xy} - \hat{y}\hat{x} = \hat{x}\hat{y}$$
$$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x}$$

So, you get the relations (A1.7) to (A1.9).

Now, imagine a non-orthonormal and non-orthogonal metric where the relations (A1.4) to (A1.6) apply and also we know that:

$$g_{xy} = 3 \quad (A1.10)$$
$$g_{yz} = 2 \quad (A1.11)$$
$$g_{zx} = 7 \quad (A1.12)$$

And we want to calculate:

$$(2\hat{y} + \hat{x})(5\hat{x}\hat{y} + 7\hat{x} + 3\hat{y}) =$$

$$(2\hat{y} + \hat{x})(5\hat{x}\hat{y} + 7\hat{x} + 3\hat{y}) = 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} + 6\hat{y}\hat{y} + 5\hat{x}\hat{x}\hat{y} + 7\hat{x}\hat{x} + 3\hat{x}\hat{y} =$$

First, we operate the squares using equation (A1.4) to (A1.6).

$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} + 6(-5^2) + 5(3^2)\hat{y} + 7(3^2) + 3\hat{x}\hat{y} =$$
$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} - 150 + 45\hat{y} + 63 + 3\hat{x}\hat{y} =$$
$$= 10\hat{y}\hat{x}\hat{y} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

Now, we reverse two vectors of the first element, so we can get a square of $\hat{y}$.But, we cannot do it as we always have done, just changing the sign. Now, we are in a non-orthogonal basis, so we have to use (A1.7) to (A1.12).

$$= 10\hat{y}\left(2g_{xy} - \hat{y}\hat{x}\right) + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

$$= 10\hat{y}(2(3) - \hat{y}\hat{x}) + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

$$= 60\hat{y} - 10\hat{y}\hat{y}\hat{x} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

Now, we use (A1.4) to (A1.6) for the square of $\hat{y}$. And we sum the elements that multiply the vector $\hat{y}$.

$$= 60\hat{y} - 10(-5^2)\hat{x} + 14\hat{y}\hat{x} - 87 + 45\hat{y} + 3\hat{x}\hat{y} =$$

$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3\hat{x}\hat{y} =$$

Now, we reverse the last element (using (A1.7) to (A1.12).), so we can sum it to the third element.

$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3\left(2g_{xy} - \hat{y}\hat{x}\right) =$$

$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 3(2(3) - \hat{y}\hat{x}) =$$

$$= 105\hat{y} + 250\hat{x} + 14\hat{y}\hat{x} - 87 + 18 - 3\hat{y}\hat{x} =$$

Now, we sum the scalars and the third and the last element.

$$= 105\hat{y} + 250\hat{x} + 11\hat{y}\hat{x} - 69 =$$

We cannot simplify more, so this would be the result. In case that for convention we should have to leave $\hat{x}\hat{y}$ instead of $\hat{y}\hat{x}$ in a certain discipline, we could have used the following equation that is another form for the equation (A1.7), to leave everything in $\hat{x}\hat{y}$ form. You can obtain the equation, just changing the side where $\hat{x}\hat{y}$ and $\hat{y}\hat{x}$ are.

$$\hat{y}\hat{x} = 2g_{xy} - \hat{x}\hat{y}$$

Another important point is the inverse of the vectors in a non-orthonormal basis. If we take (A1.1):

$$\hat{x}\hat{x} = \|\hat{x}\|^2 \qquad (A1.1)$$

And you premultiply by $\hat{x}^{-1}$both sides of the equation, you have:

$$\hat{x}^{-1}\hat{x}\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

By definition, the product of the inverse of a vector by the vector itself is 1.

$$(1)\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

$$\hat{x} = \hat{x}^{-1}\|\hat{x}\|^2$$

Now, the square of the norm is a scalar (it is a number, not a vector), so we can pass it to the other side dividing:

$$\frac{\hat{x}}{\|\hat{x}\|^2} = \hat{x}^{-1}$$

Exchanging sides:

$$\hat{x}^{-1} = \frac{\hat{x}}{\|\hat{x}\|^2} \qquad (A1.13)$$

Doing the same for the other vectors, we get:

$$\hat{x}^{-1} = \frac{\hat{x}}{\|\hat{x}\|^2} \qquad (A1.13)$$

$$\hat{y}^{-1} = \frac{\hat{y}}{\|\hat{y}\|^2} \qquad (A1.14)$$

$$\hat{z}^{-1} = \frac{\hat{z}}{\|\hat{z}\|^2} \qquad (A1.15)$$

A2. Annex A2. Time as the trivector

In this chapter I will develop a little more regarding time being the trivector. Also, how it is used when we are in a non-orthonormal basis (and/or non-Euclidean metric)

First, we will comment regarding the time vector $\hat{t}$ and its inverse $\hat{t}^{-1}$. In general, it is more practical to work and to give the original definition to $\hat{t}^{-1}$ instead of $\hat{t}$. The reason is in physics (including Quantum Mechanics) the time appears normally dividing. As in general, it is the magnitude that is used to take the derivatives. See for example equation (10) and the ones before it, in chapter 9.

So, we start defining:

$$\hat{t}^{-1} = \hat{x}\hat{y}\hat{z}$$

If we premultiply by $\hat{t}$ in both sides:

$$\hat{t}\,\hat{t}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{z}$$

By definition, the product of the inverse of a vector by the vector itself is 1.

$$1 = \hat{t}\,\hat{x}\hat{y}\hat{z}$$

Now, we postmultiply by $\hat{z}^{-1}$ both sides.

$$\hat{z}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{z}\hat{z}^{-1}$$

Again, the product of a vector by its inverse is 1.

$$\hat{z}^{-1} = \hat{t}\,\hat{x}\hat{y}$$

Now, we postmultiply by $\hat{y}^{-1}$ both sides and we operate.

$$\hat{z}^{-1}\hat{y}^{-1} = \hat{t}\,\hat{x}\hat{y}\hat{y}^{-1}$$

$$\hat{z}^{-1}\hat{y}^{-1} = \hat{t}\,\hat{x}$$

In the last step we will post multiply by $\hat{x}^{-1}$.

$$\hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{t}\,\hat{x}\hat{x}^{-1}$$

$$\hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \hat{t}$$

So,

$$\hat{t} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1}$$

In a non-orthonormal basis, we have to use the equations (A1.13) to (A1.15) to calculate the inverses:

$$\hat{t} = \hat{z}^{-1}\hat{y}^{-1}\hat{x}^{-1} = \frac{\hat{z}}{\|\hat{z}\|^2}\frac{\hat{y}}{\|\hat{y}\|^2}\frac{\hat{x}}{\|\hat{x}\|^2}$$

In an orthonormal basis, the norms are equal to 1, so we get the relation that has been commented in the paper (7.1) for orthonormal bases:

$$\hat{t} = \hat{z}\hat{y}\hat{x} = -\hat{x}\hat{y}\hat{z} = -\hat{t}^{-1}$$

One thing to comment is the time basis vector in $Cl_{1,3}$ that is commented in the literature [1][3] normally denoted as γ_0. γ_0 has positive signature and its norm is 1 so:

$$\gamma_0\gamma_0 = \|\gamma_0\|^2 = 1$$

So, its inverse is itself. We can prove it premultiplying by its inverse:

$$\gamma_0\gamma_0 = 1$$
$$\gamma_0{}^{-1}\gamma_0\gamma_0 = \gamma_0{}^{-1}$$
$$(1)\gamma_0 = \gamma_0{}^{-1}$$
$$\gamma_0 = \gamma_0{}^{-1}$$
$$\gamma_0{}^{-1} = \gamma_0$$

Our $\hat{t}$ and $\hat{t}^{-1}$ have negative signature (they are the trivector, see chapter 8 for its definition and chapter 4 to check the negative signature of the trivector).

This means, we can choose γ_0 to be $\hat{t}$ or $\hat{t}^{-1}$, what we prefer as a convention, if we keep the same definition all the time. Choosing one or another will only change the sign of $\hat{t}$ in all the subsequent equations but all of them will be coherent among them if we keep the convention in all the equations.

In the paper we have chosen to consider γ_0 to $\hat{t}^{-1}$.

Another thing I commented in Annex 3 of [5] is that time instead of being exact inverse of the spatial dimensions they could be related by a constant k that could be Ricci scalar, trace of the metric tensor, product of the diagonal elements of the metric tensor, determinant of the metric tensor… This is, a scalar related to the metric, a constant that is necessary to normalize the value of $\hat{t}$ or $\hat{t}^{-1}$ compared with the space elements.

$$\hat{t} = \frac{1}{\hat{t}^{-1}} = \frac{k}{\hat{x}\hat{y}\hat{z}} = k \cdot \hat{x}^{-1}\hat{y}^{-1}\hat{z}^{-1} = k\,\frac{\hat{x}\hat{y}\hat{z}}{||\hat{x}||^2\,||\hat{y}||^2\,||\hat{z}||^2}$$

It is important to remark, as I did in [2] and [5], that if the basis vector $\hat{t}$ is composed by the space basis vectors, it does not mean that the dimension time is not independent from the space ones. The parameter t (without hat) that multiplies the basis vector $\hat{t}$ (with hat) is completely free and independent. The dimension of time exists although its basis vector is somehow related to the space ones. In fact, in geometric algebra, having three space vectors imply the existence of 8 dimensions (scalars, 3 basis vectors, 3 bi-vectors and one pseudoscalar (the time in this approach)). You can check this in [3] for example. So, time would be just one of these 8 dimensions (the trivector/pseudoscalar) appearing from the three space dimensions.

It is somehow as the odd-grade elements of the multivector (vectors and the trivector) are the elements that we see as dimensions in our world, the three dimensions of space and the time. And the even grade (scalars and bivectors) represents other things probably related to forces or relations related to interactions. As we will see in Annex 3, the scalar probably somehow related to metric.

A3. Annex A3. Normalization of the wavefunction in Quantum Mechanics. A loss of information?

As I have commented in chapters 11 and 12, it could be that the scalar component of the electromagnetic field E_0 has an effect of escalation of all the magnitude in a frame. So, inside the local frame no change will be noted as all magnitudes (lengths, time…) change in the same proportion for all the elements of the interaction. But for a distant observer probably he could see the difference of these values compared to its own local frame.

This means, the absolute value of the magnitudes is not important in a local frame, only the relative differences between them to understand the interactions. But a distant observer could see not only the relative differences of the magnitudes in that frame but the absolute values as he can compare with its own frame values. He can see that in that far frame the things are slower or bigger than in his own frame. But the ones that are in the distant frame if they only see their own interactions, he cannot see any difference as he measures with elements inside his own frame affected also for whatever escalation is happening.

So, what does this have to do with the normalization of the wavefunction?

The standard process is to get the value of unknown constants in the wavefunction, normalizing it. So, the square of the wavefunction is always 1. And the square of the partial coefficients of the wave function have only a value between 0 and 1 representing the probability.

If the reason of normalizing is only that, it is not really necessary. You can define the probability as the square of the partial coefficient divided by the square of the wavefunction (even if it is not 1 and has whatever other value). The result will also be a number between 0 and 1 representing the probability.

When you have normalized you have lost information of the real square of the wavefunction. That you could keep there and use as denominator when you want to calculate probabilities between 0 and 1.

The answer here, would be that normally you have free constants where can select the value we want and we decide to normalize the function, so really, we have not lost information.

But if is this not really the case? Could it not be that we have a lack of equations that we still do not know (and we see that Geometric Algebra can create a lot of them) that would apply a specific value to these constants? And the square of the wavefunction instead of being always 1 have a value that represents something? For example, a kind of escalation in its own frame similar of what we have commented in chapters 11, and 12 for the scalar of the Electromagnetic Field? I keep the question open, but it is something that it should be checked in the future.