Abdelmajid Ben Hadj Salem, Richard Moeckel

Lecture Notes On Celestial Mechanics

- Elements of Central Configuration For Undergraduate Students -

Part I

Abstract: From the lectures for an advanced course on celestial mechanics which Prof. Richard Moeckel gave in Trieste in 1994 on the topic - ***Central configurations of the n-body problem*** - that was one of his favorites, I have decided to develop a part of it as an introduction course for the undergraduate students where I have added more details (gray boxes) of the proofs to be understood by undergraduate students.

It is based on the handwritten notes from the 1994 Trieste course. Part I of the notes concerns 3 chapters :

- chapter 1: introduction,
- chapter 2: the two-body problem,
- chapter 3: special solutions to the n body problem.

October 2022

Richard Moeckel (1943-2021)

Lecture Notes On Celestial Mechanics - Elements of Central Configuration For Undergraduate Students - Part I

Presented by
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Celestial Mechanics

(especially central configurations)

Lecture Notes by:

Richard Moeckel

I dedicate this work to the memory of Prof. Richard Moeckel and my beloved parents

Fig. 0.1: A Photo of Richard Moeckel (1943-2021)

CONTENTS

PREFACE

From the lectures for an advanced course on celestial mechanics which Prof. Richard Moeckel gave in Trieste in 1994 on the topic - ***Central configurations of the n-body problem*** - that was one of his favorites, I have decided to develop a part of it as an introduction course for the undergraduate students where I have added more details (gray boxes) of the proofs to be understood by undergraduate students.

It is based on the handwritten notes from the 1994 Trieste course [1]. Part I of the notes concerns 3 chapters :
- chapter 1: introduction,
- chapter 2: the two-body problem,
- chapter 3: special solutions to the n body problem.

I dedicate this first part of the monograph to the memory of Professor Richard Moeckel.

I hope that the reader will find this monograph more accessible to undergraduate students.

Tunis,
October 2022

Abdelmajid Ben Hadj Salem, Dipl.-Ing.
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CHAPTER

ONE

INTRODUCTION

Definition 1.1 *Celestial mechanics is the study of points particles in* $\mathbb{R}^3$ *moving under the influence of their mutual gravitational attraction.*

A point particle is characterized by a position:

$$\boldsymbol{q} = (X, Y, Z) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathbb{R}^3 \tag{1.1}$$

and a mass $m \in \mathbb{R}^+$. A motion of such a particle is a smooth curve $\boldsymbol{q}(t)$ where $t \in \mathbb{R}$ represents time.

Definition 1.2 *Then one defines:*

$$\begin{aligned} \textbf{\textit{Velocity :}}\ v(t) = \dot{\boldsymbol{q}}(t) = \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} \in \mathbb{R}^3 \\ \textbf{\textit{Momentum :}}\ \boldsymbol{p}(t) = mv(t) = m\dot{\boldsymbol{q}}(t) \in \mathbb{R}^3 \end{aligned} \tag{1.2}$$

Definition 1.3 *Newton's equation of motion is:*

$$\dot{\boldsymbol{p}} = \boldsymbol{F}(\boldsymbol{q}, t) \quad \text{or} \quad m\ddot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q}, t) \tag{1.3}$$

where $\boldsymbol{F}(\boldsymbol{q}, t)$ *is the force acting on the particle.*

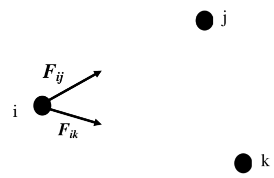
Consider n particles with positions $\boldsymbol{q}_i = \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$ and masses $m_i; i = 1, ..., n$. Introduce the notation:

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \\ \vdots \\ \boldsymbol{q}_n \end{pmatrix} \in \mathbb{R}^{3n},\ v = \dot{\boldsymbol{q}},\ \ {}_{3n}M_{3n} = \begin{pmatrix} m_1 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & m_1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & m_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & m_n & 0 & 0 \\ 0 & \cdots & \cdots & 0 & m_n & 0 \\ 0 & \cdots & \cdots & 0 & 0 & m_n \end{pmatrix} \tag{1.4}$$

$$\boxed{\boldsymbol{p} = M.v = M\dot{\boldsymbol{q}}} \tag{1.5}$$

Definition 1.4 *According to Newton, the gravitational force acting on particle i due to the presence of particle j is:*

$$\boldsymbol{F}_{ij} = \frac{m_i m_j (\boldsymbol{q}_j - \boldsymbol{q}_i)}{||\boldsymbol{q}_j - \boldsymbol{q}_i||^3} \tag{1.6}$$



and the total force on particle i is:

$$\boldsymbol{F}_i = \sum_{j \neq i} \boldsymbol{F}_{ij} \tag{1.7}$$

This can be written:

$$\boldsymbol{F}_i = \nabla_i U(\boldsymbol{q}) \tag{1.8}$$

where:

Definition 1.5 ***(Newtonian potential)***

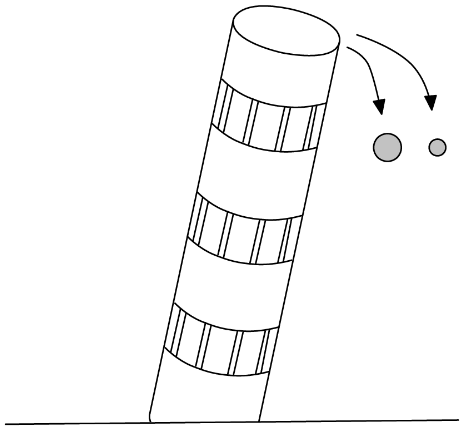
$$U(\boldsymbol{q}) = \sum_{i<j,(i,j)} \frac{m_i m_j}{||\boldsymbol{q}_i - \boldsymbol{q}_j||} \tag{1.9}$$

$$\nabla_i U(\boldsymbol{q}) = \sum_{j=1, j\neq i}^{j=n} \frac{m_i m_j (\boldsymbol{q}_j - \boldsymbol{q}_i)}{||\boldsymbol{q}_j - \boldsymbol{q}_i||^3} \tag{1.10}$$

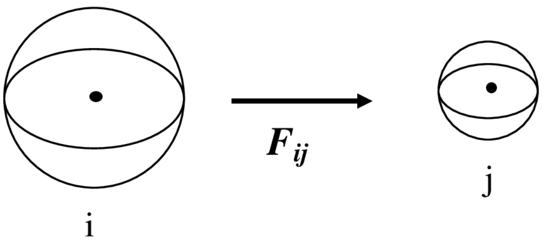
and :

$$\nabla_i = \begin{pmatrix} \frac{\partial}{\partial X_i} \\ \frac{\partial}{\partial Y_i} \\ \frac{\partial}{\partial Z_i} \end{pmatrix} \tag{1.11}$$

Note: The Newtonian inter-particle potential has two important properties. First, the factor $m_i m_j$ causes the m_i to cancel out of the equation of motion for $\boldsymbol{q}_i$. This agrees with Galileo's observation that the behaviour of a falling body is independent of its mass.



Second; the function $\dfrac{1}{||\boldsymbol{q}_i - \boldsymbol{q}_j||}$ is harmonic function of $\boldsymbol{q}_i \in \mathbb{R}^3$. It follows (Gauss) that the potential due to a spherically symmetric mass distribution is the same as if the whole mass were concentrated at the center. This is some justification for the use of point particles.



The equations of the Newtonian n-body problem are:

$$\dot{\boldsymbol{p}}_i = m_i \ddot{\boldsymbol{q}}_i = \nabla_i U(\boldsymbol{q}) \quad i = 1, ..., n \tag{1.12}$$

or:

$$\dot{\boldsymbol{p}} = M \ddot{\boldsymbol{q}} = \nabla U(\boldsymbol{q}) \tag{1.13}$$

where ∇ the gradient in $\mathbb{R}^{3n}$.

This is a system of real analytic, second order differential equations on **the configuration space**:

$$X = \mathbb{R}^{3n} \backslash \Delta$$

where:

$$\Delta = \left\{ \boldsymbol{q} : \boldsymbol{q}_i = \boldsymbol{q}_j, for\ some\ i \neq j \right\} \tag{1.14}$$

is **the collision set**.

The equation (1.13) is equivalent to the first order system:

$$\begin{aligned} \dot{\boldsymbol{q}} &= v \\ \dot{v} &= M^{-1}\nabla U(\boldsymbol{q}) \end{aligned} \tag{1.15}$$

on the **phase space** $TX = \left\{(\boldsymbol{q}, v) : \boldsymbol{q} \in X, v \in \mathbb{R}^{3n}\right\} \subset \mathbb{R}^{6n}$. ($TX$ is the tangent bundle of X).

1.1 Variational Formulations

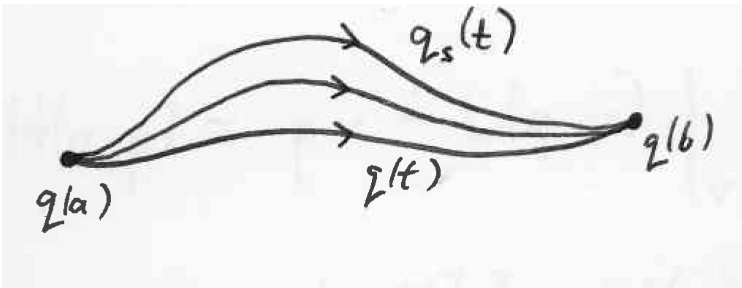
Introduce a Lagrangian function:

$$L : TX \longrightarrow \mathbb{R}/ \quad L(X, v) = \frac{1}{2}v^T M v + U(X) \tag{1.16}$$

If $\boldsymbol{q}(t)$ is a smooth curve, define the action of $\boldsymbol{q}$ on $[a, b]$:

$$Q = \int_a^b L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))dt \tag{1.17}$$

The principle of least action states that if $\boldsymbol{q}(t)$ is a motion of the Lagrangian system then Q is stationary under fixed-endpoint variations of $\boldsymbol{q}$:



$$\delta Q = \left.\frac{d}{ds}\right|_{s=0} Q(\boldsymbol{q}_s(t)) = 0 \tag{1.18}$$

The calculus of variations shows that this condition implies the Euler-Lagrange equations.

We recall the following theorem [2]:

Theorem 1.1. *(**The Euler–Lagrange Variational Principle**) A necessary condition for the functional $I(u)$ to be stationary at u is that u must satisfy the Euler–Lagrange equation :*

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \tag{1.19}$$

$$\text{with} \quad I = I(u) = \int_b^a F(x, u, u')dx, \quad u = \frac{du}{dx} \tag{1.20}$$

in $a \leq x \leq b$ with the boundary conditions $u(a) = \alpha$ and $u(b) = \beta$, where u is a twice continuously differentiable function on the interval $[a, b]$ ($u \in C^2([a, b])$), F is continuous in $x, u,$ and u, and has continuous partial derivatives with respect to u and u'.

We write the equation (1.19) in our case, we find the equation:

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \boldsymbol{q}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \tag{1.21}$$

Using the notation below, we obtain the equation (1.24):

$$\dot{\boldsymbol{p}} = \frac{\partial L}{\partial \boldsymbol{q}} = \frac{\partial L}{\partial \boldsymbol{q}}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \tag{1.22}$$

Introduce the conjugate momentum:

$$\begin{aligned} &\boldsymbol{p} = \frac{\partial L}{\partial v}(\boldsymbol{q}, v) = \left[\frac{\partial L}{\partial v_1}, \cdots, \frac{\partial L}{\partial v_{3n}}\right] \in \mathbb{R}^{3n*} \;\; (1 - forms\ or\ covectors) \\ &\boldsymbol{p}(t) = \frac{\partial L}{\partial v}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \end{aligned} \tag{1.23}$$

Then the Euler-Lagrange equation is:

$$\dot{\boldsymbol{p}}(t) = \frac{\partial L}{\partial \boldsymbol{q}}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \tag{1.24}$$

For the n-body problem Lagrangian, this becomes:

$$\dot{\boldsymbol{p}}(t) = M.\dot{v} = \nabla(U(\boldsymbol{q})) \Longrightarrow \dot{v}^T.M^T = \dot{v}^T.M = (\nabla U(\boldsymbol{q}))^T \tag{1.25}$$

because as M is a diagonal matrix, then $M^T = M$. We obtain the transpose of Newton's equation $M\dot{v} = \nabla U(\boldsymbol{q})$.

Solving the equation defining the conjugate momentum for v :

$$\dot{\boldsymbol{p}}(t) = M.\dot{v} \Longrightarrow v = M^{-1}\boldsymbol{p} \tag{1.26}$$

one can define a Hamiltonian function $H : T^*X \longrightarrow \mathbb{R}$:

$$H(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{p}.v - L(\boldsymbol{q},v)|_{v=M^{-1}\boldsymbol{p}} = \boldsymbol{p}^T.v - L(\boldsymbol{q},v)|_{v=M^{-1}\boldsymbol{p}}$$
$$H(\boldsymbol{q},\boldsymbol{p}) = v^T M^T v - \left(\tfrac{1}{2}v^T M v + U(\boldsymbol{q})\right) = \frac{1}{2}v^T M v - U(\boldsymbol{q}) \tag{1.27}$$
$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}(M^{-1}\boldsymbol{p})^T M M^{-1}\boldsymbol{p} - U(\boldsymbol{q}) = \frac{1}{2}.\boldsymbol{p}^T.M^{-1}.M.M^{-1}\boldsymbol{p} - U(\boldsymbol{q})$$
$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}.\boldsymbol{p}^T.M^{-1}.\boldsymbol{p} - U(\boldsymbol{q}) \tag{1.28}$$

as $(M^{-1})^T = M^{-1}$. Differentiating the definition of H with respect to $\boldsymbol{p}$ gives:

$$\frac{\partial H}{\partial \boldsymbol{p}} = M^{-1}\boldsymbol{p} = v = \dot{\boldsymbol{q}}$$

The Euler-Lagrange equation is:

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \tag{1.29}$$

Thus Hamilton's equations hold:

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p}) \longleftarrow (1-forms) \tag{1.30}$$
$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}(\boldsymbol{q},\boldsymbol{p}) \longleftarrow vectors\left(Note : \frac{\partial H}{\partial \boldsymbol{p}} \in (\mathbb{R}^{3n*})^* \approx \mathbb{R}^{3n}\right) \tag{1.31}$$

This implies $dH(\boldsymbol{q},\boldsymbol{p}) = \frac{\partial H}{\partial \boldsymbol{p}}.d\boldsymbol{p} + \frac{\partial H}{\partial \boldsymbol{q}}.d\boldsymbol{q} = \dot{\boldsymbol{q}}.d\boldsymbol{p} - \dot{\boldsymbol{p}}.d\boldsymbol{q} = (\dot{\boldsymbol{q}}.\dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}.\dot{\boldsymbol{q}})dt = 0$. Then:

$$H(\boldsymbol{q},\boldsymbol{p}) = constant \tag{1.32}$$

and the Hamiltonian H (1.32) is a first integral.

Since $L(\boldsymbol{q},v) = \boldsymbol{p}v - H(\boldsymbol{q},\boldsymbol{p})|_{\boldsymbol{p}=v^T M}$, the action is :

$$Q = \int_a^b [\boldsymbol{p}(t)\dot{\boldsymbol{q}}(t) - H(\boldsymbol{q}(t),\boldsymbol{p}(t))]\,dt \tag{1.33}$$

If one forgets the definition of $\boldsymbol{p}$ in terms of the Lagrangian, this can be viewed as a functional of the curve $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ in T^*X rather than the curve $\boldsymbol{q}(t)$ in X. Hamilton's equations follow from assuming this curve is stationary under variations in T^*X which fix the end points.

The variational approach facilitates changes of coordinates. Let (Q,P) denote new coordinates related to the old ones by a diffeomorphism:

$$\boldsymbol{q} = \boldsymbol{q}(Q,P) \tag{1.34}$$
$$\boldsymbol{p} = \boldsymbol{p}(Q,P) \tag{1.35}$$

Let:

$$K(Q,P) = H(\boldsymbol{q}(Q,P),\boldsymbol{p}(Q,P)) \tag{1.36}$$

and suppose that:

$$\boldsymbol{p}(Q(t),P(t)).\dot{\overline{\boldsymbol{q}(Q(t),P(t))}} = P(t)\dot{Q}(t) \tag{1.37}$$

for every curve $(Q(t), P(t))$. Another way to state this is to require equality of differential forms:

$$\boldsymbol{p}(Q,P)d\boldsymbol{q}(Q,P) = PdQ \tag{1.38}$$

or, less formally:

$$\boldsymbol{p}d\boldsymbol{q} = PdQ \tag{1.39}$$

Then the action integrals are equal on corresponding curves:

$$\int_a^b [\boldsymbol{p}(t)\dot{\boldsymbol{q}}(t) - H(\boldsymbol{q}(t), \boldsymbol{p}(t))]\, dt = \int_a^b \left[P(t)\dot{Q}(t) - K(Q(t), P(t))\right] dt \tag{1.40}$$

if: $(\boldsymbol{q}(t), \boldsymbol{p}(t)) = (\boldsymbol{q}(Q(t), P(t)), \boldsymbol{p}(Q(t), P(t)))$.

It follows that Hamilton's equations hold in the new coordinates:

$$\dot{Q}(t) = \frac{\partial K}{\partial P} \tag{1.41}$$

$$\dot{P}(t) = -\frac{\partial K}{\partial Q} \tag{1.42}$$

As an example, consider Jacobi coordinates for the 3-body problem.

Example: The 3-body problem. Let 3 particles $M_1(m_1), M_2(m_2)$ and $M_3(m_3)$. Let G be the center of mass of the three particles so we have:

$$m_1\boldsymbol{GM}_1 + m_2\boldsymbol{GM}_2 + m_3\boldsymbol{GM}_3 = 0 \Longrightarrow \boldsymbol{OG} = \frac{m_1\boldsymbol{OM}_1 + m_2\boldsymbol{OM}_2 + m_3\boldsymbol{OM}_3}{m_1 + m_2 + m_3} \tag{1.43}$$

We obtain with the notations below:

$$\overline{m} = m_1 + m_2 + m_3, \quad \bar{\boldsymbol{q}} = \boldsymbol{OG} \tag{1.44}$$

$$\bar{\boldsymbol{q}} = \frac{1}{\overline{m}}(m_1\boldsymbol{q}_1 + m_2\boldsymbol{q}_2 + m_3\boldsymbol{q}_3) \tag{1.45}$$

We verify that: $\overline{\boldsymbol{p}}d\overline{\boldsymbol{q}} + P_1dQ_1 + P_2dQ_2 = p_1d\boldsymbol{q}_1 + p_2d\boldsymbol{q}_2 + p_3d\boldsymbol{q}_3$ and: $||\boldsymbol{q}_3 - \boldsymbol{q}_1|| = ||Q_2 + \gamma_2 Q_1||; \quad ||\boldsymbol{q}_3 - \boldsymbol{q}_2|| = ||Q_2 - \gamma_1 Q_1||$. The second equation of (1.48) is verified.

Introduce new coordinates:

$$\begin{aligned}
&\overline{\boldsymbol{q}} = \frac{1}{\overline{m}}(m_1\boldsymbol{q}_1 + m_2\boldsymbol{q}_2 + m_3\boldsymbol{q}_3), \quad \textit{center of mass}, \quad \overline{m} = m_1 + m_2 + m_3 \\
&Q_1 = \boldsymbol{q}_2 - \boldsymbol{q}_1 \\
&Q_2 = \boldsymbol{q}_3 - \gamma_1\boldsymbol{q}_1 - \gamma_2\boldsymbol{q}_2 \\
&\gamma_1 = \frac{m_1}{m_1 + m_2}, \quad \gamma_2 = \frac{m_2}{m_1 + m_2}
\end{aligned} \tag{1.46}$$

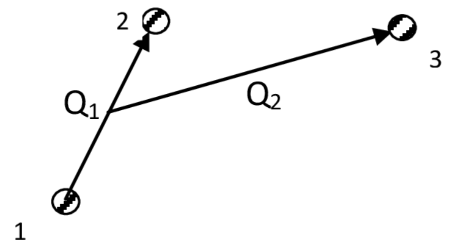


Fig. 1.1: Jacobi coordinates (Q_1, Q_2)

and **conjugate momenta**:

$$\overline{\boldsymbol{p}} = \overline{m}\dot{\overline{\boldsymbol{q}}}, \quad P_1 = \alpha \dot{Q}_1, \quad P_2 = \beta \dot{Q}_2$$
$$\alpha = \frac{m_1 m_2}{m_1 + m_2}, \quad \beta = \frac{(m_1 + m_2) m_3}{\overline{m}} \tag{1.47}$$

Then $\overline{\boldsymbol{p}} d\overline{\boldsymbol{q}} + P_1 dQ_1 + P_2 dQ_2 = p_1 d\boldsymbol{q}_1 + p_2 d\boldsymbol{q}_2 + p_3 d\boldsymbol{q}_3$ and the Hamiltonian is:

$$K(P,Q) = \frac{1}{\overline{m}}||\overline{\boldsymbol{p}}||^2 + \frac{1}{\alpha}||P_1||^2 + \frac{1}{\beta}||P_2||^2 - U(Q)$$
$$U(Q) = \frac{m_1 m_2}{||Q_1||} + \frac{m_1 m_3}{||Q_2 + \gamma_2 Q_1||} + \frac{m_2 m_3}{||Q_2 - \gamma_1 Q_1||} \tag{1.48}$$
$$P = (\overline{\boldsymbol{p}}, P_1, P_2)^T, \quad Q = (\overline{\boldsymbol{q}}, Q_1, Q_2)^T$$

1.2 Symmetries and Integrals

The Hamiltonian of the n-body problem is symmetric(invariant) under the action of the Euclidean group $Euc(3)$. An element $g \in Euc(3)$ takes the form:

$$g.X = AX + b \quad \begin{cases} A \in \Theta(3) : 3 \times 3 \ \textit{orthogonal matrix}, \ A^{-1} = A^T \ , det(A) = 1 \\ b \in \mathbb{R}^3 \ \textit{translation} \\ X \in \mathbb{R}^3 \end{cases} \tag{1.49}$$

Extend this action to $T^*X \subset \mathbb{R}^{3n} \times \mathbb{R}^{3n*}$ via:

$$g.(\boldsymbol{q}, \boldsymbol{p}) = g\left(\begin{pmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_n \end{pmatrix}, [p_1, \cdots, p_n]\right) = \left(\begin{pmatrix} g.\boldsymbol{q}_1 \\ \vdots \\ g.\boldsymbol{q}_n \end{pmatrix}, [p_1 A^T, \cdots, p_n A^T]\right) \tag{1.50}$$

Then:

$$\begin{aligned} H(g.(\boldsymbol{q},\boldsymbol{p})) &= \sum_i \frac{1}{2m_i} p_i A^T (p_i A^T)^T - \sum_{i<j} \frac{m_i m_j}{||g.\boldsymbol{q}_i - g.\boldsymbol{q}_j||} \\ &= \sum_i \frac{1}{2m_i} p_i A^T A p_i^T - \sum_{i<j} \frac{m_i m_j}{||A.(\boldsymbol{q}_i - \boldsymbol{q}_j)||} = H(\boldsymbol{q},\boldsymbol{p}) \end{aligned} \tag{1.51}$$

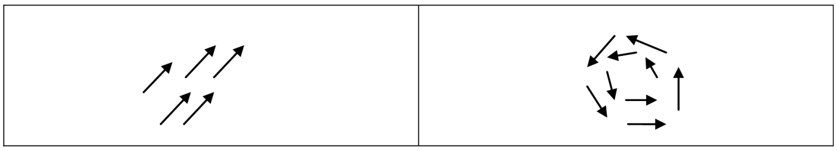
since A is orthogonal ($AA^T = I$, it verifies also $||A.X|| = ||X||$).

This action also preserves the form $\boldsymbol{p}d\boldsymbol{q}$. It follows that $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ solves the n-body problem $\iff g.(\boldsymbol{q}(t),\boldsymbol{p}(t))$ does.

Let g_s be one-parameter family of Euclidean transformations and suppose $g_0 = I$. Define a vector field:

$$\chi(\boldsymbol{q}) = \frac{d}{ds} g_s.(\boldsymbol{q})\bigg|_{s=0} \tag{1.52}$$

$$g_s X = X + s.b \qquad\qquad g_s X = \begin{pmatrix} coss & -sins & 0 \\ sins & coss & 0 \\ 0 & 0 & 1 \end{pmatrix} .X$$



Theorem 1.1 *(**E. Noether Theorem**[a]). If the system (M,L) admits the group g^α then $I = p.v$ is a first integral of the equations of motion.*

E. Noether's theorem states that if g_s is a family of symmetries of the Hamiltonian $H(\boldsymbol{q},\boldsymbol{p})$ then:

$$F(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{p}\chi(\boldsymbol{q}) \tag{1.53}$$

is an integral of Hamilton's equations, that is:

$$F(\boldsymbol{q}(t),\boldsymbol{p}(t)) = constant \tag{1.54}$$

if $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ is a solution.

[a] In this form this theorem was first stated by E. Noether in 1918. [3]

Using this, one can derive the classical integrals of the n-body problem.
Let $g_s X = X + sb$, $b \in \mathbb{R}^3$. g_s is a translation with vector translation sb. Then :

$$g_s.\boldsymbol{q} = (\boldsymbol{q}_1 + sb, \ldots, \boldsymbol{q}_n + sb)^T$$

so:

$$\chi(\boldsymbol{q}) = \frac{d}{ds} g_s.(\boldsymbol{q})\Big|_{s=0} = \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} \quad \textit{and} \quad F(\boldsymbol{q},\boldsymbol{p}) = (p_1, \cdots, p_n) \begin{pmatrix} b \\ \vdots \\ b \end{pmatrix} = (\sum_{i=1}^{n} p_i) b \tag{1.55}$$

Since this is an integral for all $b \in \mathbb{R}^3$, the quantity :

$$\overline{\boldsymbol{p}} = \sum_{i=1}^{n} p_i \in \mathbb{R}^{3*} \tag{1.56}$$

is a form-valued integral, called the total momentum. Using Noether's theorem above and from (1.53,1.54) and (1.55), we conclude:

$$\overline{\boldsymbol{p}} = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i \dot{\boldsymbol{q}}_i = \textit{constant} \tag{1.57}$$

Then (1.57) is a first integral.

The value of $\overline{\boldsymbol{p}}$ determines the motion of the center of mass:

$$\overline{\boldsymbol{q}} = \frac{1}{\overline{m}} \sum_{i=1}^{n} m_i \boldsymbol{q}_i \quad \textit{where} \quad \overline{m} = \sum_{i=1}^{n} m_i$$

Clearly $\dot{\overline{\boldsymbol{q}}} = \frac{d}{dt}\overline{\boldsymbol{q}} = \frac{1}{\overline{m}} \sum_{i=1}^{n} m_i \dot{\boldsymbol{q}}_i = \frac{1}{\overline{m}} \sum_{i=1}^{n} p_i = \frac{1}{\overline{m}} \overline{\boldsymbol{p}}$ so $\overline{\boldsymbol{q}}(t) = \overline{\boldsymbol{q}}(t_0) + \frac{1}{\overline{m}} \overline{\boldsymbol{p}}(t - t_0)$.

if $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ is any solution of the n-body problem with momentum $\overline{\boldsymbol{p}}$ then :

$$\tilde{\boldsymbol{q}}_i(t) = \boldsymbol{q}_i(t) - \overline{\boldsymbol{q}}(t), \quad \tilde{p}_i(t) = p_i(t) - \frac{m_i}{\overline{m}} \overline{\boldsymbol{p}} \tag{1.58}$$

is another solution, but with total momentum $\overline{\tilde{\boldsymbol{p}}} = \mathbf{O}$ (we verify that $\sum_i \left(p_i(t) - \frac{m_i}{\overline{m}} \overline{\boldsymbol{p}} \right) = \mathbf{O}$).

Thus one can study solutions with $\overline{\boldsymbol{p}} = \mathbf{O}$ without loss of generality.

With $\overline{\boldsymbol{p}} = \mathbf{O}$, $\overline{\boldsymbol{q}}$ becomes a vector-valued constant of motion. The solution $\tilde{\boldsymbol{q}}(t)$ has $\overline{\tilde{\boldsymbol{q}}} = \mathbf{O}$ so one may assume:

$$\overline{\boldsymbol{q}} = \overline{\boldsymbol{p}} = \mathbf{O} \tag{1.59}$$

Further integrals result from applying Noether's theorem to a one-parameter family of rotations with constant angular velocity vector $w = (w_1, w_2, w_3)^T \in \mathbb{R}^3$, $g_s X = R_s X$. Then:

$$\frac{d}{ds} R_s\Big|_{s=0} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} = W \in SO(3) \tag{1.60}$$

The action of W can be represented using cross products : $W_X = w \wedge X$ (let $X = (x, y, z)^T$, we verify that $W(X) = w \wedge X$). It follows that:

$$\chi(\boldsymbol{q}) = \begin{pmatrix} W\boldsymbol{q}_1 \\ \vdots \\ W\boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} w \wedge \boldsymbol{q}_1 \\ \vdots \\ w \wedge \boldsymbol{q}_n \end{pmatrix} \tag{1.61}$$

$$F(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{n} \boldsymbol{p}_i.(w \wedge \boldsymbol{q}_i) = \sum_{i=1}^{n} w.(\boldsymbol{q}_i \wedge \boldsymbol{p}_i) = \sum_{i=1}^{n} w^T.(\boldsymbol{q}_i \wedge \boldsymbol{p}_i) \in \mathbb{R}^3 \tag{1.62}$$

It follows that:

$$\Omega = \sum_{i=1}^{n} \boldsymbol{q}_i \wedge \boldsymbol{p}_i \in \mathbb{R}^3 \tag{1.63}$$

is a vector-valued integral. Using Noether's theorem, we find that $\Omega = constant$, then the equation (1.63) is a first integral. It is called the angular momentum.

The integrals $\overline{\boldsymbol{p}}, \overline{\boldsymbol{q}}, \Omega$ provide 3+3+3=9 constants of motion (at least if $\overline{\boldsymbol{p}} = 0$). The Hamiltonian:

$$H(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{2}\boldsymbol{p}M^{-1}\boldsymbol{p}^T - U(\boldsymbol{q}) = constant \tag{1.64}$$

is the 10th.

Integrals give rise to invariant sets in phase space. Given constants ω, h the set:

$$S_{(\omega_0,h)} = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*X : \overline{\boldsymbol{q}} = 0, \overline{\boldsymbol{p}} = 0, \Omega = \omega_0, H = h\} \tag{1.65}$$

is invariant. One expects that for most choices of ω, h this will be a $(6n-10)$-dimensional manifold. The conditions for this to be the case will now be investigated. Consider the gradients of the integrals ($\boldsymbol{p}$ will be viewed as a vector instead of a form during these computations):

$$\nabla\overline{\boldsymbol{q}} = {}_{6n}(\nabla\overline{\boldsymbol{q}})_3 = \frac{1}{m}\begin{pmatrix} m_1 I \\ \vdots \\ m_n I \\ --- \\ O \\ \vdots \\ O \end{pmatrix}, \quad \nabla\overline{\boldsymbol{p}} = {}_{6n}(\nabla\overline{p})_3 = \begin{pmatrix} O \\ \vdots \\ O \\ --- \\ I \\ \vdots \\ I \end{pmatrix} \tag{1.66}$$

$$\nabla\Omega =_{6n} (\nabla\Omega)_3 = \begin{pmatrix} -\tilde{p}_1 \\ \vdots \\ -\tilde{p}_n \\ --- \\ -\tilde{\boldsymbol{q}}_1 \\ \vdots \\ -\tilde{\boldsymbol{q}}_n \end{pmatrix}, \quad \nabla H =_{6n} (\nabla H)_1 = \begin{pmatrix} -\nabla U(\boldsymbol{q}) \\ --- \\ M^{-1}\boldsymbol{p} \end{pmatrix} \tag{1.67}$$

where:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.68}$$

$$\tilde{\boldsymbol{q}}_i = \begin{pmatrix} 0 & -Z_i & Y_i \\ Z_i & 0 & -X_i \\ -Y_i & X_i & 0 \end{pmatrix}, \quad \tilde{p}_i = \begin{pmatrix} 0 & -h_i & \eta_i \\ h_i & 0 & -\xi_i \\ -\eta_i & \xi_i & 0 \end{pmatrix} \tag{1.69}$$

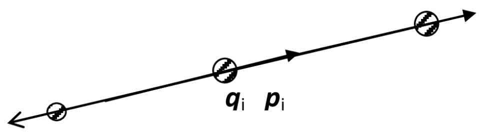
$$avec \quad \boldsymbol{q}_i = (X_i, Y_i, Z_i)^T, \quad \boldsymbol{p}_i = (\xi_i, \eta_i, h_i)^T \tag{1.70}$$

The integrals are independent at $(\boldsymbol{q}, \boldsymbol{p})$ provided the $(3n \times 10)$ matrix:

$$\left[\nabla\overline{\boldsymbol{q}} \vdots \nabla\overline{\boldsymbol{p}} \vdots \nabla\Omega \vdots \nabla H\right]$$

has rank 10 at $(\boldsymbol{q}, \boldsymbol{p})$.

It is not hard to show that the first 9 columns are independent except for points $(\boldsymbol{q}, \boldsymbol{p})$ where all the $\boldsymbol{q}_i$ and p_i are collinear. With our assumptions that $\overline{\boldsymbol{q}} = \overline{\boldsymbol{p}} = \mathbf{0}$ this situation



implies $\omega_0 = \mathbf{0}$ as well. Thus it can be ruled out by assuming $\omega_0 \neq \mathbf{0}$.

The only other way for independence to fail is if there exist constant vectors $u, v, w \in \mathbb{R}^3$ with:

$$\nabla H = \nabla\overline{\boldsymbol{q}}u + \nabla\overline{\boldsymbol{p}}v + \nabla\Omega w$$

that is:

$$-\nabla U(\boldsymbol{q}) = \frac{1}{\overline{m}} \begin{pmatrix} m_1 u \\ \vdots \\ m_n u \end{pmatrix} + \begin{pmatrix} -p_1 \wedge w \\ \vdots \\ -p_n \wedge w \end{pmatrix} \quad and \; M^{-1}\boldsymbol{p} = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_1 \wedge w \\ \vdots \\ \boldsymbol{q}_n \wedge w \end{pmatrix} \tag{1.71}$$

Summing the n 3-dimensional components of the first equation gives:

$$\mathbf{O} = u - \overline{\boldsymbol{p}} \wedge w = u \Longrightarrow u = \mathbf{O} \tag{1.72}$$

(The left side vanishes because $F_{ij} = -F_{ji}$).

Summing in the second equation with weights m_i gives :

$$\overline{\boldsymbol{p}} = \overline{m}v + \overline{m\boldsymbol{q}} \wedge w \Longrightarrow v = \mathbf{O} \tag{1.73}$$

The dependency equations become:

$$\boldsymbol{p} = -\begin{pmatrix} m_1(\boldsymbol{q}_1 \wedge w) \\ \vdots \\ m_n(\boldsymbol{q}_n \wedge w) \end{pmatrix} \tag{1.74}$$

and, substituting this into the first equation:

$$\nabla U(\boldsymbol{q}) = \begin{pmatrix} m_1(\boldsymbol{q}_1 \wedge w) \wedge w \\ \vdots \\ m_n(\boldsymbol{q}_n \wedge w) \wedge w \end{pmatrix} = -||w||^2 \begin{pmatrix} m_1 \boldsymbol{q}_1^{\perp} \\ \vdots \\ m_n \boldsymbol{q}_n^{\perp} \end{pmatrix} \tag{1.75}$$

where the superscript $\perp$ denotes orthogonal projection onto the plane, P, normal to w. It follows that $\nabla_i U \in P$ for $i = 1, ..., n$ and from this one finds $\boldsymbol{q}_i \in P$ (otherwise, consider the particle farthest from P to get a contradiction).

Thus the $\perp$ is unnecessary and we find that $\boldsymbol{q}$ satisfies an equation of the form:

$$M^{-1}\nabla U + \lambda \boldsymbol{q} = \mathbf{O} \quad \lambda \geq 0 \tag{1.76}$$

Conversely, if $\boldsymbol{q} \in X$ is a planar configuration satisfying (1.76) then one can reconstruct $w \in \mathbb{R}^3$ and $\boldsymbol{p}$ so that $(\boldsymbol{q}, \boldsymbol{p})$ is point where the 10 integrals are dependent. A configuration $\boldsymbol{q}$ of this type will be called a critical configuration. If $S_{(\omega_0,h)}$ contains no point of the form $(\boldsymbol{q}, \boldsymbol{p})$ with $\boldsymbol{q}$ a critical configuration and $\boldsymbol{p}$ the corresponding momentum, then $S_{(\omega_0,h)}$ is a smooth $(6n - 10)$-dimensional sub-manifold of T^*X.

Unfortunately, equation (1.76) is very difficult to solve. Further discussion of it will be postponed until later, where it will arise in the course of studying a very different question.

One can invoke symmetry one last time to eliminate another dimension. Note that $S_{(\omega_0,h)}$ is still invariant under orthogonal transformations which fix $\omega_0 \in \mathbb{R}^3$. In particular, there is a one parameter rotational symmetry. This means that there is a well-defined dynamical system on the quotient space $S'_{(\omega_0,h)}$, which will be a manifold of dimension $(6n - 11)$ if $S_{(\omega_0,h)}$ is a manifold of dimension $(6n - 10)$ as above.

Because the Newtonian potential is homogeneous, it is possible to obtain new solutions by scaling. If $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ is a solution in $S_{(\omega_0,h)}$ then for any constant $\sigma \neq 0$:

$$\tilde{\boldsymbol{q}}(t) = \sigma^{-2}\boldsymbol{q}(\sigma^3 t), \quad \tilde{\boldsymbol{p}}(t) = \sigma \boldsymbol{p}(\sigma^3 t) \tag{1.77}$$

is a solution in $S_{(\sigma^{-1}\omega_0,\sigma^2 h)}$. This shows that there is essentially only one parameter in the n-body problem (beside the masses) namely $||\omega_0||^2 h$. By means of this scaling one may normalize the energy h to -1,0,+1 according to sign.

Another useful rescaling allows one to normalize the masses. If $(\boldsymbol{q}(t), \boldsymbol{p}(t))$ is a solution of the n-body problem for masses $(m_1, ..., m_n)$ then for any constant $\mu \neq 0$:

$$\tilde{\boldsymbol{q}}(t) = \boldsymbol{q}(\mu t), \quad \tilde{\boldsymbol{p}}(t) = \mu^3 \boldsymbol{p}(\mu t) \tag{1.78}$$

is a solution for masses $(\mu^2 m_1, ..., \mu^2 m_n)$. It is common to normalize the masses so that:

$$\overline{m} = \sum_{i=1}^{n} m_i = 1$$

CHAPTER

TWO

THE TWO-BODY PROBLEM

For $n = 2$, the reductions described above can be used to pass from the $6n = 12$ dimensional phase space to a manifold of dimension $6n - 11 = 1$, which completely solves the problem.

$$X = \left\{(\boldsymbol{q}_1, \boldsymbol{q}_2) \in \mathbb{R}^6 : \boldsymbol{q}_1 \neq \boldsymbol{q}_2\right\}, \quad T^*X = \{(\boldsymbol{q}_1, \boldsymbol{q}_2, p_1, p_2)\} \subset \mathbb{R}^6 \times \mathbb{R}^{6*}$$

The equations of motion are:

$$\dot{\boldsymbol{q}}_1 = \frac{1}{m_1} p_1^T, \qquad \dot{\boldsymbol{q}}_2 = \frac{1}{m_2} p_2^T \tag{2.1}$$

$$\dot{p}_1^T = \frac{m_1 m_2(\boldsymbol{q}_2 - \boldsymbol{q}_1)}{||\boldsymbol{q}_2 - \boldsymbol{q}_1||^3}, \qquad \dot{p}_2^T = \frac{m_1 m_2(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{||\boldsymbol{q}_1 - \boldsymbol{q}_2||^3} \tag{2.2}$$

Scale the masses so $\bar{m} = m_1 + m_2 = 1$ and introduce new coordinates:

$$\begin{cases} \bar{\boldsymbol{q}} = m_1 \boldsymbol{q}_1 + m_2 \boldsymbol{q}_2 = \textit{ center of mass} \\ \bar{p} = p_1 + p_2 = \textit{ total momentum} \\ Q = \boldsymbol{q}_2 - \boldsymbol{q}_1 = \textit{ relative position} \\ P = \dfrac{p_2}{m_2} - \dfrac{p_1}{m_1} = \textit{ relative velocity} \end{cases} \tag{2.3}$$

then the differential equations become:

$$\dot{\bar{\boldsymbol{q}}} = \bar{p}, \quad \dot{Q} = P$$

$$\dot{\bar{p}} = 0, \quad \dot{P} = \frac{-Q}{||Q||^3}$$

Elimination of the total momentum and center of mass are accomplished by ignoring $\bar{\boldsymbol{q}}$ and $\bar{p}$, leaving a Hamiltonian system on $T^*(\mathbb{R}^3 \setminus \{0\}) \approx (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{3*}$:

$$H(Q, P) = \frac{1}{2}||P||^2 - \frac{1}{||Q||} \tag{2.4}$$

This is called the Kepler problem.

Assuming that $\bar{p} = 0$, the angular momentum is:

$$\Omega(Q,P) = m_1 m_2 Q \wedge P \tag{2.5}$$

The integral sets are (dropping the $m_1 m_2$ factor):

$$S_{(\omega_0,h)} = \left\{(Q,P) \in (\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^{3*} :\ Q \wedge P = \omega_0,\ H(Q,P) = h\right\} \tag{2.6}$$

Using the rotation symmetry, any $\omega_0 \neq 0$ could be rotated to a vertical vector . So one may assume that $\omega_0 = (0,0,K)^T$. Then equation $Q \wedge P = \omega_0$ becomes (with $Q = (Q_1, Q_2, Q_3)^T; P = (P_1, P_2, P_3)^T$):

$$\begin{array}{l} P_3Q_2 - P_2Q_3 = 0 \\ P_1Q_3 - P_3Q_1 = 0 \end{array} \Longrightarrow \begin{pmatrix} P_2 & Q_2 \\ P_1 & Q_1 \end{pmatrix} \cdot \begin{pmatrix} Q_3 \\ -P_3 \end{pmatrix} = A \begin{pmatrix} Q_3 \\ -P_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ det(A) = K = P_1Q_2 - Q_1P_2$$
$$P_2Q_1 - P_1Q_2 = K \tag{2.7}$$

If $K \neq 0$ these imply $Q_3 = P_3 = 0$, that is , the position and momentum are in the (X,Y)-plane. This is an invariant set for the equations of motion, so one can eliminate 2 more variables by ignoring Q_3, P_3. So view $(Q,P) \in T^*(\mathbb{R}^2 \backslash 0) \approx (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}^{2*}$:

$$S_{(\omega_0,h)} = \left\{(Q,P) \in (\mathbb{R}^2 \backslash \{0\}) \times \mathbb{R}^{2*} :\ P_2Q_1 - P_1Q_2 = K,\ H(Q,P) = h\right\} \tag{2.8}$$

The rest of the angular momentum reduction is easiest to accomplish in polar coordinates. Introduce new variables (reusing letter ρ):

$$r = ||Q||, \quad tg\theta = \frac{Q_2}{Q_1} \tag{2.9}$$

$$\rho = \dot{r} = \frac{P.Q}{||Q||}, \quad \Omega = Q_1P_2 - P_1Q_2 \tag{2.10}$$

The equations of motion are:

$$\dot{\theta} = \frac{\Omega}{r^2}, \quad \dot{r} = \rho \quad \dot{\Omega} = 0, \quad \dot{\rho} = -\frac{1}{r^2} + \frac{\Omega^2}{r^3} \tag{2.11}$$

From $tg\theta = Q_2/Q_1$, we obtain:

$$\left(1 + \frac{Q_2^2}{Q_1^2}\right)\dot{\theta} = \frac{Q_1\dot{Q}_2 - Q_2\dot{Q}_1}{Q_1^2} \Rightarrow (Q_1^2 + Q_2^2)\dot{\theta} = Q_1.P_2 - Q_2.P_1 = \Omega \Rightarrow$$
$$\dot{\theta} = \frac{\Omega}{Q_1^2 + Q_2^2} = \frac{\Omega}{||Q||^2} = \frac{\Omega}{r^2} \Rightarrow \boxed{\dot{\theta} = \frac{\Omega}{r^2}}$$

From $\rho = \frac{P.Q}{||Q||} = \frac{P.Q}{r}$, we obtain $\dot{\rho} = \frac{\dot{P}.Q + P.\dot{Q}}{r} - \frac{P.Q}{r^2}\dot{r}$, but $\dot{P} = -Q/||Q||^3 = -Q/r^3$, then:

$$\dot{\rho} = \frac{\dot{P}.Q + P.\dot{Q}}{r} - \frac{P.Q}{r^2}\dot{r} = -\frac{Q^2}{r^4} + \frac{P^2}{r} - \frac{(P.Q)^2}{r^3} = \frac{-1}{r^2} + \frac{||P||^2||Q||^2 - (P.Q)^2}{r^3}$$

Let φ be the angle of the two vectors P, Q, we have $P.Q = ||P||||Q||cos\varphi$, we obtain:

$$\dot{\rho} = \frac{-1}{r^2} + \frac{||P||^2||Q||^2 - ||P||^2||Q||^2cos^2\varphi}{r^3} = \frac{-1}{r^2} + \frac{||P||^2||Q||^2}{r^3}(1 - cos^2\varphi) \Rightarrow$$
$$\dot{\rho} = \frac{-1}{r^2} + \frac{||P||^2||Q||^2sin^2}{r^3} = \frac{-1}{r^2} + \frac{||P \wedge Q||^2}{r^3} = \frac{-1}{r^2} + \frac{\Omega^2}{r^3} \Rightarrow \boxed{\dot{\rho} = \frac{-1}{r^2} + \frac{\Omega^2}{r^3}}$$

As $\Omega = K = const. \Rightarrow \boxed{\dot{\Omega} = 0}$. From (2.11), we have $\boxed{\dot{r} = \rho}$.

The angular momentum equation is just $\omega = K$ and then the angle θ represents the symmetry that one would like to eliminate in passing from $S_{(\omega,h)}$ to the quotient space $S'_{(\omega_0,h)}$. Thus fixing Ω and passing to the quotient amounts to ignoring Ω and θ and setting $\Omega = K$ in the $\dot{\rho}$ equation. Thus we have the reduced Hamiltonian system:

$$H(r,\rho) = \frac{1}{2}\rho^2 + \frac{1}{2}\frac{K^2}{r^2} - \frac{1}{r} \tag{2.12}$$

From (2.4), we have:

$$H(Q,P) = \frac{1}{2}||P||^2 - \frac{1}{||Q||} = \frac{1}{2}||\dot{Q}||^2 - \frac{1}{r} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{r} = \frac{1}{2}\left(\rho^2 + \frac{K^2}{r^2}\right) - \frac{1}{r} = H(r,\rho)$$

We find the expression above.

on $T^*\mathbb{R}+ \approx \mathbb{R}^+ \times \mathbb{R}$. This one degree of freedom problem can be understood using phase portrait analysis.

The first graph concerns the function $E_1(r) = \frac{K^2}{2r^2} - \frac{1}{r}$. We obtain :

$$E_1(r) = 0 \Rightarrow r_1 = \frac{K^2}{2}$$
$$E_1'(r) = 0 \Rightarrow r_2 = K^2 \Rightarrow E_1(r_2) = -\tfrac{1}{2K^2} = minE_1(r)$$
$$lim_{r \longrightarrow +\infty}E_1(r) = 0^-; lim_{r \longrightarrow 0}E_1(r) = +\infty$$

The second graph concerns $E_2(r) = \frac{\rho^2}{2} = H(r,\rho) - E_1(r) = h - \left(\frac{K^2}{2r^2} - \frac{1}{r}\right)$

Thus the reduced level sets can be classified topologically as:

$$S'_{(\omega_0,h)} \approx \mathbb{R}, \quad h||\omega_0||^2 \geq 0 \quad or \quad h = 0, ||\omega_0|| \neq 0 \tag{2.13}$$
$$S'_{(\omega_0,h)} \approx S^1, \quad -\frac{1}{2} \leq h||\omega_0||^2 \leq 0 \tag{2.14}$$
$$S'_{(\omega_0,h)} \approx point, \quad h||\omega_0||^2 = -\frac{1}{2} \tag{2.15}$$
$$S'_{(\omega_0,h)} = \emptyset, \quad h||\omega_0||^2 \leq -\frac{1}{2} \tag{2.16}$$

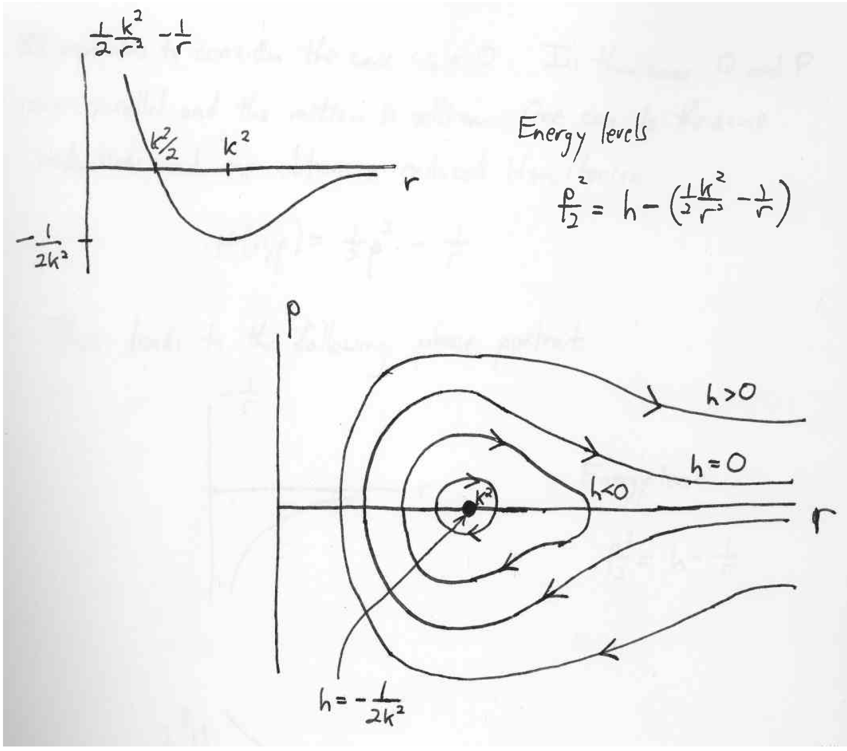


Fig. 2.1: Graphs of energy levels

The unreduced sets either $\mathbb{R} \times S^1, T^2 = S^1 \times S^1, S^1$ or $\emptyset$.

It remains to consider the case $\omega_0 = Q \wedge P = 0$. In this case Q and P are parallel and the motion is collinear. One can do the same reduction and one obtains a reduced Hamiltonian:

$$H(r,\rho) = \frac{1}{2}\rho^2 - \frac{1}{r} \tag{2.17}$$

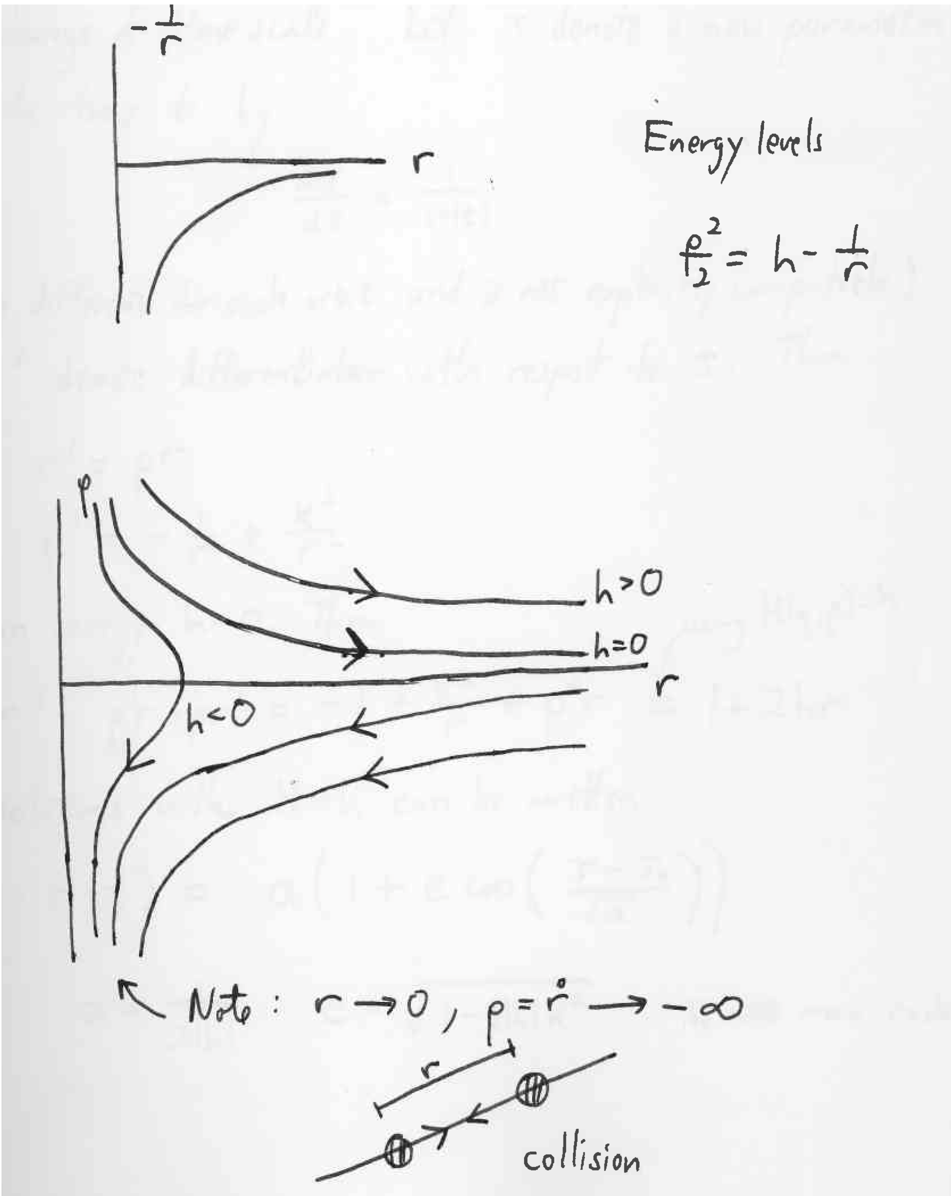
This leads to the following phase portrait. To get a more detailed understanding of the solutions one must find the time parametrization of the orbits, that is, solve:

$$\begin{aligned}\dot{r} &= \rho \\ \dot{\rho} &= -\frac{1}{r^2} + \frac{K^2}{r^3} \\ H &= \frac{1}{2}\rho^2 + \frac{K^2}{2r^2} - \frac{1}{r} = h \end{aligned} \tag{2.18}$$

We will consider only the case $h < 0$. We obtain the differential equation of the first order (2.18):

$$\dot{r}^2 + \frac{K^2}{r^2} - \frac{2}{r} = 2h \tag{2.19}$$

The above equation is not solvable in terms of elementary functions, but can nevertheless be well-understood after a change of time scale. Let τ denote a new parameter related to time t by:



$$\frac{d\tau}{dt} = \frac{1}{r(t)} \tag{2.20}$$

(this is different for each orbit and is not explicitly computable). Let $'$ denote differentiation with respect to τ. Then:

$$r' = \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \dot{r}r = \rho r \tag{2.21}$$

$$\rho' = \frac{d\rho}{d\tau} = \frac{d\rho}{dt}\frac{dt}{d\tau} = \dot{\rho}r = r\left(-\frac{1}{r^2} + \frac{K^2}{r^3}\right) = -\frac{1}{r} + \frac{K^2}{r^2} \tag{2.22}$$

Fix an energy $h < 0$. Then using $H(\boldsymbol{q}, \boldsymbol{p}) = h$:

$$r" = \rho' r + \rho r' = r\left(-\frac{1}{r} + \frac{K^2}{r^2}\right) + \rho^2 r = -1 + \frac{K^2}{r} + r\left(2h + \frac{2}{r} - \frac{K^2}{r^2}\right) \Longrightarrow$$

$$r" - 2hr = 1 \tag{2.23}$$

The above equation is a differential equation of the second order with constant coefficients, it is a known equation with the right member is not null of the type $y" + \alpha^2 y = 1$. The particular solution of (2.23) is $r = \dfrac{1}{-2h} = \dfrac{1}{2|h|} = a$ where a is the major semi-axis of the ellipse of revolution. The general solution of (2.23) is :

$$r = \frac{1}{2|h|} + Acos(\sqrt{2|h|}(\tau - \tau_0)) = a + Acos\left(\frac{\tau - \tau_0}{\sqrt{a}}\right)$$

Let τ_0 the instant of the passage of the second body at the perigee, then $r_0 = r(\tau_0) = a(1+e)$ is maximum. We obtain $r_0 = a + A = a(1+e) \Rightarrow A = ae \Longrightarrow r = a\left(1 + ecos\left(\frac{\tau-\tau_0}{\sqrt{a}}\right)\right)$. We have also $\dot{r}(\tau_0) = 0 \Rightarrow \rho(\tau_0) = 0$. From $H = h$ we get:

$$2hr_0^2 + 2r_0 - K^2 = 0 \Longrightarrow e^2 = 1 - 2|h|K^2 \Rightarrow \boxed{e = \sqrt{1 - 2|h|K^2}}$$

The solutions with $H = h$ can be written:

$$r(\tau) = a\left(1 + ecos\left(\frac{\tau - \tau_0}{\sqrt{a}}\right)\right) \tag{2.24}$$

$$a = \frac{1}{2|h|}, \quad e = \sqrt{1 - 2|h|k^2}, \quad \tau_0 \Longleftrightarrow max\, radius \tag{2.25}$$

Knowing $r(\tau)$, one can integrate $\dfrac{dt}{d\tau} = r(\tau)$ to find:

$$S = \frac{t - t_0}{a^{\frac{3}{2}}} = \sigma + esin\sigma, \quad where\ \sigma = \frac{\tau - \tau_0}{\sqrt{a}} \tag{2.26}$$

which is Kepler's equation.

From $\dfrac{dt}{d\tau} = r(t) = r(\tau) \Longrightarrow dt = r(\tau)d\tau$, then:

$$t - t_0 = \int_{\tau_0}^{\tau}\left(a + aecos\frac{\xi - \tau_0}{\sqrt{a}}\right)d\xi = a(\tau - \tau_0) + ae\int_{\tau_0}^{\tau} cos\frac{\xi - \tau_0}{\sqrt{a}}d\xi \Longrightarrow$$

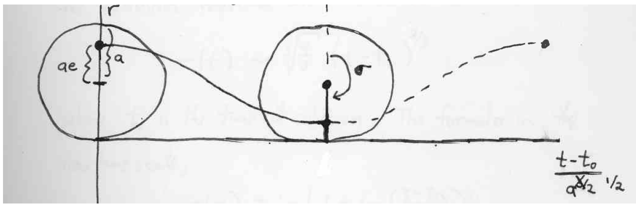
$$t - t_0 = a^{3/2}\left(\frac{\tau - \tau_0}{\sqrt{a}} + esin\frac{\tau - \tau_0}{\sqrt{a}}\right) \Longrightarrow \frac{t - t_0}{a^{3/2}} = \frac{\tau - \tau_0}{\sqrt{a}} + esin\frac{\tau - \tau_0}{\sqrt{a}} \Rightarrow$$

$$S = \frac{1}{a^{3/2}}(t - t_0) = \sigma + esin\sigma, \quad \sigma = \frac{\tau - \tau_0}{\sqrt{a}} \tag{2.27}$$

We find the Kepler equation $S = \sigma - esin\sigma$, S is called the mean anomaly.

The inability to invert this equation $S = \sigma - esin\sigma$ to get $\tau(t)$ is the only barrier to a complete formula for the solution to the Kepler problem.

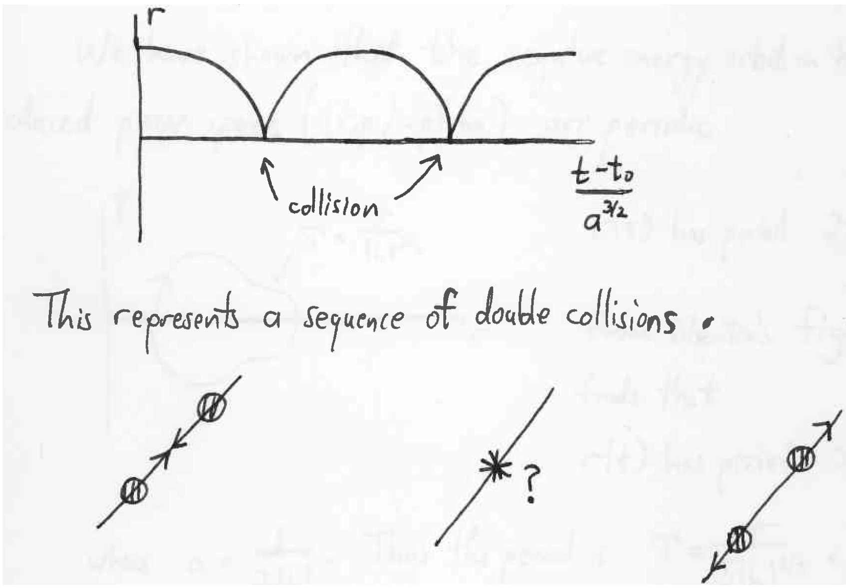
Newton found a geometrical way to construct the graph of $r(s)$. Let a circle of radius a roll without slipping along the S axis in the (S, r) plane and let σ denote the number of radians it has rolled. Then the path of a point at radius ae is the graph of $r(t)$.



Newton proves it without using any formulas!

Note: Period satisfies $\dfrac{T}{\sqrt{a}} = 2\pi a =$ circumference of circle. That implies $T = 2\pi a^{3/2}$.

When the angular momentum is $K = 0, e = 1, ea = a$ and we obtain a cycloid.



The behaviour near a collision is found to be : $r(t) \approx \sqrt[3]{\dfrac{\boldsymbol{q}}{2}}(t - t_c)^{\frac{2}{3}}$.

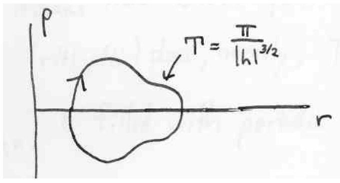
where t_c is the time of collision. The formula in the new time scale :

$$r(\tau) = a\left(1 + cos\left(\frac{\tau - \tau_0}{\sqrt{a}}\right)\right) \tag{2.28}$$

has no singularity at all. These formulas suggest that it may be possible to "regularize" the double collision singularities, that is, to eliminate them by means of changes of coordinates and time scales. It is important to regularize the whole system, rather than simply studying individual collision orbits, however.

Before proceeding with regularization, it is worthwhile to point out another curious property of the Kepler problem.

We have shown that the negative energy orbits in the reduced phase space ((r,ρ)-plane) are periodic. $r(\tau)$ has period $2\pi\sqrt{a}$. From Newton's figure one finds that $r(t)$ has period



$2\pi a^{\frac{3}{2}}$, where $a = \dfrac{1}{2|h|}$. Thus the period is $T = \dfrac{\pi}{\sqrt{a}|h|^{\frac{3}{2}}}$.

Now consider the unreduced integral set $S_{(\omega_0,h)} \approx T^2$. The angular variables describing the torus are some angle in the (r,ρ)-plane together with the variable Θ which was ignored during reduction. To understand the dynamics on the torus we need to know:

$$\Delta\Theta = \int_0^T \dot{\Theta}(t)dt = \int_0^{2\pi\sqrt{a}} \Theta'(\tau)d\tau \tag{2.29}$$

Recall that Θ satisfies :

$$\dot{\Theta} = \frac{k}{r^2} \Longrightarrow \Theta'(\tau) = \frac{k}{r(\tau)} \tag{2.30}$$

Using the formula for $r(\tau)$ and setting $\sigma = \dfrac{\tau}{\sqrt{a}}$ gives:

$$\Delta\Theta = \frac{k}{\sqrt{a}}\int_0^{2\pi} \frac{d\sigma}{1+ecos\sigma} = \frac{k}{\sqrt{a}}.\frac{2\pi}{\sqrt{1-e^2}} = \pm 2\pi \tag{2.31}$$

according to the sign of k (recall $e = \sqrt{1-2|h|k^2} = \sqrt{1-\frac{k^2}{a}}$). This means that $\Theta(t)$ completes one cycle in exactly same time that $(r(t),\rho(t))$ does, namely, $T = 2\pi a^{\frac{3}{2}}$. So the torus $S_{(\omega_0,h)}$ if filled with periodic orbits.

In the configuration space (reduced to a plane) we have :

the remarkable thing that this frequency-locking occurs on all the tori, $S_{(\omega_0,h)}$. A "generic" Hamiltonian with symmetry will reduced to tori with independent frequencies with solution curves typically dense in the torus.

We will now make some coordinate changes which will simultaneously regularize the double collision singularity and give an explanation for the persistent frequency locking.

Returning to unreduced Kepler problem, introduce the new time scale τ and let $'$ denote $\dfrac{d}{d\tau} = r\dfrac{d}{dt}$ where $r = ||Q||$. Then:

$$Q' = rP \tag{2.32}$$

$$P' = -\frac{Q}{r^2} \tag{2.33}$$

$$H = \frac{1}{2}\rho^2 - \frac{1}{r} = h < 0 \tag{2.34}$$

Starting from these and the equation $t' = r$ relating the two time scales, one finds:

$$t' = r, \quad Q' = rP \tag{2.35}$$

$$t'' = r' = PQ, \quad Q'' = r'P - \tfrac{Q}{r} \tag{2.36}$$

$$t''' = r'' = 1 + 2rh \ (using H = h), \quad Q''' = (r'' - 1)P = 2hrP = 2hQ' \tag{2.37}$$

$$t'''' = 2hr' = 2ht'', \quad Q'''' = 2hQ'' \tag{2.38}$$

Now let $X = (t'', \sqrt{a}Q'')^T = (P.Q, \sqrt{a}((PQ)P - \dfrac{Q}{r}))^T \in \mathbb{R}^4$ and $Y = X' = (1 - \dfrac{r}{a}, -\dfrac{rP}{\sqrt{a}})^T \in \mathbb{R}^4$ where $a = \dfrac{1}{2|h|}$. Then one can check that:

$$X' = Y, \quad ||X|| = \sqrt{a} \tag{2.39}$$

$$Y' = -\frac{1}{a}X, \quad ||Y|| = 1, \quad X^T.Y = 0 \tag{2.40}$$

These are the equations for the geodesic flow on the three-sphere $S^3_{\sqrt{a}} = \{||X|| = \sqrt{a}\} \subset \mathbb{R}^4$!!

We have constructed an embedding taking the five-dimensional fixed energy manifold

$$\left\{(Q_j P) \in T^*(\mathbb{R}^3 \backslash 0) : H = h = -\frac{1}{2a}\right\}$$

into the five-dimensional unit tangent bundle $T_1 S^3_{\sqrt{a}}$. The inverse can be written:

$$Q = -\sqrt{a}(X_0 \hat{Y} + (1 - Y_0)\hat{X}) \tag{2.41}$$

$$P = -\frac{\hat{Y}^T}{\sqrt{a}(1 - Y_0)} \tag{2.42}$$

where $X, Y \in \mathbb{R}^4$ have been split as:

$$X = (X_0, \hat{X})^T \tag{2.43}$$

$$Y = (Y_0, \hat{Y})^T, \quad \hat{X}, \hat{Y} \in \mathbb{R}^3 \tag{2.44}$$

This is well defined as long as the unit vector $Y \neq (1, 0, 0, 0)$. Note that as $Y \longrightarrow (1, 0, 0, 0)$ we have :

$$Q \longrightarrow 0, \quad ||P|| \longrightarrow \infty$$

which corresponds to the collision singularity.

This embedding neatly regularizes the collision singularity of the Kepler problem. The non singular Kepler orbits of energy $h = -\frac{1}{2a}$ map to great circles in $S^3_{\sqrt{a}}$ which do not pass through the "poles" on the X_0-axis!. On these orbits, the embedding is invertible. The singular Kepler orbits map to great circles which do pass through these poles. In the geodesic flow, these are non-singular orbits so they can be followed for all time. We take the pre-images to find the "regularized" orbits in $T^*(\mathbb{R}^3\backslash 0)$.

Because the geodesic flow behaves smoothly with respect to initial conditions, this regularization extends the collision orbits smoothly with respect to initial conditions (except at the actual moment of collision when $P(t)$ is undefined).

This change of coordinates also reveals a hidden symmetry of the Kepler problem. Whereas the Kepler problem is obviously $SO(3)$ invariant, the geodetic flow on $S^3_{\sqrt{a}}$ is obviously $SO(4)$ invariant. Using the inverse embedding we obtain a not so obvious $SO(4)$ symmetry for the Kepler problem. Since $SO(4)$ is 6-dimensional (versus 3 dimensions for $SO(3)$) we can use Noether's theorem to produce 6 "angular momenta". The quantity :

$$F(X,Y) = Y_0\hat{X} - X_0\hat{Y} \in \mathbb{R}^3 \tag{2.45}$$

is a vector-valued integral leading to a new integral:

$$F(Q,P) = \sqrt{a}\left(\frac{Q}{r} - \frac{Q}{a} - (P.Q)P\right) = \sqrt{a}\left(\Omega \wedge P - \frac{Q}{||Q||}\right) \in \mathbb{R}^3 \tag{2.46}$$

for the Kepler problem. $\Omega \wedge P - \frac{Q}{||Q||}$ is the Laplace vector.

This extra integral can be used to explain the frequency locking on the tori $S_{(\omega_0,h)}$ and to find the orbits shapes as well. Let :

$$\Lambda = \Omega \wedge P - \frac{Q}{||Q||} \tag{2.47}$$

the Laplace vector. Then :

$$\Lambda^T.Q = (\Omega \wedge P)^T.Q - ||Q|| = ||Q||^2 - ||Q|| = ||Q||||\Lambda||cos\varphi \tag{2.48}$$

where φ is the angle between Q and Λ. Thus $r = ||Q||$ satisfies:

$$r = \frac{||\Omega||^2}{1 + ||\Lambda||cos\varphi} \tag{2.49}$$

which is the familiar conic section.

This forces $\Delta\varphi = \Delta\Theta = 2\pi$ and makes all orbits periodic.

CHAPTER

THREE

SPECIAL SOLUTIONS OF THE N-BODY PROBLEM

The reduction of the 2-body problem is one of the few success stories in celestial mechanics. For $n \geq 3$ one cannot give a complete description of the flow. Instead one tries to prove existence or non-existence of orbits with prescribed behaviour. We will consider the simplest possible kinds of solutions first.

Euler was able to generalize the zero angular momentum collision solutions of the Kepler problem to $n = 3$. He sought solutions of the collinear 3-body problem which collapse homothetically to triple collision, that is, the size of the configuration tends to 0 while the shape remains the same. For example, if two masses are equal then there is an obvious symmetrical solution of this type with the third mass remaining motionless at the origin.

It is not obvious whether such solutions exist for non equal masses.

We will consider the following further generalization.

Definition 3.1 ***((Central configuration)*** *Consider the n-body problem and let $\boldsymbol{q}_0 \in \mathbb{R}^{3n} \backslash \Delta$.*

Call $\boldsymbol{q}_0$ a <u>central configuration</u> if there is some function $r(t) > 0$ such that :

$$\boldsymbol{q}(t) = r(t)\,\boldsymbol{q}_0 \tag{3.1}$$

is a solution to n-body problem. Thus $\boldsymbol{q}_0$ represents the constant shape of the configuration and $r(t)$ the size.

Substituting into Newton equation gives:

$$\ddot{r} M \boldsymbol{q}_0 = \nabla U(r\boldsymbol{q}_0) = r^{-2} \nabla U(\boldsymbol{q}_0) \tag{3.2}$$

using homogeneity of the potential. Multiplying both sides by $\boldsymbol{q}_0^T$ and using homogeneity again gives:

$$\ddot{r} = -\frac{\lambda}{r^2}, \quad \lambda = \frac{U(\boldsymbol{q}_0)}{\boldsymbol{q}_0^T M \boldsymbol{q}_0} \tag{3.3}$$

as the equation of motion for the size and:

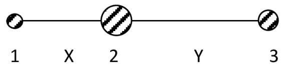
$$\boxed{M^{-1}.\nabla U(\boldsymbol{q}_0) + \lambda \boldsymbol{q}_0 = 0} \tag{3.4}$$

for the constant shape. The differential equation is the reduced zero-angular momentum Kepler problem whose solutions are represented by Newton's cycloid.

The equation (3.4) is exactly the same as the equation for critical configurations which we derived when studying integral manifolds (however, a critical configuration had to be planar).

Thus we can construct solutions of the n-body problem by first solving (3.4), then multiplying by any solution of the collinear Kepler problem. As noted ahead, (3.4) is not so easy to solve. Once again, discussion of this equation will be postponed, but to get some idea of the complexity we will look at Euler's case.

Let $X = q_2 - q_1$, $Y = q_3 - q_1$, and assume the particles are in the order $q_1 < q_2 < q_3 \in \mathbb{R}$.



Equation (3.4) reduces to:

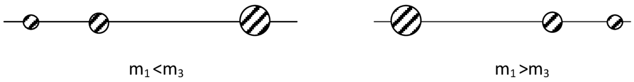
$$\lambda X - \frac{(m_1 + m_2)}{X^2} + \frac{m_3}{Y^2} - \frac{m_3}{(X+Y)^2} = 0 \tag{3.5}$$

$$\lambda Y - \frac{(m_2 + m_3)}{Y^2} + \frac{m_1}{X^2} - \frac{m_1}{(X+Y)^2} = 0 \tag{3.6}$$

Eliminating λ gives a single homogeneous equation for (X, Y). Introducing the ratio $Z = Y/X$ one finds:

$$f(Z) = (m_1+m_2)Z^5+(3m_1+2m_2)Z^4+(3m_1+m_2)Z^3-(3m_3+m_2)Z^2-(3m_3+2m_2)Z-(m_3+m_2) = 0 \tag{3.7}$$

A positive root of this equation determines a collinear 3-body central configuration. It is an exercise to show that this equation always has exactly one positive real root $\overline{Z}$. If $m_1 < m_3$ then $\overline{Z} > 1$ while for $m_1 > m_3, \overline{Z} < 1$ (smaller masses closer together).



Given any $Z > 0$ there is a plane in mass space consisting of triples (m_1, m_2, m_3) for which the Eulerian central configuration has shape given by Z.

These were the first explicit solutions of the three-body problem.

A similar problem was taken up by Lagrange who sought solutions of the form:

$$\boldsymbol{q}(t) = g(t)\boldsymbol{q}_0, \quad g(t) \in Euc(3) \tag{3.8}$$

Such a solution would have constant shape and size, changing configuration by a rigid motion. Assuming that the center of mass remains at the origin we are reduced to the case:

$$\boldsymbol{q}(t) = A(t)\boldsymbol{q}_0, \quad A(t) \in SO(3), \quad A\boldsymbol{q}_0 = \begin{pmatrix} Aq_{01} \\ \vdots \\ Aq_{0n} \end{pmatrix} \tag{3.9}$$

A configuration $\boldsymbol{q}_0 \in \mathbb{R}^{3n} \backslash \Delta$ which admits a solution of this form is called a relative equilibrium of the n-body problem.

Substituting into Newton's equation gives:

$$\ddot{A}M\boldsymbol{q}_0 = A\nabla U(\boldsymbol{q}_0) \tag{3.10}$$

Or:

$$A^{-1}\ddot{A}M\boldsymbol{q}_0 = \nabla U(\boldsymbol{q}_0)$$

Consider the antisymmetric matrix $W(t) = A^{-1}\dot{A} \in so(3)$. As usual, W can be viewed as a cross product with an angular velocity vector $w(t) \in \mathbb{R}^3$. Now:

$$\dot{W} = A^{-1}\ddot{A} - (A^{-1}\dot{A}A^{-1})\dot{A}$$

so:

$$A^{-1}\ddot{A} = \dot{W} + W^2 = \begin{pmatrix} -(w_2^2 + w_3^2) & -\dot{w}_3 + w_1w_2 & \dot{w}_2 + w_1w_3 \\ \dot{w}_3 + w_1w_2 & -(w_1^2 + w_3^2) & -\dot{w}_1 + w_2w_3 \\ -\dot{w}_2^2 + w_1w_3 & \dot{w}_1 + w_2w_3 & -(w_1^2 + w_3)^2 \end{pmatrix}$$

It will be shown that $w(t)$ is a constant vector. In other words the rotation $A(t)$ is a uniform rotation around a fixed axis. First we will show that $A^{-1}\ddot{A}$ is a constant matrix. Then since $A^{-1}\ddot{A} = \dot{W} + W^2$ is the decomposition into antisymmetric and symmetric parts, it follows that $W(t)^2$ is constant. From this one finds easily that $w(t)$ is constant.

We have :

$$A^{-1}\ddot{A}\boldsymbol{q}_{oi} = m_i^{-1}\nabla_i U(\boldsymbol{q}_i) = const \quad i = 1, 2, ..., n$$

There are 3 cases to consider according to whether the vectors $\boldsymbol{q}_{oi}, ..., \boldsymbol{q}_{on}$ span $\mathbb{R}^n$, a two-dimensional subspace $(\mathbb{R}^2, .)$, M a one-dimensional subspace $(\mathbb{R}^1, .)$. In the first case, it is immediate that $A^{-1}\ddot{A}$ is constant since it is linear and constant on some basis of $\mathbb{R}^3$. Next suppose the $\boldsymbol{q}_{oi}$ span $\mathbb{R}^2$. Then the 3×3 matrix $A^{-1}\ddot{A}$ is constant on $\mathbb{R}^2$ and maps $\mathbb{R}^2$ into itself. Hence it takes the form:

$$A^{-1}\ddot{A} = \begin{pmatrix} c_{11} & c_{12} & * \\ c_{21} & c_{22} & * \\ 0 & 0 & * \end{pmatrix}$$

Using the formula on the previous page this gives:

$$\left.\begin{matrix} w_2^2 + w_3^2 = -c_{11}, & -\dot{w}_3 + w_1w_2 = c_{12} \\ w_1^2 + w_3^2 = -c_{22}, & \dot{w}_3 + w_1w_3 = c_{21} \end{matrix}\right\} \Longrightarrow \begin{cases} w_1^2 - w_2 = c_{11} - c_{22} \\ w_1w_2 = c_{12} + c_{21} \end{cases}$$
$$-\dot{w}_2^2 + w_1w_3 = \dot{w}_1 + w_2w_3 = 0$$

Hence w_1 and w_2 are constant and $w_1w_3 = w_2w_3 = 0$.

It follows that:

$$A^{-1}\ddot{A} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} = const$$

as required.

Finally, if the $q_{0i} \in \mathbb{R}$ are collinear, one assume $w(0) = (w_{01}, 0, w_{02})^T$. Then the initial velocities $\dot{\boldsymbol{q}}_i(0) = w(0) \wedge \boldsymbol{q}_{0i}$ all lie in the plane $\mathbb{R}^2$, so the motion remains in $\mathbb{R}^2$ for all time. It follows that $v(t) = (0, 0, w_3(t))$ and so:

$$A^{-1}\ddot{A} = \begin{pmatrix} -w_3^2 & -\dot{w}_3 & 0 \\ \dot{w}_3 & -w_3^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.11}$$

Since this must map $\mathbb{R}^1$ to itself we have $\dot{w}_3 = 0$, $w_3 = const$ as required.

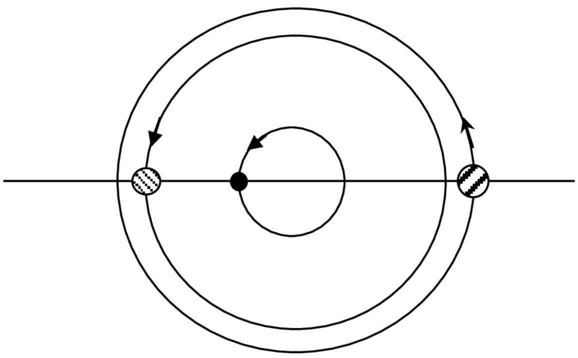
Now that we know $w(t) = w_0 = const.$, we may assume without loss of generality that $w_0 = (0, 0, k)^T$. Then in all cases:

$$A^{-1}\ddot{A} = \begin{pmatrix} -k^2 & 0 & 0 \\ 0 & -k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

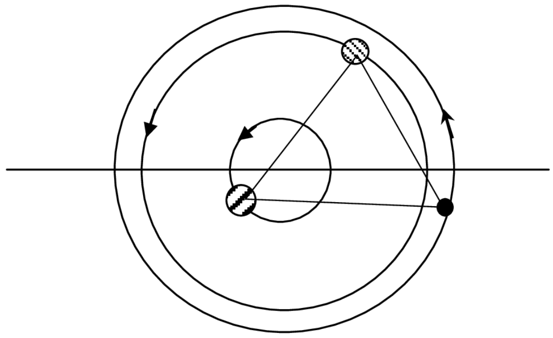
From $A^{-1}\ddot{A} = \dfrac{1}{m_i}\nabla_i U(\boldsymbol{q})$ it follows that the mutual accelerations are all planar and so in fact $\boldsymbol{q}_{0i} \in \mathbb{R}^2$; $i = 1, \cdots, n$. Thus the first case considered above is impossible. Given that $\boldsymbol{q}_{0i} \in \mathbb{R}^2$, we can replace $A^{-1}\ddot{A}$ by $-k^2 I$ to find that $\boldsymbol{q}_0$ satisfies:

$$M^{-1}\nabla U(\boldsymbol{q}_0) + \lambda \boldsymbol{q}_0 = O; \quad \lambda = k^2 > 0 \tag{3.12}$$

Thus a relative equilibrium is planar solution of (3.12), just like a critical configuration. Given a relative equilibrium, we get a periodic orbit of the n-body problem which rotates rigidly about the center of mass with period $\dfrac{2\pi}{\sqrt{\lambda}}$. For example, the collinear central configurations of Euler are certainly planar, so there are periodic orbits of the form:



Lagrange found that the equilateral triangle is a relative equilibrium of the three-body problem for all choices of the masses (this is easy for equal masses). We will prove this later. For now we just take note of Lagrange's periodic orbits:



One could introduce a uniformly rotating coordinates system in which such orbits appear fixed. This is the reason for the term "relative equilibrium" (it is easy to see that the n-body problem has no real equilibria!).

Combining the ideas of Euler and Lagrange one could look for more general "homographic" solutions:

$$\boldsymbol{q}(t) = r(t)A(t)\boldsymbol{q}_0, \quad r(t) > 0, A(t) \in SO(3)$$

Substitution into Newton's equation gives:

$$r^2(\ddot{r}I + 2\dot{r}W + rA^{-1}\ddot{A})\boldsymbol{q}_0 = M^{-1}\nabla U(\boldsymbol{q}_0) \tag{3.13}$$

where $W = A^{-1}\dot{A}$. A central configuration leads to a solution with $A(t) = I, W(t) = \mathbf{O}$ while a relative equilibrium leads to a solution with $r(t) = 1$. The matrix on the left is:

$$B(t) = r^2 \begin{pmatrix} \ddot{r} - r(w_2^2 + w_3^2) & -r\dot{w}_3 - 2\dot{r}w_3 + rw_1w_2 & r\dot{w}_3 + 2\dot{r}w_3 + rw_1w_3 \\ r\dot{w}_3 + 2\dot{r}w_3 + rw_1w_3 & \ddot{r} - r(w_1^2 + w_3^2) & -r\dot{w}_1 - 2\dot{r}w_1 + rw_2w_3 \\ -r\dot{w}_3 - 2\dot{r}w_2 + rw_1w_3 & r\dot{w}_1 + 2\dot{r}w_1 + rw_2w_3 & \ddot{r} - r(w_1^2 + w_2^2) \end{pmatrix} \tag{3.14}$$

where $w(t) \in \mathbb{R}^2$ is the angular velocity vector associated to the antisymmetric matrix $W(t)$. Assuming the $\boldsymbol{q}_{0i}$ span $\mathbb{R}^2$ or $\mathbb{R}^3$ (non-collinear case), the restriction of B to $\mathbb{R}^2$ must be constant so :

$$B = \begin{pmatrix} c_{11} & c_{12} & * \\ c_{21} & c_{22} & * \\ * & * & * \end{pmatrix}$$

Then:

$$\left.\begin{array}{l} c_{22} - c_{11} = r^3w_1^2 - r^3w_2^2 \\ c_{12} + c_{21} = r2^3w_1w_2 \end{array}\right\} \Longrightarrow r^{3/2}w_1 \text{ and } r^{3/2}w_2 \quad \text{are constant}$$

It will be shown that, in fact, $w_1(t) = w_2(t) = 0$. In the case where $\boldsymbol{q}_{0i}$ span $\mathbb{R}^3$ one can assume without loss of generality, that $w_1(0) = w_2(0) = 0$ and then $r^{3/2}w_i(t) = r^{3/2}w_i(0) = 0, i = 1, 2$.

The planar case needs more work since the rotational freedom is already used up in putting the particle in $\mathbb{R}^2$. In this case the matrix entries $B_{31} = B_{32} = 0$ since $B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Using $r^{3/2}w_i = const$, one can put these equations into the form:

$$\begin{pmatrix} \frac{1}{2}\dot{r} & rw_3 \\ rw_3 & -\frac{1}{2}\dot{r} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Either $w_1 = w_2 = 0$ or $\dot{r} = w_3 = 0$. The latter is the relative equilibrium problem considered above so again $w(t)$ is at the required form. In the collinear the same trick as for relative equilibrium reduces as to the same form.

Now that we have $w(t) = (0, 0, w_3(t))^T$, it follows that:

$$A(t) = \begin{pmatrix} cos\theta(t) & -sin\theta(t) & 0 \\ sin\theta(t) & cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $w_3(t) = \dot{\theta}(t)$. It follows that:

$$B(t) = r^2 \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 & -r\ddot{\theta} - 2\dot{r}\dot{\theta} & 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} & \ddot{r} - r\ddot{\theta}^2 & 0 \\ 0 & 0 & \ddot{r} \end{pmatrix}$$

One can use the angular momentum integral to show that the off diagonal entries are 0. Namely, for a solution of this type:

$$\boldsymbol{p}_i = m_i\dot{\boldsymbol{q}}_i = m_i\dot{r}A\boldsymbol{q}_{oi} + m_ir\dot{A}\boldsymbol{q}_{oi} = m_iA(\dot{r}\boldsymbol{q}_{oi} + rw \wedge \boldsymbol{q}_{oi})$$

So:

$$\Omega = \sum_{i=1}^{n}(rAq_{oi} \wedge A(\dot{r}\boldsymbol{q}_{oi} + rw \wedge \boldsymbol{q}_{oi}) = r^2A\sum_{i=1}^{n}\boldsymbol{q}_{oi} \wedge (w \wedge \boldsymbol{q}_{oi})$$

The third component is:

$$\Omega_3 = r^2\dot{\theta}\left(\sum_{i=1}^{n} m_i(X_{oi}^2 + Y_{oi}^2)\right) = I_3r^2\dot{\theta}$$

Now $I_3 \neq 0$ so $r^2\dot{\theta} = c$ for some constant c. Differentiation shows that the off diagonal terms of $B(t)$ vanish.

Now in all these cases, $B(t)\boldsymbol{q}_0 = M^{-1}\nabla U(\boldsymbol{q}_0)$ implies that the entry $B_{11}(t)$ is constant. But $B_{11} = B_{22}$. Thus there is a constant λ such that:

$$B(t) = r^2 \begin{pmatrix} \ddot{r} - r\dot{\theta}^2 & -r\ddot{\theta} - 2\dot{r}\dot{\theta} & 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} & \ddot{r} - r\ddot{\theta}^2 & 0 \\ 0 & 0 & \ddot{r} \end{pmatrix} = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & * \end{pmatrix}$$

The equations:

$$\begin{cases} r^2\dot{\theta} = c \\ \ddot{r} - r\dot{\theta} = \dfrac{-\lambda}{r^2} \end{cases} \tag{3.15}$$

are exactly the polar coordinate version of the Kepler problem!

In the case where $\boldsymbol{q}_{oi}$ span $\mathbb{R}^3$, the entire matrix $B(t)$ is constant and we have:

$$r^2\ddot{r} = c_{33}, \quad r^3\dot{\theta}^2 = c'$$

Since we already have $r^2\dot{\theta} = c$, the last equations show that either $c = c' = \dot{\theta} = 0$ or $r(t) = const$. The first case is the central configuration problem discussed above. We well show the second is impossible. It implies $\ddot{r} = 0$ so the lower right entry $B_{33} = 0$. Thus $B : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and it follows from $B(t)\boldsymbol{q}_0 = M^{-1}\nabla U(\boldsymbol{q}_0)$ that $\boldsymbol{q}_0$ is planar, a contradiction. Thus the only homographic solutions with non-planar configurations are the homothetic collapse solutions of central configurations.

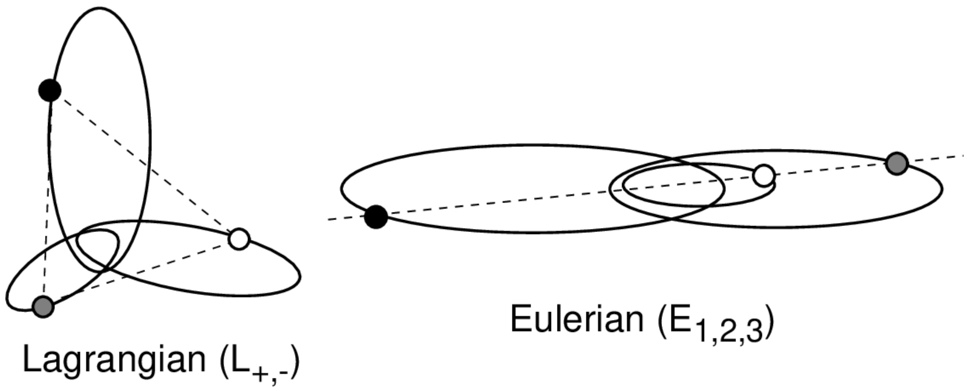
In the planar (and collinear) cases we have $B(t)\boldsymbol{q}_0 = -\lambda\boldsymbol{q}_0$ so, once again:

$$M^{-1}\nabla U(\boldsymbol{q}_0) + \lambda\boldsymbol{q}_0 = \mathbf{O} \tag{3.16}$$

Given any planar solution of (3.16) and any solution $r(t), \theta(t)$ of the Kepler problem we have a solution of the n-body problem with:

$$\boldsymbol{q}_i(t) = r(t)\begin{pmatrix} cos\theta(t) & -sin\theta(t) & 0 \\ sin\theta(t) & cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{q}_{0i}$$

Choosing elliptical periodic solutions and using Euler and Lagrange solutions of (3.16) we get [4]:



The configuration always maintains the same shape but changes in size and orientation as the particles move around their ellipses.

CHAPTER
FOUR

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LIST OF FIGURES

LIST OF TABLES